Barker College

Student Number

Year 12, 2021 HSC Mathematics Advanced Assessment Task 1 (26 Marks)
In-class Component:

Sequences and Series

Series and Finance

Staff Involved: 8:40 AM

PDJ PJR ARP DZP

DXC LAK* LZM AYG RJW WEDNESDAY 10 FEBRUARY

145 copies

General Instructions:

- Working time – 45 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided separately
- In Questions 11 - 12, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale

Section I – 10 marks

- Attempt Questions 1 - 10
- Circle the most correct answer

Section II – 16 marks

- Attempt Questions 11 - 12
- Write you answers in the spaces provided

Questions 1 to 10 are multiple choice questions. Circle the correct answer.

1. The first three terms of an arithmetic sequence are 12, 9, and 6. What is the 11th term of this sequence?

A	-21	B	-18
C	$708\ 588$	D	$-2\ 123\ 764$

2. Calculate the sum of the arithmetic series $103 + 110 + 117 + \ldots + 152$.

A	1275	B	2040
C	612	D	1020

3. $15, x, 135$ are the first three terms of an **arithmetic** sequence.
$15, y, 135$ are the first three terms of a **geometric** sequence.
Find x and y:

A	$x = \pm 75$ $y = \pm 45$	B	$x = 75$ $y = \pm 45$
C	$x = \pm 75$ $y = 45$	D	$x = 75$ $y = 45$

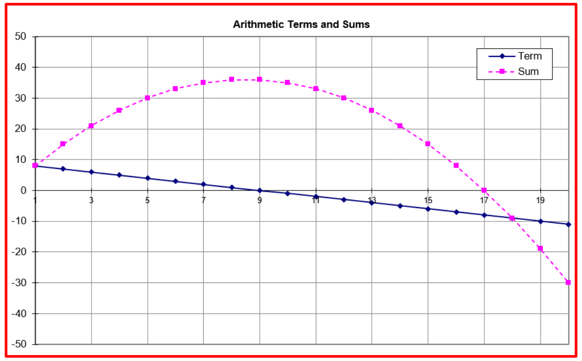
4. Find the limiting sum for $5 + 2 + \frac{4}{5} + \cdots$

A	$\frac{25}{3}$	B	$\frac{3}{25}$
C	$\frac{7}{25}$	D	$\frac{25}{7}$

5. The recurring decimal $0.\dot{2}\dot{7}$ can be converted to a fraction by evaluating the limiting sum of a geometric series that has:

A	first term= $\frac{2}{10}$ ratio= $\frac{7}{10}$	B	first term= $\frac{27}{10}$ ratio= $\frac{1}{10}$
C	first term= $\frac{27}{100}$ ratio= $\frac{1}{100}$	D	first term= $\frac{27}{100}$ ratio= $\frac{1}{10}$

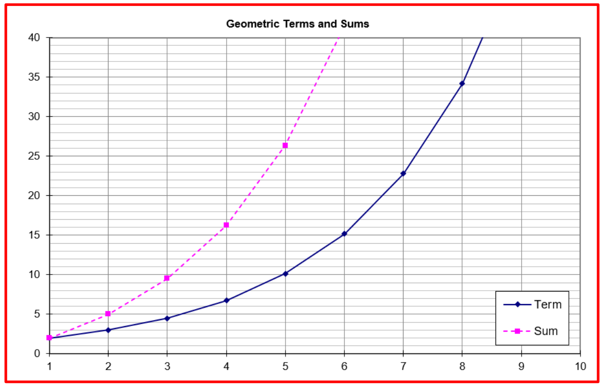
6. The following graph was produced using the spreadsheet **Arithmetic Series.xls** from the Investigation.



What could you say about the series represented by this curve?

A	$a > 0 \text{ and } d > 0$	B	$a < 0 \text{ and } d > 0$
C	$a > 0 \text{ and } d < 0$	D	$a < 0 \text{ and } d < 0$

7. The following graph was produced using the **Geometric series.xls** spreadsheet from the Investigation.



Given that the first term in the series is 2, what is the common ratio?

A	ratio $= 3$	B	ratio $= 5$
C	ratio $= \frac{3}{2}$	D	ratio $= \frac{5}{2}$

8. For an arithmetic series, the sum of the first ten terms, $S_{10} = 212$, and $S_1 = 14$. What is the value of the common difference in this series?

A	1.44	B	1.6
C	2.1	D	3.2

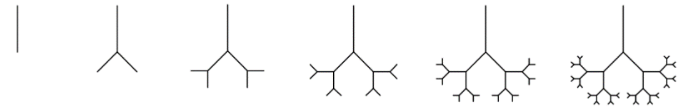
9. If $x, x+3, x+9, \ldots$ are the first three terms in a **geometric progression**, find x and the common ratio.

A	$x = 1,$ $ratio = 3$	B	$x = 3,$ $ratio = 3$
C	$x = 3,$ $ratio = 2$	D	$x = 9,$ $ratio = \frac{4}{3}$

10. Part of the fractal root problem in the investigation is shown below:

Fractals can be used as a model of coastlines, the surface of the lungs, the system of arteries or of ferns.

This problem models the roots of a tree.



Stage 0	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5

The initial root (at Stage 0) is vertical and has length 1. At Stage 1, from the end of the initial root, two branches, each half as long, grow at right angles to each other, each a 45° rotation from the initial root. This process is repeated at the end of each root, from Stage 2 onwards.

a) Complete this table for the pattern above.

Stage	Number of new roots	Total number of roots	Length of each new root	Total length of all roots
0	1	1	1	1
1	2	3	$\frac{1}{2}$	2
2	4	7	$\frac{1}{4}$	3
3				

Consider the root system at Stage 5. If an ant were to walk along the length of the roots, following one path until the end, how far would it travel, given that the length of Stage 0 is 1 unit.

A	$1\frac{28}{32}\ units$	B	$1\frac{29}{32}\ units$
C	$1\frac{30}{32}\ units$	D	$1\frac{31}{32}\ units$

End of multiple choice questions

Questions 11 and 12 are written responses.

11. In the investigation, a home loan scenario was considered in the *Paying off a Home Loan* article by Jim Stamell.

(i) Using your understanding of loan repayments **explain** the trend shown in Fig 4 i.e. why is a higher proportion of each repayment made up of interest at the beginning of the loan than at the end of the loan? **2**

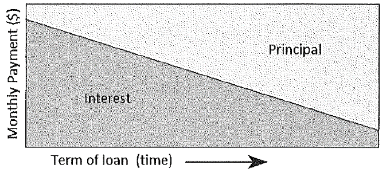


FIG. 4. SIMPLISTIC DIAGRAM SHOWING HOW INTEREST CONSTITUTES A LESSER PROPORTION OF THE MONTHLY REPAYMENT AS TIME PROGRESSES

(ii) Fig 2 shows the amount owed on a loan with the standard repayments. Ian Trest makes repayments greater than the standard repayments. Explain the impact of this on the loan and sketch the new 'curve' on the graph. **2**

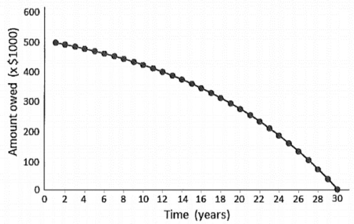


FIG. 2. PROGRESSIVE AMOUNT OWED OVER THE PERIOD OF THE LOAN

(iii) Fig 2 shows the amount owed on a loan with the standard repayments. Owen Munny makes a single lump sum repayment 10 years into the loan. Explain the impact of this on the loan and sketch the new 'curve' on the graph. **2**

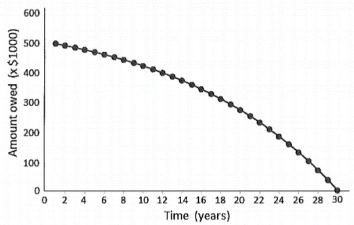


FIG. 2. PROGRESSIVE AMOUNT OWED OVER THE PERIOD OF THE LOAN

12. Janice borrowed \$425 000 to buy a house. The interest rate was 7.5% per annum, compounded monthly. She agreed to repay the loan in 35 years with equal monthly repayments of \$2866.

(i) Calculate how much Sarah owed after her first monthly repayment. **2**

(ii) Calculate how much Sarah owed after her second monthly repayment. **2**

(iii) After making her repayment at the end of 2 **years**, Janice decides that she would rather pay the loan off over the next 26 years (from this point). The amount, $\$A_n$, owing on the loan after the nth monthly repayment, is now calculated using the formula:

$$\$A_n = 419587 \times 1.00625^n - 1.00625^{n-1}M - \cdots - 1.00625M - M$$

where $\$M$ is the monthly repayment, and $n = 1, 2, \ldots, 312$. (Do NOT prove this formula.)

Calculate the monthly repayment if the loan is to be repaid over the remaining 26 years (312 months). **3**

(iv) Janice gets a new job and chooses to increase her monthly repayments to \$3500. Use the formula in part (iii) to calculate how long it will take her to repay the \$419587. **3**

End of paper

Questions 1 to 10 are multiple choice questions. Circle the correct answer.

1. The first three terms of an arithmetic sequence are 12, 9, and 6. What is the 11th term of this sequence?

A	-21	B	-18
C	$708\ 588$	D	$-2\ 123\ 764$

$T_n = a + (n-1)d$

$n = 11$
$a = 12$
$d = -3$

$\therefore T_{11} = 12 + (11-1) \times -3$
$= 12 - 30$
$= -18$

2. Calculate the sum of the arithmetic series $103 + 110 + 117 + \ldots + 152$.

A	1275	B	2040
C	612	D	1020

$a = 103$
$d = 7$
$T_n = 152 = 103 + (n-1)7$
$49 = 7(n-1)$
$7 = n - 1$
$\therefore n = 8$

$S_8 = \frac{8}{2}(103 + 152)$
$= 1020$

3. $15, x, 135$ are the first three terms of an **arithmetic** sequence.
$15, y, 135$ are the first three terms of a **geometric** sequence.
Find x and y:

A	$x = \pm 75$ $y = \pm 45$	B	$x = 75$ $y = \pm 45$
C	$x = \pm 75$ $y = 45$	D	$x = 75$ $y = 45$

$x - 15 = 135 - x$
$2x = 150$
$\therefore x = 75$

$\frac{y}{15} = \frac{135}{y}$
$y^2 = 2025$
$\therefore y = \pm 45$

$a = 5$

$r = \frac{2}{5}$

4. Find the limiting sum for $5 + 2 + \frac{4}{5} + \cdots$

A	$\frac{25}{3}$	B	$\frac{3}{25}$
C	$\frac{7}{25}$	D	$\frac{25}{7}$

$\therefore S_\infty = \frac{5}{1-\frac{2}{5}} = \frac{5}{\frac{3}{5}} = \frac{25}{3}$

5. The recurring decimal $0.\dot{2}\dot{7}$ can be converted to a fraction by evaluating the limiting sum of a geometric series that has:

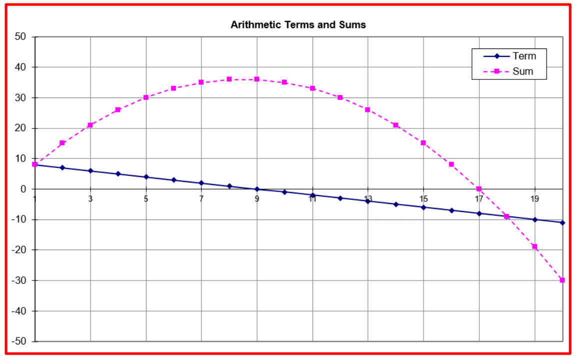
A	first term= $\frac{2}{10}$ ratio= $\frac{7}{10}$	B	first term= $\frac{27}{10}$ ratio= $\frac{1}{10}$
C	first term= $\frac{27}{100}$ ratio= $\frac{1}{100}$	D	first term= $\frac{27}{100}$ ratio= $\frac{1}{10}$

$0.\dot{2}\dot{7} = 0.27272727\ldots$

$= \frac{27}{100} + \frac{27}{10000} + \frac{27}{1000000} + \ldots$

$\therefore a = \frac{27}{100}, \quad r = \frac{1}{100}$

6. The following graph was produced using the spreadsheet **Arithmetic Series.xls** from the Investigation.



T_n starts positive, so $a > 0$

T_n is decreasing which implies $d < 0$.

What could you say about the series represented by this curve?

A	$a > 0$ *and* $d > 0$	B	$a < 0$ *and* $d > 0$
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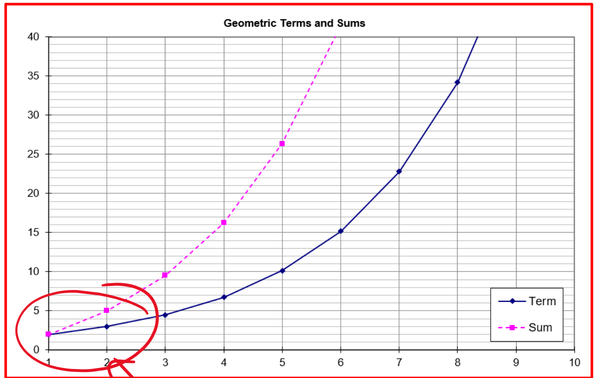

Given that the first term in the series is 2, what is the common ratio?

Compare term 1 & Term 2.

A	ratio = 3	B	ratio = 5
C	ratio = $\frac{3}{2}$	D	ratio = $\frac{5}{2}$

8. For an arithmetic series, the sum of the first ten terms, $S_{10} = 212$, and $S_1 = 14$. What is the value of the common difference in this series?

A	1.44	B	1.6
C	2.1	D	3.2

$S_1 = a = 14$

$\therefore S_{10} = 212 = \frac{10}{2}(2\times14 + (10-1)d)$

$212 = 140 + 45d$

$72 = 45d$

$\therefore d = 1.6$

9. If $x, x+3, x+9, \ldots$ are the first three terms in a **geometric progression**, find x and the common ratio.

A	$x = 1$, $ratio = 3$	B	$x = 3$, $ratio = 3$
C	$x = 3$, $ratio = 2$	D	$x = 9$, $ratio = \frac{4}{3}$

$\frac{x+3}{x} = \frac{x+9}{x+3}$

$(x+3)^2 = x^2 + 9x$

$x^2 + 6x + 9 = x^2 + 9x$

$9 = 3x$

$\therefore x = 3$

$\therefore$ GP is

$3, 6, 12, \ldots$

$\therefore r = 2$.

10. Part of the fractal root problem in the investigation is shown below:

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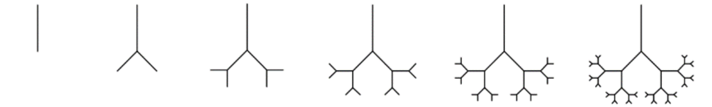

Stage 0	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5

The initial root (at Stage 0) is vertical and has length 1. At Stage 1, from the end of the initial root, two branches, each half as long, grow at right angles to each other, each a 45° rotation from the initial root. This process is repeated at the end of each root, from Stage 2 onwards.

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Consider the root system at Stage 5. If an ant were to walk along the length of the roots, following one path until the end, how far would it travel, given that the length of Stage 0 is 1 unit.

A	$1\frac{28}{32}$ *units*	B	$1\frac{29}{32}$ *units*
C	$1\frac{30}{32}$ *units*	D	$1\frac{31}{32}$ *units*

sum this column up to stage 6.

$\therefore 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^6} = \frac{1(1 + \frac{1}{2^7})}{1 - \frac{1}{2}} = 1\frac{31}{32}$

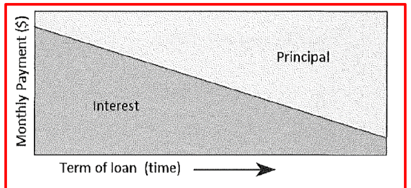
End of multiple choice questions

Questions 11 and 12 are written responses.

11. In the investigation, a home loan scenario was considered in the *Paying off a Home Loan* article by Jim Stamell.

(i) Using your understanding of loan repayments **explain** the trend shown in Fig 4 i.e. why is a higher proportion of each repayment made up of interest at the beginning of the loan than at the end of the loan? **2**

At the start of the loan, there is a lot of interest as interest is charged on the amount owing and the loan has not had any repayments yet. As time goes on, there is less interest as the amount owing on the loan has been reduced through repayments.



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(ii) Fig 2 shows the amount owed on a loan with the standard repayments. Ian Trest makes repayments greater than the standard repayments. Explain the impact of this on the loan and sketch the new 'curve' on the graph. **2**

The curve is steeper than the original curve, showing that the time to repay the loan is quicker. Moreover, the total amount repaid is less.

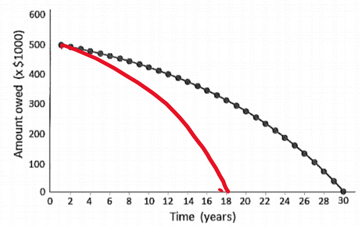

FIG. 2. PROGRESSIVE AMOUNT OWED OVER THE PERIOD OF THE LOAN

(iii) Fig 2 shows the amount owed on a loan with the standard repayments. Owen Munny makes a single lump sum repayment 10 years into the loan. Explain the impact of this on the loan and sketch the new 'curve' on the graph. **2**

The curve is the same until the 10 year mark, at which there is a vertical 'drop' and then the curve will be steeper as there is less interest.

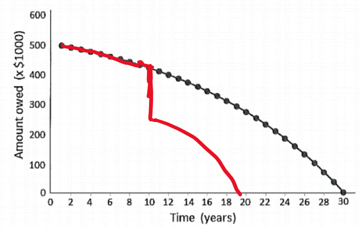

FIG. 2. PROGRESSIVE AMOUNT OWED OVER THE PERIOD OF THE LOAN

12. Janice borrowed $425 000 to buy a house. The interest rate was 7.5% per annum, compounded monthly. She agreed to repay the loan in 35 years with equal monthly repayments of $2866.

(i) Calculate how much Sarah owed after her first monthly repayment. **2**

$$r = \frac{7.5\%}{12} = 0.00625$$

$$\therefore A_1 = 425\,000(1.00625) - 2866 = 424790.25$$

(ii) Calculate how much Sarah owed after her second monthly repayment. **2**

$$A_2 = A_1 \times (1.00625) - 2866$$
$$= 424790.25 \times 1.00625 - 2866$$
$$\doteqdot 424579.19$$

(iii) After making her repayment at the end of 2 **years**, Janice decides that she would rather pay the loan off over the next 26 years (from this point). The amount, $\$A_n$, owing on the loan after the nth monthly repayment, is now calculated using the formula:

$\$A_n = 419587 \times 1.00625^n - 1.00625^{n-1}M - \cdots - 1.00625M - M$

where $\$M$ is the monthly repayment, and $n = 1, 2, \ldots, 312$. (Do NOT prove this formula.)

Calculate the monthly repayment if the loan is to be repaid over the remaining 26 years (312 months). **3**

$$A_{312} = 419587 \times 1.00625^{312} - 1.00625^{311}M - \ldots - M$$

$$\therefore 0 = 419587 \times 1.00625^{312} - M(1 + 1.00625 + \ldots + 1.00625^{311})$$

$$0 = 419587 \times 1.00625^{312} - \frac{M(1.00625^{312} - 1)}{1.00625 - 1}$$

$$M\left(\frac{1.00625^{312} - 1}{1.00625 - 1}\right) = 419587 \times 1.00625^{312}$$

$$\therefore M = \frac{419587 \times 1.00625^{312}}{\left(\frac{1.00625^{312} - 1}{1.00625 - 1}\right)} \doteqdot \$3060.50$$

(iv) Janice gets a new job and chooses to increase her monthly repayments to \$3500. Use the formula in part (iii) to calculate how long it will take her to repay the \$419587. **3**

$$A_n = 419587 \times 1.00625^n - 3500(1.00625)^{n-1} - \ldots - 3500$$

$$\therefore 0 = 419587(1.00625)^n - 3500\left(\frac{1.00625^n - 1}{1.00625 - 1}\right)$$

$$0 = 419587(1.00625)^n - 560000(1.00625^n - 1)$$

$$0 = 419587(1.00625)^n - 560000(1.00625)^n + 560000$$

$$140413(1.00625)^n = 560000$$

$$1.00625^n = \frac{560000}{140413}$$

$$n\ln(1.00625) = \ln\left(\frac{560000}{140413}\right)$$

$$n = \frac{\ln\left(\frac{560000}{140413}\right)}{\ln(1.00625)} \doteqdot 222.026\ldots$$

$\therefore n \doteqdot 223$ months (need to round up)

End of paper