Barker College

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**Student Number**

# Year 12, 2020 HSC Mathematics Advanced Assessment Task 1 (28 Marks)
# In-class Component:

## Sequences and Series

## Series and Finance

Staff Involved: 9:40AM

**ESP AXD* ALY JZT**

**ARP RAS RJW AYG LMD** WEDNESDAY 12 FEBRUARY

145 copies

General Instructions:

- Working time – 45 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided separately
- In Questions 14 – 15, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale

Section I – 13 marks

- Attempt Questions 1 - 13

Section II – 15 marks

- Attempt Questions 14 - 15

**Questions 1 to 13 are multiple choice questions. Circle the correct answer.**

**1.** The first three terms of an arithmetic sequence are 3, 7, and 11. What is the 15th term of this sequence?

| | | | |
|---|---|---|---|
| A | 59 | B | 63 |
| C | 465 | D | 495 |

**2.** Which expression is a term of the geometric series $3x - 6x^2 + 12x^3 - \cdots$?

| | | | |
|---|---|---|---|
| A | $3072x^{10}$ | B | $-3072x^{10}$ |
| C | $3072x^{11}$ | D | $-3072x^{11}$ |

**3.** Given the arithmetic sequence $28, 25, 22, 19,$ …. calculate the sum of the first 12 terms.

| | | | |
|---|---|---|---|
| A | $138$ | B | 276 |
| C | $-138$ | D | 198 |

**4.** $3, x, 75$ are the first three terms of an **arithmetic** sequence.
$3, y, 75$ are the first three terms of a **geometric** sequence.
Find $x$ and $y$:

| | | | |
|---|---|---|---|
| A | $x = \pm 39$<br>$y = \pm 15$ | B | $x = \pm 39$<br>$y = 15$ |
| C | $x = 39$<br>$y = 15$ | D | $x = 39$<br>$y = \pm 15$ |

**5**. If $a, b, c$ are part of an arithmetic sequence then the relationship between $a, b,$ and $c$ is:

| | | | |
|---|---|---|---|
| A | $2b = 2a + 3c$ | B | $2a = b + c$ |
| C | $2b = a + c$ | D | $2c = a + c$ |

**6**. Find the limiting sum for $1 + 0.1 + 0.01 + \cdots$

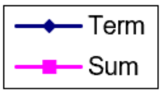
| | | | |
|---|---|---|---|
| A | $\frac{9}{10}$ | B | $\frac{10}{11}$ |
| C | $\frac{10}{9}$ | D | $\frac{11}{10}$ |

**7**. The only time an arithmetic series has a limiting sum is when the:

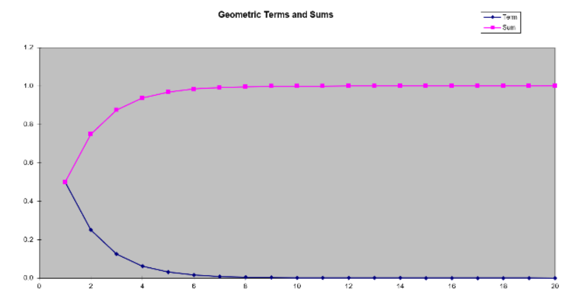
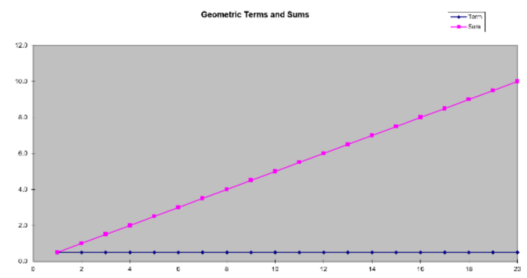
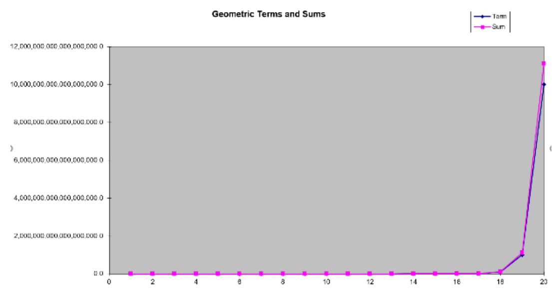
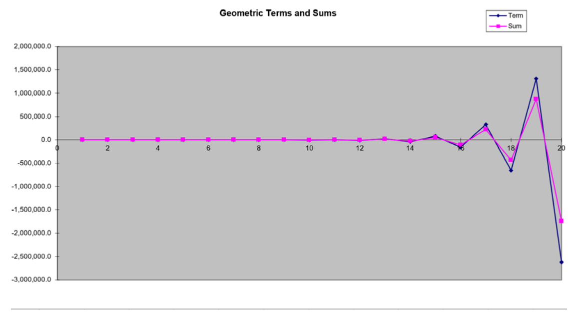
| | | | |
|---|---|---|---|
| A | first term= $1$<br>difference= $-1$ | B | first term= $0$<br>difference= $0$ |
| C | first term= $1$<br>difference= $1$ | D | first term= $0$<br>difference= $1$ |

**8**. The following graphs were produced using the spreadsheet Geometric Series.xls from the Investigation.

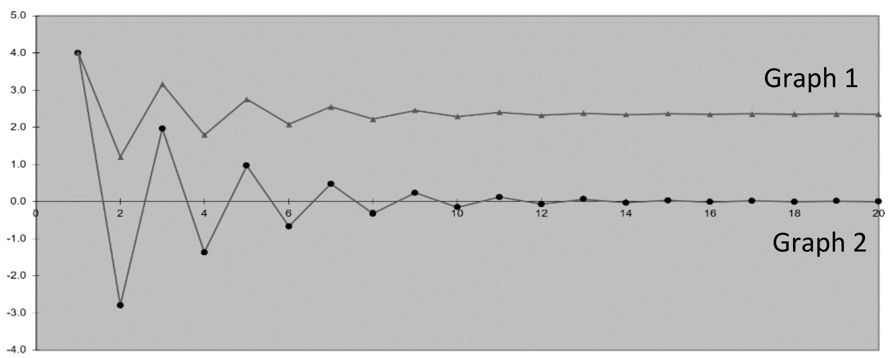
The key and heading given are: **Geometric Terms and Sums**




Which of the following 'curves' is formed by both the **terms** and the **sum of the terms** of a geometric series which has a limiting sum:

| | | | |
|---|---|---|---|
| A |   | B |   |
| C |   | D |   |

**9**. The following graph was produced using the spreadsheet from the Investigation.




One of the graphs above shows the **terms of a series** and the other graph shows the **sum of the terms** of the same series.
Is this series arithmetic or geometric?
Which of the graphs shows the sum of the terms of the series?

| | | | |
|---|---|---|---|
| A | Series: Arithmetic<br>Graph: 1 | B | Series: Arithmetic<br>Graph: 2 |
| C | Series: Geometric<br>Graph: 1 | D | Series: Geometric<br>Graph: 2 |

**10.** An arithmetic sequence starts with 15 and ends with 75 and has 11 terms. Evaluate the related series (ie the sum of the sequence).

| | | | |
|---|---|---|---|
| A | 420 | B | 210 |
| C | 495 | D | 480 |

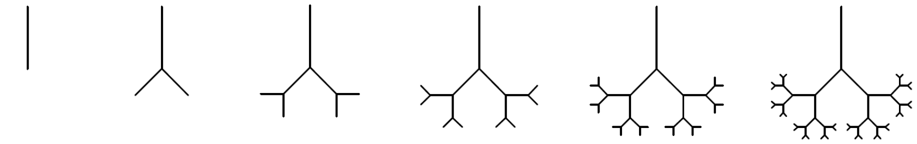
**11**. The series $(x+1)+3(x+1)^2+9(x+1)^3+\cdots$ has a limiting sum. Find the range of values for $x$.

| | | | |
|---|---|---|---|
| A | $-\frac{4}{3}\leq x\leq -\frac{2}{3}$ | B | $-\frac{4}{3}< x< -\frac{2}{3}$ |
| C | $\frac{2}{3}\leq x\leq \frac{4}{3}$ | D | $\frac{2}{3}< x< \frac{4}{3}$ |

**12**. Part of the fractal root problem in the investigation is shown below:

Fractals can be used as a model of coastlines, the surface of the lungs, the system of arteries or of ferns.

This problem models the roots of a tree.



Stage 0 Stage 1 Stage 2 Stage 3 Stage 4 Stage 5

The initial root (at Stage 0) is vertical and has length 1. At Stage 1, from the end of the initial root, two branches, each half as long, grow at right angles to each other, each a 45° rotation from the initial root. This process is repeated at the end of each root, from Stage 2 onwards.

| Stage | Number of new roots | Total number of roots | Length of each new root | Total length of all roots |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 2 | 3 | $\frac{1}{2}$ | 2 |
| 2 | 4 | 7 | $\frac{1}{4}$ | 3 |
| 3 | | | | |

At Stage 3 the **Total number of roots** and the **Length of each new root** are:

| | | | |
|---|---|---|---|
| A | $9; \frac{1}{6}$ | B | $15; \frac{1}{8}$ |
| C | $16; \frac{1}{8}$ | D | $11; \frac{1}{8}$ |

**13.** Continuing with the fractal root problem in the investigation as shown above.

Let ***h*= total depth of the root system**. Which expression summarises the total **width** of the root system?

| | | | |
|---|---|---|---|
| A | $h-1$ | B | $h+1$ |
| C | $2(h-1)$ | D | $2(h+1)$ |

**End of multiple choice questions**

**Questions 14 and 15 are written responses**

**14.** In the investigation, a home loan scenario was considered in the *Paying off a Home Loan* article by Jim Stamell.

(i) Using your understanding of loan repayments **explain** the trend shown in Fig 3 ie. why is more interest paid off per repayment at the beginning of the loan than at the end of the loan? **2**

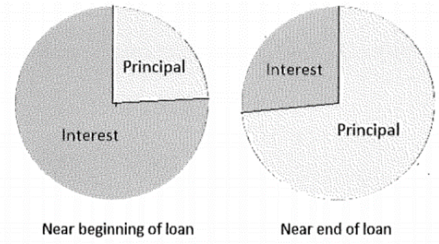



FIG. 3. COMPARING THE PROPORTION OF EACH REPAYMENT THAT GOES TOWARDS PAYING INTEREST AND REDUCING THE PRINCIPAL LOAN AMOUNT

(ii) Fig 2 shows the amount owed on a loan with the standard repayments. Ian Trest makes repayments greater than the standard repayments. Explain the impact of this on the loan and sketch the new 'curve' on the graph. **2**

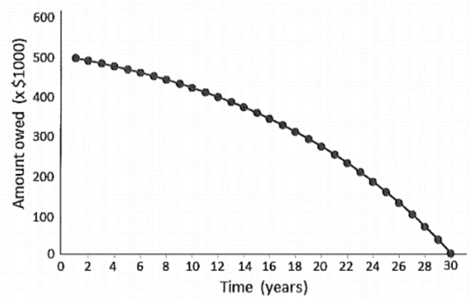



FIG. 2. PROGRESSIVE AMOUNT OWED OVER THE PERIOD OF THE LOAN

(iii) Fig 2 shows the amount owed on a loan with the standard repayments. Owen Munny makes a single lump sum repayment 10 years into the loan. Explain the impact of this on the loan and sketch the new 'curve' on the graph. **2**

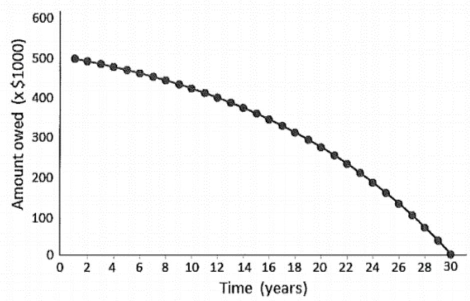



FIG. 2. PROGRESSIVE AMOUNT OWED OVER THE PERIOD OF THE LOAN

**15**. One year ago Sarah borrowed \$350 000 to buy a house. The interest rate was 9% per annum, compounded monthly. She agreed to repay the loan in 25 years with equal monthly repayments of \$2937.

(i) Calculate how much Sarah owed after her first monthly repayment. **1**

(ii) Sarah has just made her 12th monthly repayment. She now owes \$346 095. The interest rate now decreases to 6% per annum, compounded monthly.

The amount, $\$A_n$, owing on the loan after the $nth$ monthly repayment is now calculated using the formula $\$A_n = 346\ 095 \times 1.005^n - 1.005^{n-1}M - \cdots - 1.005M - M$ where $\$M$is the monthly repayment, and $n = 1,2, \dots ,288$. (Do NOT prove this formula.)

Calculate the monthly repayment if the loan is to be repaid over the remaining 24 years (288 months). **3**

(iii) Sarah chooses to keep her monthly repayments at \$2937. Use the formula in part (ii) to calculate how long it will take her to repay the \$346 095. **3**

(iv) How much will Sarah save over the term of the loan by keeping her monthly repayments at \$2937, rather than reducing her repayments to the amount calculated in part (ii)? **2**

**End of paper**

**Barker College**

| S | O | L | U | T | I | O | N |
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Student Number

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**Questions 1 to 13 are multiple choice questions. Circle the correct answer.**

**1.** The first three terms of an arithmetic sequence are 3, 7, and 11. What is the 15th term of this sequence?

| | | | |
|---|---|---|---|
| A | 59 | B | 63 |
| C | 465 | D | 495 |

$a = 3$ $d = 4$

$n = 15$

$T_{15} = 3 + (15 - 1)4$

$= 59$

**2.** Which expression is a term of the geometric series $3x - 6x^2 + 12x^3 - \cdots$?

| | | | |
|---|---|---|---|
| A | $3072x^{10}$ | B | $-3072x^{10}$ |
| C | $3072x^{11}$ | D | $-3072x^{11}$ |

$a = 3x$

$r = -2x$

$T_{11} = (3x)(-2x)^{11-1}$

$= 3x(1024x^{10})$

$= 3072x^{11}$

**3.** Given the arithmetic sequence 28, 25, 22, 19, .... calculate the sum of the first 12 terms.

| | | | |
|---|---|---|---|
| A | 138 | B | 276 |
| C | $-138$ | D | 198 |

$a = 28$ $d = -3$

$n = 12$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{12} = \frac{12}{2}[2 \times 28 + 11 \times (-3)]$

$= 138$

**4.** $3, x, 75$ are the first three terms of an **arithmetic** sequence.
$3, y, 75$ are the first three terms of a **geometric** sequence.
Find $x$ and $y$:

| | | | |
|---|---|---|---|
| A | $x = \pm 39$<br>$y = \pm 15$ | B | $x = \pm 39$<br>$y = 15$ |
| C | $x = 39$<br>$y = 15$ | D | $x = 39$<br>$y = \pm 15$ |

$$x = \frac{3+75}{2} = 39$$

$$y = \pm\sqrt{3 \times 75} = \pm 15$$

**5**. If $a, b, c$ are part of an arithmetic sequence then the relationship between $a, b,$ and $c$ is:

| | | | |
|---|---|---|---|
| A | $2b = 2a + 3c$ | B | $2a = b + c$ |
| C | $2b = a + c$ | D | $2c = a + c$ |

$$b - a = c - b$$

$$2b = c + a$$

$$2b = a + c$$

**6**. Find the limiting sum for $1 + 0.1 + 0.01 + \cdots$

| | | | |
|---|---|---|---|
| A | $\frac{9}{10}$ | B | $\frac{10}{11}$ |
| C | $\frac{10}{9}$ | D | $\frac{11}{10}$ |

$a = 1$

$r = 0 \cdot 1$

$$S_\infty = \frac{1}{1 - 0 \cdot 1} = \frac{10}{9}$$

**7**. The only time an arithmetic series has a limiting sum is when the:

| | | | |
|---|---|---|---|
| A | first term= 1<br>difference= $-1$ | B | first term= 0<br>difference= 0 |
| C | first term= 1<br>difference= 1 | D | first term= 0<br>difference= 1 |

**8**. The following graphs were produced using the spreadsheet Geometric Series.xls from the Investigation.

The key and heading given are:


Which of the following ‘curves’ is formed by both the **terms** and the **sum of the terms** of a geometric series which has a limiting sum:

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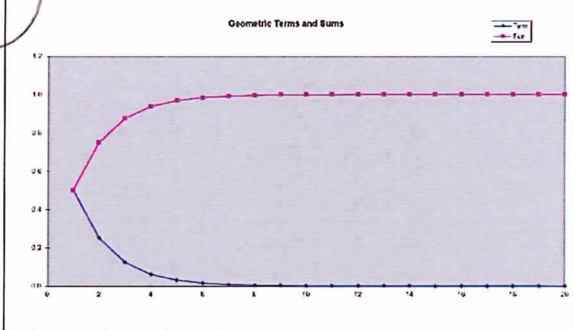
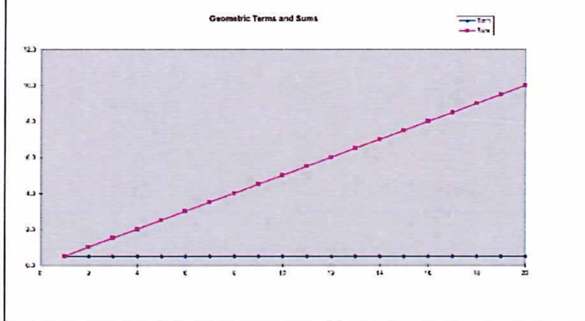
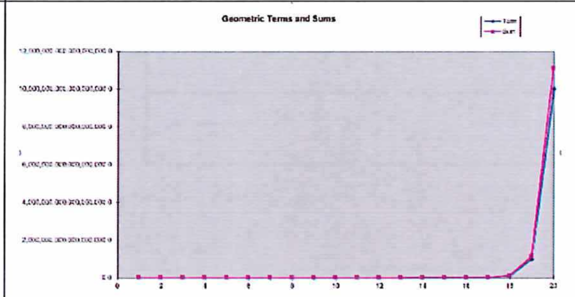
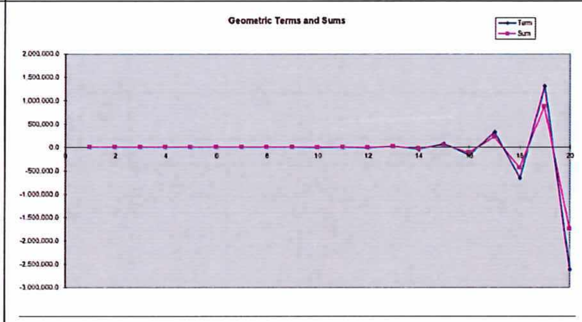
| A |  | B |  |
| C |  | D |  |

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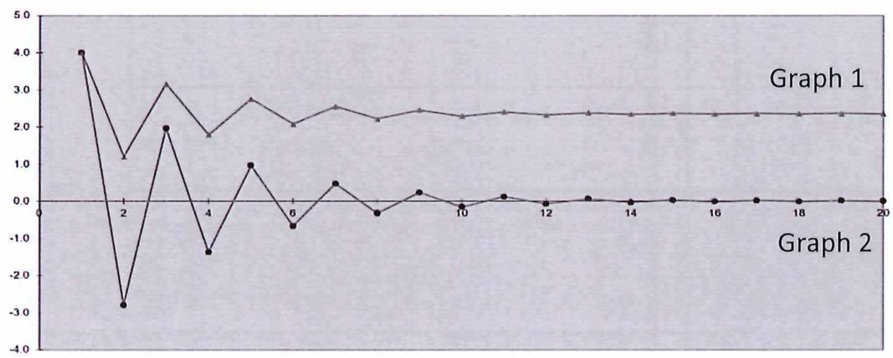


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Is this series arithmetic or geometric?
Which of the graphs shows the sum of the terms of the series?

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| C | Series: Geometric<br>Graph: 1 | D | Series: Geometric<br>Graph: 2 |

**10.** An arithmetic sequence starts with 15 and ends with 75 and has 11 terms. Evaluate the related series (ie the sum of the sequence).

| | | | |
|---|---|---|---|
| A | 420 | B | 210 |
| C | 495 | D | 480 |

$a = 15$

$l = 75$

$n = 11$

$$S_n = \frac{n}{2}(a+l)$$

$$= \frac{11}{2}(15+75)$$

$$= 495$$

**11**. The series $(x+1) + 3(x+1)^2 + 9(x+1)^3 + \cdots$ has a limiting sum. Find the range of values for $x$.

| | | | |
|---|---|---|---|
| A | $-\frac{4}{3} \leq x \leq -\frac{2}{3}$ | B | $-\frac{4}{3} < x < -\frac{2}{3}$ |
| C | $\frac{2}{3} \leq x \leq \frac{4}{3}$ | D | $\frac{2}{3} < x < \frac{4}{3}$ |

$$r = \frac{3(x+1)^2}{x+1}$$

$$r = 3(x+1)$$

for limiting sum to exist $-1 < r < 1$

$$-1 < 3(x+1) < 1$$

$$-1 < 3x + 3 < 1$$

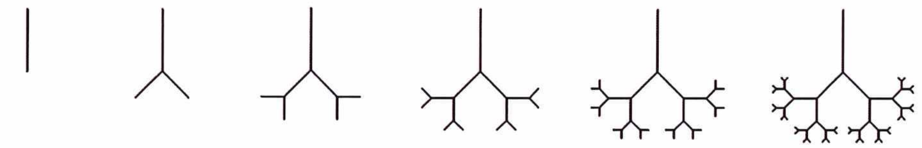
$$-4 < 3x < -2$$

$$-\frac{4}{3} < x < -\frac{2}{3}$$

**12**. Part of the fractal root problem in the investigation is shown below:

Fractals can be used as a model of coastlines, the surface of the lungs, the system of arteries or of ferns.

This problem models the roots of a tree.



Stage 0 Stage 1 Stage 2 Stage 3 Stage 4 Stage 5

The initial root (at Stage 0) is vertical and has length 1. At Stage 1, from the end of the initial root, two branches, each half as long, grow at right angles to each other, each a 45° rotation from the initial root. This process is repeated at the end of each root, from Stage 2 onwards.

| Stage | Number of new roots | Total number of roots | Length of each new root | Total length of all roots |
|---|---|---|---|---|
| 0 | $2^0$ 1 | 1 | 1 | 1 |
| 1 | $2^1$ 2 | 3 | $\frac{1}{2}$ | 2 |
| 2 | $2^2$ 4 | 7 | $\frac{1}{4}$ | 3 |
| 3 | $2^3$ 8 | $(2^{n+1}-1)$ 15 | $\left(\frac{1}{2^n}\right)$ $\frac{1}{8}$ | $(n+1)$ 4 ← not required |

At Stage 3 the **Total number of roots** and the **Length of each new root** are:

| | | | |
|---|---|---|---|
| A | $9; \frac{1}{6}$ | B | $15; \frac{1}{8}$ |
| C | $16; \frac{1}{8}$ | D | $11; \frac{1}{8}$ |

**13.** Continuing with the fractal root problem in the investigation as shown above.

Let ***h*= total depth of the root system**. Which expression summarises the total **width** of the root system?

| | | | |
|---|---|---|---|
| A | $h-1$ | B | $h+1$ |
| C | $2(h-1)$ | D | $2(h+1)$ |

$w = 2(h-1)$

**End of multiple choice questions**

**15.** One year ago Sarah borrowed \$350 000 to buy a house. The interest rate was 9% per annum, compounded monthly. She agreed to repay the loan in 25 years with equal monthly repayments of \$2937.

(i) Calculate how much Sarah owed after her first monthly repayment. **1**

(b) (i) $P = 350\,000$

$$r = \frac{9}{12}\% = 0.0075$$

$$A_1 = 350\,000(1.0075) - 2937$$
$$= \$349\,688.$$

(ii) Sarah has just made her 12th monthly repayment. She now owes \$346 095. The interest rate now decreases to 6% per annum, compounded monthly.

The amount, $\$A_n$, owing on the loan after the $nth$ monthly repayment is now calculated using the formula $\$A_n = 346\,095 \times 1.005^n - 1.005^{n-1}M - \cdots - 1.005M - M$ where $\$M$is the monthly repayment, and $n = 1,2, \ldots, 288$. (Do NOT prove this formula.)

Calculate the monthly repayment if the loan is to be repaid over the remaining 24 years (288 months). **3**

(ii) $A_{12} = 346\,095$

$$r = \frac{6}{12}\% = 0.005$$

$$n = 24 \times 12 = 288$$

$$A_{288} \equiv 0$$

Substituting into the give formula and rearranging,

$$A_{288} = 346\,095 \times 1.005^{288} - M\left(1 + 1.005 + \ldots + 1.005^{287}\right)$$

$$\therefore 0 = 346\,095(1.005)^{288} - \frac{M\left(1.005^{288} - 1\right)}{0.005}$$

$$\frac{M\left(1.005^{288} - 1\right)}{0.005} = 346\,095(1.005)^{288}$$

$$\therefore M = \frac{346\,095(1.005)^{288}(0.005)}{1.005^{288} - 1}$$
$$= \$2270.31 \text{ (nearest cent).}$$

**Questions 14 and 15 are written responses**

**14.** In the investigation, a home loan scenario was considered in the *Paying off a Home Loan* article by Jim Stamell.

(i) Using your understanding of loan repayments **explain** the trend shown in Fig 3 ie. why is more interest paid off per repayment at the beginning of the loan than at the end of the loan? **2**

At the beginning of the loan the amount owing is greater, ∴ the interest charged is greater. When the standard repayment is made more of this pays off interest than at the end of the loan.

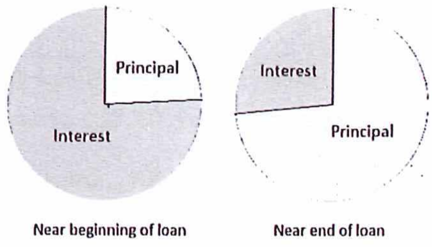


FIG. 3. COMPARING THE PROPORTION OF EACH REPAYMENT THAT GOES TOWARDS PAYING INTEREST AND REDUCING THE PRINCIPAL LOAN AMOUNT

(ii) Fig 2 shows the amount owed on a loan with the standard repayments. Ian Trest makes repayments greater than the standard repayments. Explain the impact of this on the loan and sketch the new 'curve' on the graph. **2**

Paying more than the standard repayments means the amount owing will decrease more quickly. Hence the loan will be paid off sooner. Hence the line is steeper.

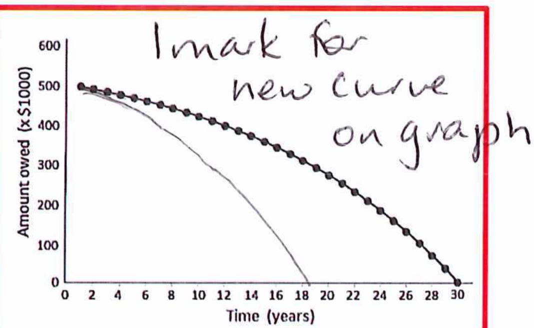


FIG. 2. PROGRESSIVE AMOUNT OWED OVER THE PERIOD OF THE LOAN

(iii) Fig 2 shows the amount owed on a loan with the standard repayments. Owen Munny makes a single lump sum repayment 10 years into the loan. Explain the impact of this on the loan and sketch the new 'curve' on the graph. **2**

A single lump sum at 10 years has a sudden impact upon the graph. There will be a sudden drop ie. vertical line and then a steeper gradient. If the repayments remain the same as before the new curve will be steeper.

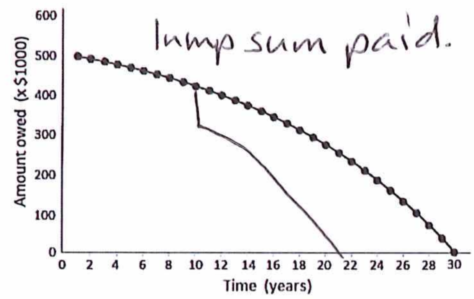


FIG. 2. PROGRESSIVE AMOUNT OWED OVER THE PERIOD OF THE LOAN

(iii) Sarah chooses to keep her monthly repayments at \$2937. Use the formula in part (ii) to calculate how long it will take her to repay the \$346 095. **3**

(iii) $M = 2937, A_n = 0$

From (ii),

$$A_n = 346\,095(1.005)^n - \frac{M(1.005^n - 1)}{0.005}$$

$$\therefore 0 = 346\,095(1.005)^n - \frac{2937(1.005^n - 1)}{0.005}$$

$$\frac{2937(1.005^n - 1)}{0.005} = 346\,095(1.005)^n$$

$$2937(1.005)^n - 2937 = 1730.475(1.005)^n$$

$$(1.005)^n(2937 - 1730.475) = 2937$$

$$(1.005)^n = 2.4342\ldots$$

$$n \ln 1.005 = \ln 2.4342\ldots$$

$$n = \frac{\ln 2.4342\ldots}{\ln 1.005}$$

$$\doteqdot 178.373\,317\,5$$

$\therefore$ It will take 179 months to repay the \$346 095.

(iv) How much will Sarah save over the term of the loan by keeping her monthly repayments at \$2937, rather than reducing her repayments to the amount calculated in part (ii)? **2**

(iv) Total repayments for (ii)

$\doteqdot 288 \times \$2270.31$

$= \$653\ 849.28$

Total repayments for (iii)

$\doteqdot 178.373\ 317\ 5 \times \$2937$

$= \$523\ 882.43$ (nearest cent)

$\therefore$ Saving over term of loan

$= \$653\ 849.28 - \$523\ 882.43$

$= \$129\ 966.85$.

**End of paper**