Student Number: ________________________________

Student Name: ________________________________

Teacher: ________________________________

Fort Street High School

2021

YEAR 12 ASSESSMENT TASK 2

Mathematics Advanced

Outcomes	Assessment Area Description and Marking Guidelines
MA12-3	applies calculus techniques to model and solve problems
MA12-6	applies appropriate differentiation methods to solve problems
MA12-10	Constructs arguments to prove and justify results and provides reasoning to support conclusions which are appropriate to the context

Time allowed: 75 minutes (plus 5 minutes reading time)

Total Marks 62

Section I 6 marks

- Multiple Choice, attempt all questions.
- Allow about 10 minutes for this section.

Section II 56 Marks

- Attempt Questions 7-13.
- Allow about 65 minutes for this section.

General Instructions:

- Questions 7-13 are to be started in a new sheet.
- The marks allocated for each question are indicated.
- In Questions 7 – 13, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.

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Section I

6 marks
Attempt Questions 1 to 6
Allow about 10 minutes for this section
Use the multiple-choice answer sheet for Questions 1 to 6

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1. If $f(x)=3x^5+2x^4-x^3-2$, find $f''(-1)$.

   (A) -90

   (B) -30

   (C) -6

   (D) 2

2. Calculate $\dfrac{d}{dx}\left[\tan\left(1-x^2\right)\right]$.

   (A) $2x\sec^2\left(1-x^2\right)$

   (B) $\sec^2\left(1-x^2\right)$

   (C) $-2x\sec^2\left(1-x^2\right)$

   (D) $2x^2\sec^2\left(1-x^2\right)$

3. Given $y=\log_3 x$, which statement is incorrect?

   (A) $y=\dfrac{\ln x}{\ln 3}$

   (B) $y=\dfrac{1}{\ln 3}\times\ln x$

   (C) $y'=\dfrac{1}{x\ln 3}$

   (D) $y'=\dfrac{1}{3\ln x}$

4. Differentiate $y = \dfrac{\sin 2x}{x}$ with respect to x.

(A) $\dfrac{dy}{dx} = \dfrac{x\cos 2x - \sin 2x}{x^2}$

(B) $\dfrac{dy}{dx} = \dfrac{\sin 2x - 2x\cos 2x}{x^2}$

(C) $\dfrac{dy}{dx} = \dfrac{2x\cos 2x - \sin 2x}{x^2}$

(D) $\dfrac{dy}{dx} = \dfrac{2x\cos 2x - x\sin x}{x^2}$

5. The gradient of the tangent to the curve $y = e^{3x}$ at the point where $x = 0$ is:

(A) 3

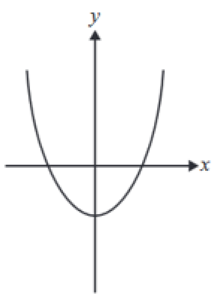
(B) 1

(C) $-\dfrac{1}{3}$

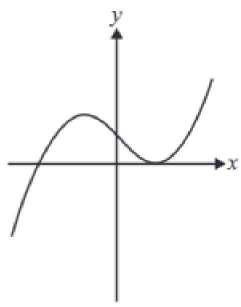
(D) -1

6. If $f(x) = x^3 - 4x$, which of the following diagrams could represent the graph of $y = f''(x)$?

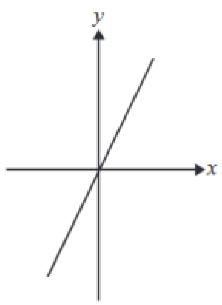
(A)



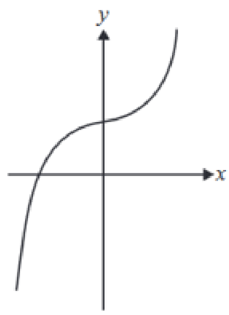
(B)



(C)



(D)



Section II

56 marks
Attempt Questions 7 to 13
Allow about 65 minutes for this section

Start each question in a new writing sheet. Extra writing sheets are available.
In Questions 7 – 13, your responses should include relevant mathematical reasoning and calculations.

Question 7 (7 marks) **Start a new writing sheet.**

(a) Find $\dfrac{d^2y}{dx^2}$ if $y=\sqrt{4x+2}$. **2**

(b) Given $f(x)=\dfrac{4}{x}$ find the value of $f''(2)$. **2**

(c) Find the value/s of k if $f(x)=4e^{kx}$ is a solution of $f''(x)-4f'(x)-5f(x)=0$. **3**

End of Question 7

Question 8 (7 marks) **Start a new writing sheet.**

(a) If $y = 12x^3 - 6x^2 + 2$ find the value of x for which $\frac{d^2y}{dx^2} = 0$. **2**

(b) Show that the equation of the normal to the curve $y = x^3 - 5x$ at the point $(1, -4)$ is given by $x - 2y - 9 = 0$. **2**

(c) Prove that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ given $y = xe^{-x}$. **3**

End of Question 8

Question 9 (7 marks) **Start a new writing sheet.**

(a) Find $\frac{d}{dx}\left(\sin^2 x\right)$. **1**

(b) Differentiate the following expressions with respect to x:

(i) $xe+\frac{1}{3}e^{3x}$. **1**

(ii) $3\tan\frac{\pi x}{3}$. **1**

(iii) $\ln\left(xe^{-x}\right)$. **2**

(c) Show that $\frac{d}{dx}\left[\ln\left(\tan x\right)\right]=\tan x+\cot x$. **2**

End of Question 9

Question 10 (7 marks) **Start a new writing booklet.**

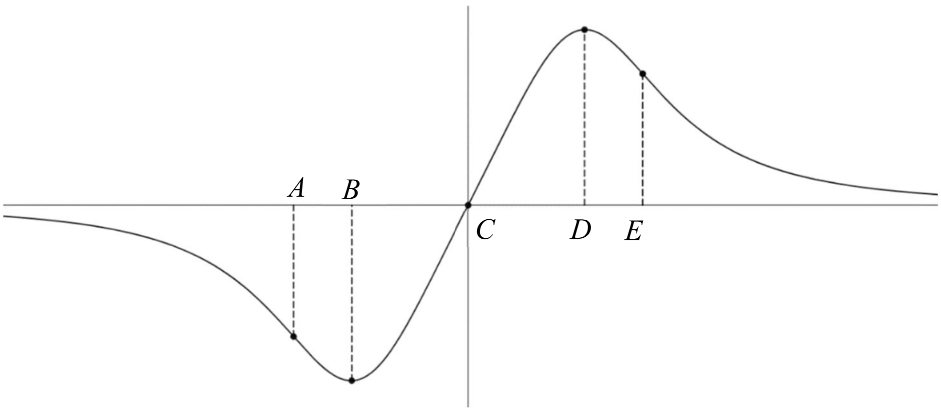
(a) Simplify $\log_e e^{2ax}$. **1**

(b) **(i)** Factorise the expression $2a^2 - 7a + 3$. **1**

(ii) Hence, solve the following equation for x: **2**

$$2\left(\log_2 x\right)^2 - 7\left(\log_2 x\right) + 3 = 0$$

(c) The graph of the function $y = f(x)$ is shown below. There are points of inflexion at $x = A$, $x = C$, and $x = E$, and stationary points at $x = B$ and $x = D$.



Draw the graph of $y = f'(x)$ on a separate axis showing all relevant and significant points. (Use the attached answer sheet on page 13) **3**

End of Question 10

Question 11 (14 marks) **Start a new writing sheet.**

Consider the curve $y = (x-1)(x^2-1)$.

(i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. **2**

(ii) Find the coordinates of the two stationary points. **2**

(iii) Determine the nature of the stationary points. **2**

(iv) Show that there is a point of inflexion at $\left(\dfrac{1}{3}, \dfrac{16}{27}\right)$. **3**

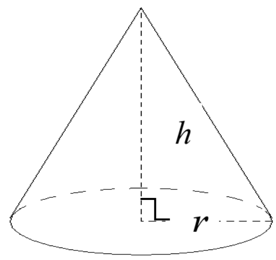
(v) Hence, sketch the curve showing the above features and any intercepts. **3**

(iv) Find the global maximum and minimum of the curve over the domain $[-1, 3]$. **2**

End of Question 11

Question 12 (7 marks) **Start a new writing sheet.**

A right circular cone has a perpendicular height of h cm and a slant edge of 9 cm.



(i) Write an expression for r in terms of h. **1**

(ii) Show that the volume of the cone can be expressed as: $V = \frac{\pi}{3}\left(81h - h^3\right)$ **2**

(iii) Find the value of h that maximises the volume of the cone. **2**

(iv) Find the maximum possible volume (Give your answer as an exact value). **2**

End of Question 12

Question 13 (7 marks) **Start a new writing sheet.**

A length of wire $120 cm$ long is to be cut into two pieces. One is bent into the shape of a circle and the other into the shape of an equilateral triangle.

(i) If x is the circumference of the circle, what is the perimeter of the equilateral triangle in terms of x? **1**

(ii) Show that the sum of the areas (A) formed by the two pieces is given by: **3**

$$A = \frac{x^2}{4\pi} + \frac{\sqrt{3}}{36}(120 - x)^2$$

(iii) Find the value of x so that the sum of areas is a minimum. **3**

End of Exam

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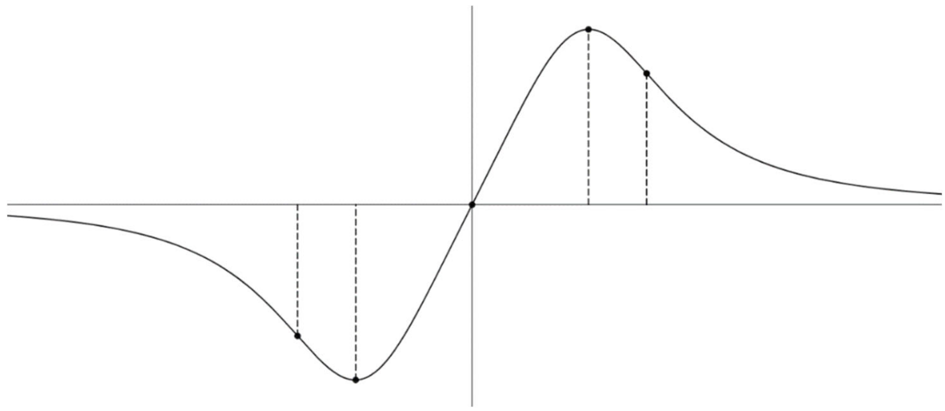
Student Name:______________________________________ Class: _____________________

Teacher's Name: _____________________________

Question 10

(c) The graph of the function $y = f(x)$ is shown below. There are points of inflexion at $x = A$, $x = C$, and $x = E$, and stationary points at $x = B$ and $x = D$.



Draw the graph of $y = f'(x)$ on a separate axis showing all relevant and significant points. **3**

Student Number: ________________________________

Student Name: ________________________________

Teacher: ________________________________

Fort Street High School

2021

YEAR 12 ASSESSMENT TASK 2

SOLUTIONS

Mathematics Advanced

Outcomes	Assessment Area Description and Marking Guidelines
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Total Marks 62

Section I 6 marks

- Multiple Choice, attempt all questions.
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Section I

6 marks
Attempt Questions 1 to 6
Allow about 10 minutes for this section
Use the multiple-choice answer sheet for Questions 1 to 6

1. If $f(x)=3x^5+2x^4-x^3-2$, find $f''(-1)$.

$$\begin{aligned} f'(x) &= 15x^4+8x^3-3x^2 \\ f''(x) &= 60x^3+24x^2-6x \\ f''(-1) &= 60(-1)^3+24(-1)^2-6(-1) \\ &= -60+24+6 \\ &= -30 \end{aligned}$$

(A) -90

(B) -30

(C) -6

(D) 2

2. Calculate $\frac{d}{dx}\left[\tan\left(1-x^2\right)\right]$.

$$\begin{aligned} \frac{d}{dx}\left[\tan\left(1-x^2\right)\right] &= \sec^2\left(1-x^2\right).-2x \\ &= -2x\sec^2\left(1-x^2\right) \end{aligned}$$

(A) $2x\sec^2\left(1-x^2\right)$

(B) $\sec^2\left(1-x^2\right)$

(C) $-2x\sec^2\left(1-x^2\right)$

(D) $2x^2\sec^2\left(1-x^2\right)$

3. Given $y=\log_3 x$, which statement is incorrect?

Use change of base and log laws

(A) $y=\frac{\ln x}{\ln 3}$

(B) $y=\frac{1}{\ln 3}\times \ln x$

(C) $y'=\frac{1}{x\ln 3}$

(D) $y'=\frac{1}{3\ln x}$

4. Differentiate $y = \dfrac{\sin 2x}{x}$ with respect to x.

(A) $\dfrac{dy}{dx} = \dfrac{x\cos 2x - \sin 2x}{x^2}$

(B) $\dfrac{dy}{dx} = \dfrac{\sin 2x - 2x\cos 2x}{x^2}$

(C) $\dfrac{dy}{dx} = \dfrac{2x\cos 2x - \sin 2x}{x^2}$

(D) $\dfrac{dy}{dx} = \dfrac{2x\cos 2x - x\sin x}{x^2}$

$$\frac{dy}{dx} = \frac{x.\cos 2x.2 - \sin 2x.1}{x^2} = \frac{2x\cos 2x - \sin 2x}{x^2}$$

5. The gradient of the tangent to the curve $y = e^{3x}$ at the point where $x = 0$ is:

(A) 3

(B) 1

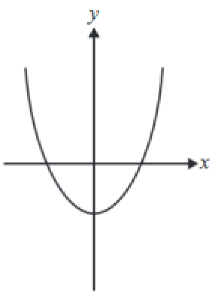
(C) $-\dfrac{1}{3}$

(D) -1

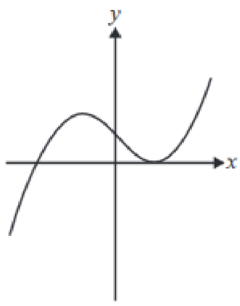
$$y' = 3e^{3x} \quad at \ x = 0$$
$$y' = 3e^{3(0)}$$
$$= 3$$

6. If $f(x) = x^3 - 4x$, which of the following diagrams could represent the graph of $y = f''(x)$?

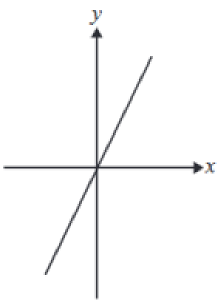
(A)



(B)

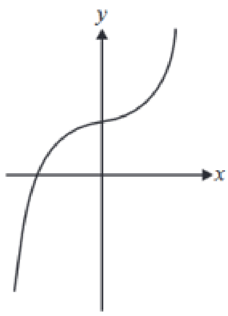


(C)



$$f(x) = x^3 - 4x$$
$$f'(x) = 3x^2 - 4$$
$$f''(x) = 6x$$

(D)



Section II   56 marks   Attempt Questions 7 to 13

Question 7 (7 marks)

(a) Find $\dfrac{d^2y}{dx^2}$ if $y=\sqrt{4x+2}$.

$$y=\sqrt{4x+2}=(4x+2)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}(4x+2)^{-\frac{1}{2}}.4$$

$$=\frac{2}{\sqrt{4x+2}}$$

$$\frac{d^2y}{dx^2}=-\left(\frac{1}{2}\right).2(4x+2)^{\frac{-3}{2}}.4$$

$$=\frac{-4}{\sqrt{(4x+2)^3}} \quad or \quad \frac{-4}{(4x+2)\sqrt{(4x+2)}}$$

Teacher's comments:

Mostly answered well.

Main Error was forgetting to multiply by

.

Final answer should be expressed in the same form as the question.

(b) Given $f(x)=\dfrac{4}{x}$ find the value of $f''(2)$. **2**

$$f(x)=\frac{4}{x}=4x^{-1}$$

$$f'(x)=-4x^{-2}$$

$$f''(x)=8x^{-3}$$

$$f''(2)=8(2)^{-3}$$

$$=\frac{8}{2^3}$$

$$=1$$

Teacher's comments:

Answered well.

The few students who couldn't differentiate the function forgot to express it in index form first.

(c)Find the value/s of k if $f(x)=4e^{kx}$ is a solution of $f''(x)-4f'(x)-5f(x)=0$. **3**

$$f'(x)=4e^{kx}.k$$

$$=4ke^{kx}$$

$$f''(x)=4ke^{kx}.k$$

$$=4k^2e^{kx}$$

$$\therefore f''(x)-4f'(x)-5f(x)=0$$

$$4k^2e^{kx}-4\left[4ke^{kx}\right]-5\left(4e^{kx}\right)=0$$

$$4e^{kx}\left[k^2-4k-5\right]=0$$

$$4e^{kx}\left[(k-5)(k+1)\right]=0$$

$$\therefore k=5,-1$$

Teacher's comments:

Mostly well answered.

Problems came from the inability to factorise correctly.

Question 8 (7 marks)

(a) If $y = 12x^3 - 6x^2 + 2$ find the value of x for which $\frac{d^2y}{dx^2} = 0$. **2**

$$\frac{dy}{dx} = 36x^2 - 12x$$
$$\frac{d^2y}{dx^2} = 72x - 12$$
$$72x - 12 = 0$$
$$72x = 12$$
$$x = \frac{12}{72} = \frac{1}{6}$$

Teacher's comments

Mostly well done.

(b) Show that the equation of the normal to the curve $y = x^3 - 5x$ at the point $(1, -4)$ is given by $x - 2y - 9 = 0$. **2**

$$y = x^3 - 5x$$
$$y' = 3x^2 - 5$$
$$at\ x = 1,\ m_T = 3(1)^2 - 5 \quad and \quad m_N = \frac{-1}{m_T}$$
$$= -2 \qquad\qquad = \frac{-1}{-2} = \frac{1}{2}$$
$$\therefore equation\ of\ normal\ at\ (1, -4)$$
$$y - -4 = \frac{1}{2}(x - 1)$$
$$2y + 8 = x - 1$$
$$0 = x - 2y - 9$$

Teacher's comments

Mostly well done.

(c) Prove that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ given $y = xe^{-x}$. **3**

$$\frac{dy}{dx} = x.e^{-x}.-1 + e^{-x}.1$$
$$= -x.e^{-x} + e^{-x}$$
$$\frac{d^2y}{dx^2} = -x.e^{-x}.-1 + e^{-x}.-1 + e^{-x}.-1$$
$$= xe^{-x} - e^{-x} - e^{-x}$$
$$= xe^{-x} - 2e^{-x}$$
$$LHS = \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y$$
$$= xe^{-x} - 2e^{-x} + 2(-xe^{-x} + e^{-x}) + xe^{-x}$$
$$= 2xe^{-x} - 2e^{-x} - 2xe^{-x} + 2e^{-x}$$
$$= 0$$
$$= RHS$$

Teacher's comments

Many students did not apply the product rule to differentiate. By coincidence this still worked when substituting into the final equation. If this error was made students received 1 out of 3 marks since they were unable to identify the error.

Question 9 (7 marks)

(a) Find $\dfrac{d}{dx}\left(\sin^2 x\right)$.

$$\frac{d}{dx}\left(\sin^2 x\right) = 2\sin x\cos x$$

> **Teacher's comments**
>
> Overall well done.
>
> Students that experienced difficulty should use the product or chain rule.

(b) Differentiate the following expressions with respect to x:

(i) $$\frac{d}{dx}\left(xe + \frac{1}{3}e^{3x}\right) = e + e^{3x}$$

(ii)

$$\frac{d}{dx}\left(3\tan\frac{\pi x}{3}\right) = 3\sec^2\frac{\pi x}{3}.\frac{\pi}{3}$$

$$= \pi\sec^2\frac{\pi x}{3}$$

> **Teacher's comments** i) Mostly, well done. However, some students do not know that e is a constant. The derivative of 'ex' is simply 'e'. Furthermore, it is a waste of time using the product rule to differentiate this.
>
> ii) Mostly, well done.
>
> iii) Differentiation of this function becomes much easier if the log law log(mn)=logm+logn is used firstly to split the function into 2 terms. Then differentiate each term.

(iii)

$$\frac{d}{dx}\left(\ln\left(xe^{-x}\right)\right) = \frac{d}{dx}\left(\ln x + \ln e^{-x}\right)$$

$$= \frac{1}{x} + \frac{1}{e^{-x}}.e^{-x}.-1$$

$$= \frac{1}{x} - 1$$

OR

$$\frac{d}{dx}\left(\ln\left(xe^{-x}\right)\right) = \frac{1}{xe^{-x}}\left(xe^{-x}.-1 + e^{-x}.1\right)$$

$$= \frac{e^{-x}\left(-x+1\right)}{xe^{-x}}$$

$$= \frac{-x+1}{x} = -1 + \frac{1}{x}$$

(c)

$$LHS = \frac{d}{dx}\left[\ln\left(\tan x\right)\right]$$

$$= \frac{1}{\tan x}.\sec^2 x$$

$$= \frac{\tan^2 x + 1}{\tan x}$$

$$= \frac{\tan^2 x}{\tan x} + \frac{1}{\tan x}$$

$$= \tan x + \cot x$$

$$= RHS$$

> Teacher's comment:
>
> This is a 'SHOW' question!! All steps in working should be given. The easiest method is the solution given opposite.

Question 10 (7 marks)

(a) Simplify

$$\log_e e^{2ax} = 2ax\log_e e$$
$$= 2ax(1)$$
$$= 2ax$$

Teacher's comments:

Mostly well done with a few students who need to revise this concept.

(b)

(i) $2a^2-7a+3=(2a-1)(a-3)$

$$2(\log_2 x)^2-7(\log_2 x)+3=0$$
$$Let\ \log_2 x = a$$
$$\therefore 2a^2-7a+3=0$$
$$(2a-1)(a-3)=0$$
$$a=\frac{1}{2},\ a=3$$
$$i.e.\ \ \log_2 x=\frac{1}{2},\ \log_2 x = 3$$
$$2^{\frac{1}{2}}=x,\ \ 2^3=x$$

(ii) $\therefore \quad x=\sqrt{2},\ x=8$

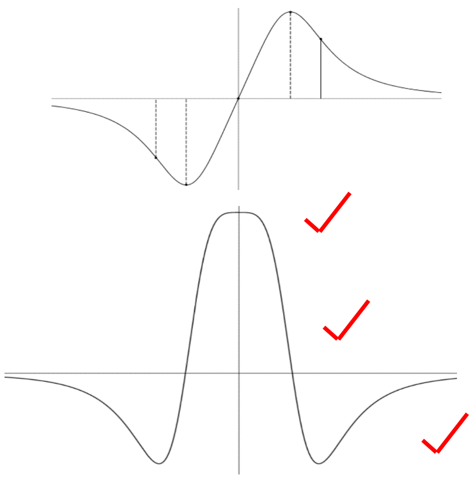
Teacher's comments:

- Many students could not reduce the question to a quadratic.
- Those that could made errors in solving $\log_2 x = \dfrac{1}{2}$ etc.
- Please leave answers in exact for rather than approximating. E.g. write $\sqrt{2}$ rather than 1.41

(c)

The graph of the function $y=f(x)$ is shown below. There are points of inflexion at $x=A$, $x=C$, and $x=E$, and stationary points at $x=B$ and $x=D$.

Copy the graph in your answer booklet and draw the graphs of $f'(x)$ on a separate axis showing all relevant significant points.



Teacher's comments:

3 marks for correct solution **or**
2 marks for one error in the graph **or**
1 mark for one correct feature

The graph was poorly drawn and in many cases unclear. Students would benefit from using a ruler to draw 1/3 of the page graphs, labelling the axes and critical points as identified in the question.

Question 11 (14 marks)

Consider the curve $y=(x-1)(x^2-1)$

$$y=x^3-x^2-x+1$$

$$\frac{dy}{dx}=3x^2-2x-1$$

(i)

$$\frac{d^2y}{dx^2}=6x-2$$

Teacher's comments:

Answered well by all students.

Student are encouraged to check the function to see if it can be simplified before differentiating.

(ii) Stationary points occur at $\frac{dy}{dx}=0$

$$3x^2-2x-1=0$$

$$(3x+1)(x-1)=0$$

$$3x+1=0,\ x-1=0$$

$$\therefore 3x=-1$$

$$x=\frac{-1}{3},\ x=1$$

$when\ x=1,\ y=?\ \ and\ \ \ when\ x=\frac{-1}{3}\ \ y=?$

$$y=(1-1)(1^2-1)$$

$$=0$$

$$y=\left(\frac{-1}{3}-1\right)\left(\left(\frac{-1}{3}\right)^2-1\right)$$

$$=\left(\frac{-4}{3}\right)\left(\frac{1}{9}-1\right)$$

$$=1\frac{5}{27}$$

$$\therefore Stationary\ \ po\text{int}\,s\ are:(1,0)\,and\left(\frac{-1}{3},1\frac{5}{27}\right)$$

Teacher's comments:

Answered well by most students. Students are reminded to take care as an error here could compromise the rest of the question.

Since the question is asking for stationary points, then both the x and y values of the point must be found.

(iii) Nature of the stationary points

$$when\ x=1,\ \frac{d^2y}{dx^2}=6(1)-2=4>0$$

$$\therefore the\ curve\ is\ concave\ up\,at\,x=1$$

$$and\,(1,\ 0)\ is\ a\ \text{min}\,imum\,turning\ point$$

$$when\ x=\frac{-1}{3},\ \frac{d^2y}{dx^2}=6\left(\frac{-1}{3}\right)-2=-4<0$$

$$\therefore the\ curve\ is\ concave\ down\,at\,x=\frac{-1}{3}$$

$$and\left(\frac{-1}{3},1\frac{5}{27}\right)\ is\ a\ \text{max}\,imum\,turning\ point$$

Teacher's comments:

While generally well answered, some students omitted calculating the value of the second derivative. One mark was deducted for this. Many students omitted stating concave up and concave down. Marks may be deducted for this omission in future assessments including the trial exam and the HSC.

If using 2nd derivative you must

- Find the value of the derivative
- State >0 or <0
- State Concave up or down
- Hence conclude max or min

(iv) For points of inflexion;

As the curve changes concavity on either side of this point i.e. maximum at $x=-\frac{1}{3}$ and minimum turning point at $x=1$, and

$$\frac{d^2y}{dx^2}=0$$
$$6x-2=0$$
$$6x=2$$
$$x=\frac{1}{3}$$

There is a point of inflexion at:
y-co-ordinate of point of inflexion:

$$x=\frac{1}{3},\ y=\left(\frac{1}{3}-1\right)\left(\left(\frac{1}{3}\right)^2-1\right)$$
$$=\left(\frac{-2}{3}\right)\left(\frac{1}{9}-1\right)$$
$$=\left(\frac{-2}{3}\right)\left(\frac{8}{9}\right)$$
$$=\frac{16}{27}$$

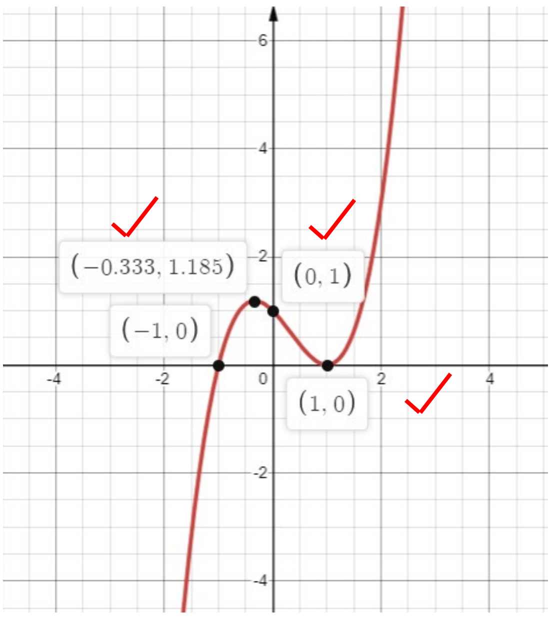
$\therefore$ there is a point of inflexion at $\left(\frac{1}{3},\frac{16}{27}\right)$.

Teacher's comments:

Most students achieved 2 marks for this question as they did not attempt to find the y-value of the point of inflection.

This is a show question and therefore it was necessary for students to show all relevant work to show their understanding.

(v) Hence, sketch the curve showing the above features and any intercepts.



Teacher's comments:

To achieve full marks students had to clearly identify the x and y intercepts, the maximum and minimum points and the point of inflection as well as their graphs being reasonably to scale.

Teacher's comments:

Part (vi) was generally answered well. Students are advised to take care comparing the local max and min due to stationary points with the y-values at the limits of their domain, prior to giving final answers.

(vi) Global maximum when $x=3, y=(3-1)(3^2-1)=16$

and global minimum when $x=-1, y=(-1-1)(-1^2-1)=0$

Question 12 (7 marks)

A right circular cone has a perpendicular height of h cm and a slant edge of 9 cm.

(i) An expression for r in terms of h

$$h^2 + r^2 = 9^2$$
$$\therefore r^2 = 81 - h^2$$
$$r = \pm\sqrt{81 - h^2}$$

Teacher's comments:

Mostly well done.

(ii) The volume of the cone can be expressed as:

$$V_{cone} = \frac{1}{3}\pi r^2 h, \; where \; r = \sqrt{81 - h^2}$$
$$= \frac{\pi}{3}\left(\sqrt{81 - h^2}\right)^2 \times h$$
$$= \frac{\pi}{3}\left(81 - h^2\right)h$$
$$V = \frac{\pi}{3}\left(81h - h^3\right)$$

Teacher's comments:

Mostly well done. Although it is good practice to avoid skipping steps in show questions.

(iii) Volume of the cone

$$V(h) = \frac{\pi}{3}\left(81h - h^3\right)$$
$$\frac{dV}{dh} = \frac{\pi}{3}\left(81 - 3h^2\right)$$

To maximise volume set $\frac{dV}{dh} = 0$

$$\frac{\pi}{3}\left(81 - 3h^2\right) = 0$$
$$81 = 3h^2$$
$$h^2 = 27$$
$$h = \pm\sqrt{27}$$
$$\therefore h = \sqrt{27}$$
$$\left(taking \; the \; +ve \, value \; for \; length\right)$$

Teacher's comments:

Most common error was forgetting to show that it is a maximum. Most students identified that due to it being a distance.

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}\left(-6h^2\right)$$

$$when\ h = \sqrt{27}, \frac{d^2V}{dh^2} = \frac{\pi}{3}\left(-6\left(\sqrt{27}\right)^2\right)$$

$$= \frac{\pi}{3}(-6 \times 27)$$

$$< 0$$

$$\therefore h = \sqrt{27} = 3\sqrt{3}\ gives\ a\ \max imum\ volume$$

(iv) The maximum possible volume is:

$$V\left(\sqrt{27}\right) = \frac{\pi}{3}\left(81\sqrt{27} - \left(\sqrt{27}\right)^3\right)$$

$$= 27\sqrt{27}\pi - 9\sqrt{27}\pi$$

$$= 18\sqrt{27}\pi$$

$$= 54\sqrt{3}\pi$$

(as an exact value)

Teacher's comments:

Mostly well done. A few errors simplifying the surds. The final answer did not necessarily have to be in simplest form, but any incorrect arithmetic was worthy of a mark deduction.

Question 13 (7 marks)

A length of wire $120\,cm$ long is to be cut into two pieces. One is bent into the shape of a circle and the other into the shape of an equilateral triangle.

(i) Given x is the perimeter of the circle, then perimeter of triangle is

$$perimeter_{\Delta} = length\ of\ wire - perimeter_{circle}$$
$$= 120 - 2\pi r$$
$$= 120 - x$$

(ii) Sum of the areas (A) formed by the two pieces:

$$x = 2\pi r$$
$$r = \frac{x}{2\pi}$$
$$area\ of\ circle = \pi\left(\frac{x}{2\pi}\right)^2$$
$$area\ of\ triangle = \frac{1}{2}\left(\frac{120-x}{3}\right)^2 \sin 60^{\circ}$$
$$= \frac{1}{2}\left(\frac{120-x}{3}\right)^2 \frac{\sqrt{3}}{2}$$
$$= area\ of\ circle + area\ of\ triangle$$
$$Total\ Area\,(A) = \frac{x^2}{4\pi} + \frac{\sqrt{3}}{36}(120-x)^2$$

Teacher's comments:

i) Well done.

ii) This is a 'show' question. This means the area of the circle and triangle must be found with ALL working shown. It is easier to find the area of the triangle by using the trigonometry formula A=½ absinC.

iii) Many students need to improve their solutions by explaining why certain steps are needed e.g dy/dx=0 is solved as that is where stationary points occur. This time, marks were not deducted if you did not say this. Students forgot to test for minimum sum of areas occurring -1 mark deducted. Also, 1 mark deducted if values were not calculated when considering sign of 1st or 2nd derivative.

(iii)

$$\frac{dA}{dx} = \frac{x}{2\pi} - \frac{\sqrt{3}}{18}(120-x)$$
$$stationary\ points\ occur\ when\ \frac{dA}{dx} = 0$$
$$\left(\frac{1}{2\pi} + \frac{\sqrt{3}}{18}\right)x - \frac{20\sqrt{3}}{3} = 0$$
$$x = \frac{\frac{20\sqrt{3}}{3}}{\frac{1}{2\pi} + \frac{\sqrt{3}}{18}}$$
$$= \frac{120\pi\sqrt{3}}{9 + \pi\sqrt{3}}$$
$$\frac{d^2A}{dx^2} = \frac{1}{2\pi} + \frac{\sqrt{3}}{18}$$
$$> 0 \qquad \therefore minimum\ when\ \frac{120\pi\sqrt{3}}{9+\pi\sqrt{3}}$$

End of Exam

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