GIRRAWEEN HIGH SCHOOL

YEAR 12 - TASK 1

2021

MATHEMATICS ADVANCED

Time allowed: 90 minutes

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- Board-approved calculators may be used.
- The BOSTES Reference Sheet may be used (see separate laminated sheet).
- **DIAGRAMS ARE NOT DRAWN TO SCALE**.
- **TOTAL MARKS: 65**

Use the space provided to show your answers for each question including all relevant working out.

Extra working pages are attached at the end of this paper. Clearly write the question numbers.

- **Section I (5 marks): Questions 1 – 5**
  - Select A, B, C or D that best answers the question.
- **Section II (60 marks): Questions 6 – 9**
  - All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.

SECTION I: MULTIPLE CHOICE QUESTIONS 1 – 5

Shade A, B, C or D that best answers the questions 1 – 5 on page 2.

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1. The fourth term of an arithmetic series is 27 and the seventh term is 12.

   What is the common difference?

   (A) -5

   (B) 5

   (C) 13

   (D) 42

2. What is the domain for $y = \frac{2x}{\sqrt{x+2}}$?

   (A) $[-2, \infty)$

   (B) $(-\infty, -2]$

   (C) $(-2, \infty)$

   (D) $(-\infty, -2)$

3. The discrete random variable X has the following probability distribution.

$\boldsymbol{x}$	0	1	2	3
$\boldsymbol{P(X = x)}$	a	$2a$	$5a$	$3a$

   The value of a is:

   (A) $\frac{1}{21}$

   (B) $\frac{1}{11}$

   (C) 11

   (D) 21

4. A geometric sequence has a first term of 20 and a common ratio of $-\frac{3}{2}$, find the 5^{th} term:

(A) 12.5

(B) $-151\frac{7}{8}$

(C) $101\frac{1}{4}$

(D) $-101\frac{1}{4}$

5. The graph of $y = f(x)$ passes through $(2, -3)$.

Where will the equivalent point be on the graph of $y = -f\left(\frac{x}{2}\right) - 5$?

(A) $(1, -2)$

(B) $(1, 2)$

(C) $(4, -2)$

(D) $(4, 2)$

Answers to Multiple Choice Questions

Select the alternative A, B, C or D that best answers the question.

1.	A	○	B	○	C	○	D	○
2.	A	○	B	○	C	○	D	○
3.	A	○	B	○	C	○	D	○
4.	A	○	B	○	C	○	D	○
5.	A	○	B	○	C	○	D	○

SECTION II: QUESTIONS 6 – 9

Show all necessary working for each question.

Marks may be deducted for careless or badly arranged work.

QUESTION 6 (14 Marks)

(a) A bag contains three red balls and four black balls. Two balls are selected at random without replacement from the bag. Let X be the number of black balls drawn.

(i) Using a tree diagram or otherwise, fill in the 2nd and the 3rd rows in the following table to find $E(X)$. (2 marks)

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$\boldsymbol{x}$	0	1	2	Sum
$\boldsymbol{P(X=x)}$				

(ii) Fill in the 4th row in the table above to find $E(X^2)$ (2 marks)

(iii) Hence find $Var(X)$ and standard deviation σ. (2 marks)

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(b) The discrete random variable X has probability distribution shown in the table below.

$\boldsymbol{x}$	-1	0	1	2	3
$\boldsymbol{P(X = x)}$	a	b	0.2	0.15	0.13

(i) Find the values of a and b, given that $E(X) = 0.55$. (3 marks)

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(ii) Determine $Var(X)$. (1 mark)

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(c) Evaluate:

(i) (2 marks)

$$\sum_{r=1}^{6} r^r$$

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(ii) (2 marks)

$$\sum_{k=3}^{12} 2 \times 3^{k-1}$$

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QUESTION 7 (16 Marks)

(a) In an arithmetic series, the third term is 5 and the tenth term is 26. Find the sum of the first 14 terms. (2 marks)

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(b) Kloe has taken up golf, each year she plays 30 rounds of golf. In 2011 she totalled 3300 shots, in 2021 she totalled 2760 shots. The number of shots she took each year decreased by the same amount. To make it as a professional golfer, she needs to get under 2115 shots in her 30 rounds. If this trend continues, in what year can she become a professional golfer? (3 marks)

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(c) Tom planted a silky oak tree seed. At the end of the first year after planting, the seedling grew to the height of 50 cm. Tom recorded the growth of the seedling at the end of each year and he found that its growth in that year was always 90% of the previous year's growth.

(i) What was the total growth of the silky oak tree in the first 3 years? (2 marks)

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(ii) Assuming that it maintains the same growth pattern, explain why the tree will never grow taller than the height of 5 m. (2 marks)

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(iii) In which year will the silky oak tree reach a height of 4 m (3 marks)

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(d) The function f is defined by $f(x) = 2 + \sqrt{x-3}$.

The function g is defined by $g(x) = \frac{12}{x} + 2$ for $x > 0$.

(i) Write the domain and range of the function f using the bracket interval notation. (2 marks)

(ii) Write an expression for the composite function $h(x) = g\big(f(x)\big)$ and hence find a value for $g\big(f(12)\big)$. (2 marks)

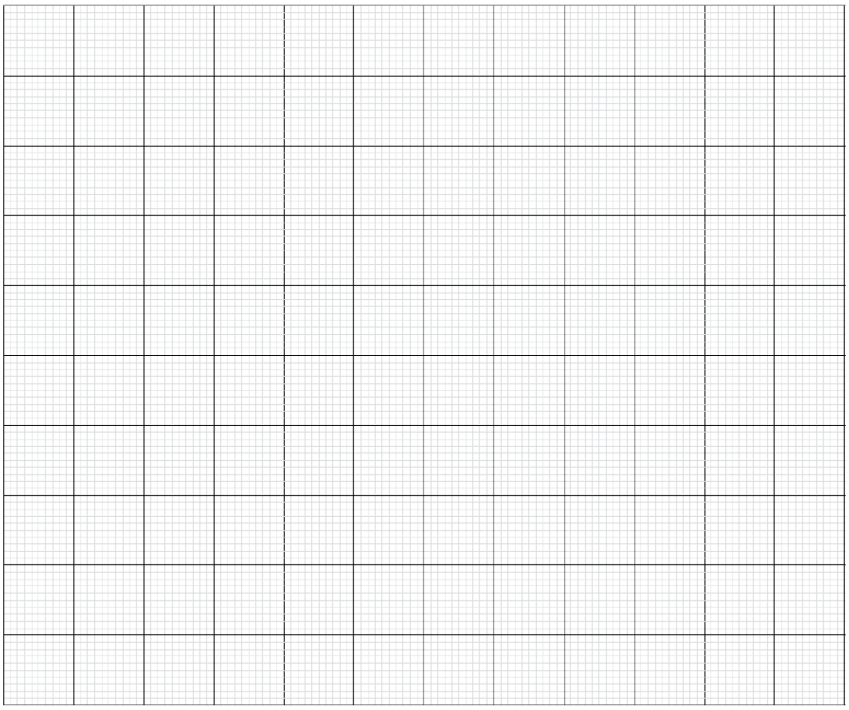
QUESTION 8 (15 Marks)

(a) Solve $|7x - 12| \geq 16$. (2 marks)

(b)

(i) Sketch the graphs of $f(x) = 2x - 2x^2$ and $g(x) = x - 1$ on the same set of axes below.

(2 marks)

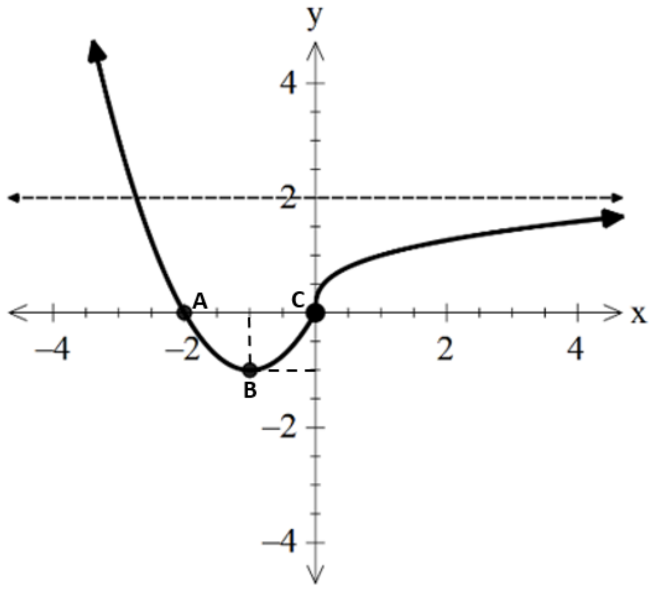


(ii) Using your graphs from part (i), or otherwise solve the inequality: (2 marks)

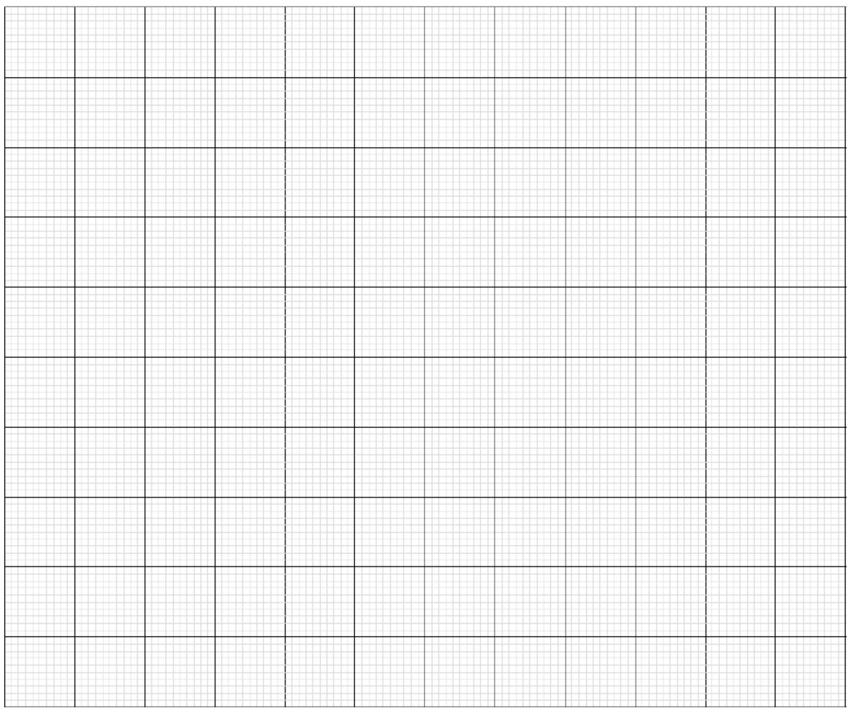
$$x - 1 < 2x - 2x^2$$

~~(c)~~ The curve of $y = f(x)$ is given below. There is an asymptote at $y = 2$.

Sketch $y = -2f(2x)$, indicating the equivalent points to A, B and C on the new graph. (3 marks)



~~(d)~~ Sketch the graph of $y = \frac{1}{2}\cos 2x$ for $0 \leq x \leq 2\pi$ showing the zeroes and the amplitude. (2 marks)



(e) Consider $y = 3\sin\left(x - \frac{\pi}{2}\right)$.

(i) State the amplitude and period. (2 marks)

(ii) Hence sketch the graph of $y = 3\sin\left(x - \frac{\pi}{2}\right)$ for $0 \le x \le 2\pi$ showing the zeroes and the amplitude. (2 marks)

QUESTION 9 (15 Marks)

(a) Consider $y = \frac{x+1}{x-4}$.

(i) Determine if the function is even or odd or neither. (2 marks)

(ii) Find any vertical asymptote(s). (1 mark)

(iii) Find the x and y intercepts. (2 marks)

(iv) Find any horizontal asymptote(s) and describe the behaviour of y for very large and small values of x. (2 marks)

~~(v)~~ Hence, sketch the graph of showing the features found in parts (i) to (iv). (2 marks)

(b) The first term of a geometric series is e^x and the fifth term is $81e^{9x}$.

(i) Show that the common ratio can be expressed by $r = 3e^{2x}$. (2 marks)

(ii) Find an expression for the 8^{th} term of the series. (1 mark)

(iii) Find the exact value of x if the limiting sum is 3. (3 marks)

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End of examination

Extra working pages – Clearly write the question numbers

GIRRAWEEN HIGH SCHOOL

YEAR 12 - TASK 1

2021

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DIRECTIONS TO CANDIDATES:

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- **TOTAL MARKS:** 65

Use the space provided to show your answers for each question including all relevant working out.

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- **Section I (5 marks): Questions 1 – 5**
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- **Section II (60 marks): Questions 6 – 9**
  - All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.

SECTION I: MULTIPLE CHOICE QUESTIONS 1 – 5

Shade A, B, C or D that best answers the question on page 2.

1. The fourth term of an arithmetic series is 27 and the seventh term is 12.

   What is the common difference?

   $T_4 = 27$

   $T_7 = 12$

   $T_n = a + (n-1)d$

   $27 = a + 3d$

   $a = 27 - 3d$

   $12 = a + 6d$

   $12 = 27 - 3d + 6d$

   $3d = -15$

   $d = -5$

   (A) -5
   (B) 5
   (C) 13
   (D) 42

2. What is the domain for $y = \frac{2x}{\sqrt{x+2}}$?

   $x + 2 > 0$

   $x > -2$

   $(-2, \infty)$

   (A) $[-2, \infty)$
   (B) $(-\infty, -2]$
   (C) $(-2, \infty)$
   (D) $(-\infty, -2)$

3. The discrete random variable X has the following probability distribution.

x	0	1	2	3
$P(X = x)$	a	$2a$	$5a$	$3a$

   The value of a is:

   $a + 2a + 5a + 3a = 1$

   $11a = 1$

   $a = \frac{1}{11}$

   (A) $\frac{1}{21}$
   (B) $\frac{1}{11}$
   (C) 11
   (D) 21

4. A geometric sequence has a first term of 20 and a common ratio of $-\frac{3}{2}$, find the 5^{th} term:

(A) 12.5

(B) $-151\frac{7}{8}$

(C) $101\frac{1}{4}$

(D) $-101\frac{1}{4}$

$a = 20$

$r = -\frac{3}{2}$

$T_5 = ?$

$T_n = ar^{n-1}$

$T_5 = 20\left(-\frac{3}{2}\right)^4$

$= \frac{20(3)^4}{2^4}$

$= \frac{405}{4}$

5. The graph of $y = f(x)$ passes through $(2, -3)$.

Where will the equivalent point be on the graph of $y = -f\left(\frac{x}{2}\right) - 5$?

(A) $(1, -2)$

(B) $(1, 2)$

(C) $(4, -2)$

(D) $(4, 2)$

point

$y = f(x)$ $(2, -3)$

$y = -f(x)$ $(2, 3)$

$y = -f\left(\frac{x}{2}\right)$ $(4, 3)$

$y = -f\left(\frac{x}{2}\right) - 5$ $(4, -2)$

Answers to Multiple Choice Questions

Select the alternative A, B, C or D that best answers the question.

1.	A	○	B	○	C	○	D	○
2.	A	○	B	○	C	○	D	○
3.	A	○	B	○	C	○	D	○
4.	A	○	B	○	C	○	D	○
5.	A	○	B	○	C	○	D	○

SECTION II: QUESTIONS 6 – 9

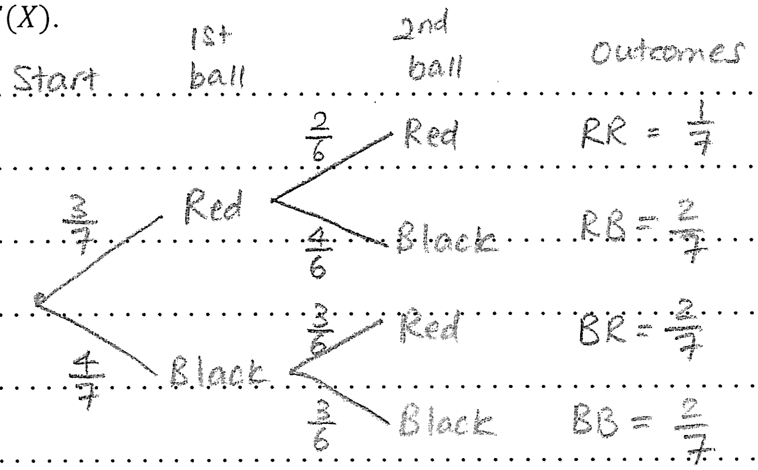
Show all necessary working for each question.

Marks may be deducted for careless or badly arranged work.

QUESTION 6 (14 Marks)

(a) A bag contains three red balls and four black balls. Two balls are selected at random without replacement from the bag. Let X be the number of black balls drawn.

(i) Using a tree diagram or otherwise, fill in the 2nd and the 3rd rows in the following table to find $E(X)$. (2 marks)



x	0	1	2	Sum
$P(X=x)$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$	1
$xP(X=x)$	0	$\frac{4}{7}$	$\frac{4}{7}$	$\sum xP(X=x) = E(X) = \frac{8}{7}$
$x^2P(X=x)$	0	$\frac{4}{7}$	$\frac{8}{7}$	$\sum x^2P(X=x) = E(X^2) = \frac{12}{7}$

(ii) Fill in the 4th row in the table above to find $E(X^2)$ (2 marks)

(iii) Hence find $Var(X)$ and standard deviation σ. (2 marks)

$Var(X) = E(X^2) - \mu^2 = \left(\frac{12}{7}\right) - \left(\frac{8}{7}\right)^2 = \frac{20}{49}$

$\sigma = \sqrt{Var(X)} = 0.6389$

(b) The discrete random variable X has probability distribution shown in the table below.

x	-1	0	1	2	3
$P(X = x)$	a	b	0.2	0.15	0.13

(i) Find the values of a and b, given that $E(X) = 0.55$. (3 marks)

$\Sigma x P(X=x) = E(X) = 0.55$

$(-a) + (0) + (0.2) + (0.3) + (0.39) = 0.55$

$-a + 0.89 = 0.55$

$\therefore a = 0.34$

$\Sigma P(X=x) = 1$

$a + b + 0.2 + 0.15 + 0.13 = 1$

$a + b + 0.48 = 1$

$b = 0.52 - a \quad \therefore b = 0.18$

(ii) Determine $Var(X)$. (1 mark)

$Var(X) = 2.0075$

(c) Evaluate:

(i) (2 marks)

$$\sum_{r=1}^{6} r^r = 1^1 + 2^2 + 3^3 + 4^4 + 5^5 + 6^6$$

$= 50069$

(ii) (2 marks)

$$\sum_{k=3}^{12} 2 \times 3^{k-1} = \underbrace{2(3)^2 + 2(3)^3 + 2(3)^4 + \ldots + 2(3)^{11}}_{\text{G.P with } a=18,\ r=3,\ n=10}$$

$S_{10} = \dfrac{18(3^{10}-1)}{3-1} = 531432$

QUESTION 7 (16 Marks)

(a) In an arithmetic series, the third term is 5 and the tenth term is 26. Find the sum of the first 14 terms.

(2 marks)

A.P with

$T_3 = 5$

$T_{10} = 26$

$T_n = a + (n-1)d$

$5 = a + 2d$

$\therefore a = 5 - 2d$ ... ①

$T_{10} = 26$

$26 = a + 9d$

Sub ①,

$26 = 5 - 2d + 9d$

$7d = 21$

$\therefore d = 3$

Sub. $d = 3$,

$\therefore a = -1$

$S_{14} = \frac{14}{2}\left(-2 + (14-1)(3)\right)$

$= 259$

(b) Kloe has taken up golf, each year she plays 30 rounds of golf. In 2011 she totalled 3300 shots, in 2021 she totalled 2760 shots. The number of shots she took each year decreased by the same amount. To make it as a professional golfer, she needs to get under 2115 shots in her 30 rounds. If this trend continues, in what year can she become a professional golfer?

(3 marks)

A.P. with

$a = 3300$

$T_{11} = 2760$

$T_n = a + (n-1)d$

$T_{11} = 2760$

$2760 = 3300 + (10)d$

$\therefore d = -54$

$T_n < 2115$

$3300 + (n-1)(-54) < 2115$

$3300 - 54n + 54 < 2115$

$54n > 1239$

$n > 22.94$

$\therefore T_{23}$ is the first term less than 2115

$\therefore$ In 2033, she can become a professional golfer.

(c) Tom planted a silky oak tree seed. At the end of the first year after planting, the seedling grew to the height of 50 cm. Tom recorded the growth of the seedling at the end of each year and he found that its growth in that year was always 90% of the previous year's growth.

(i) What was the total growth of the silky oak tree in the first 3 years? (2 marks)

Total growth in 3 years $= 50 + (50 \times 0.9) + (50 \times 0.9^2) = 135.5\text{ cm}$

(ii) Assuming that it maintains the same growth pattern, explain why the tree will never ~~reach a height of 6 m~~ grow taller than the height of 5m. (2 marks)

Total growth in n years $= \underbrace{50 + (50\times0.9) + (50\times0.9^2) + \dots + (50\times0.9^{n-1})}_{\text{G.P with } a=50,\ r=0.9}$

since $|r|<1$, it converges into a limiting sum

$$S = \frac{a}{1-r} = \frac{50}{1-0.9} = 500\text{cm}$$

$\therefore$ max. growth is 5m, which means it will never taller than 5m.

(iii) In which year will the silky oak tree reach a height of 4 m (3 marks)

Total growth = 400

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$400 = \frac{50(1-0.9^n)}{1-0.9}$$

$$1-0.9^n = 0.8$$

$$0.9^n = 0.2$$

$$n = \frac{\ln 0.2}{\ln 0.9}$$

$$n = 15.3$$

$\therefore$ in the 16^{th} year the tree will reach 4m.

(d) The function f is defined by $f(x) = 2 + \sqrt{x-3}$.

The function g is defined by $g(x) = \frac{12}{x} + 2$ for $x > 0$.

(i) Write the domain and range of the function f using the bracket interval notation. (2 marks)

Domain : $x \geqslant 3$ $[3, \infty)$

Range : $y \geqslant 2$ $[2, \infty)$

(ii) Write an expression for the composite function $h(x) = g(f(x))$ and hence find a value for $g(f(12))$. (2 marks)

$h(x) = \dfrac{12}{2+\sqrt{x-3}} + 2$

$g(f(12)) = h(12) = \dfrac{12}{2+\sqrt{12-3}} + 2$

$= \dfrac{22}{5} = 4.4$

QUESTION 8 (15 Marks)

(a) Solve $|7x - 12| \geq 16$. (2 marks)

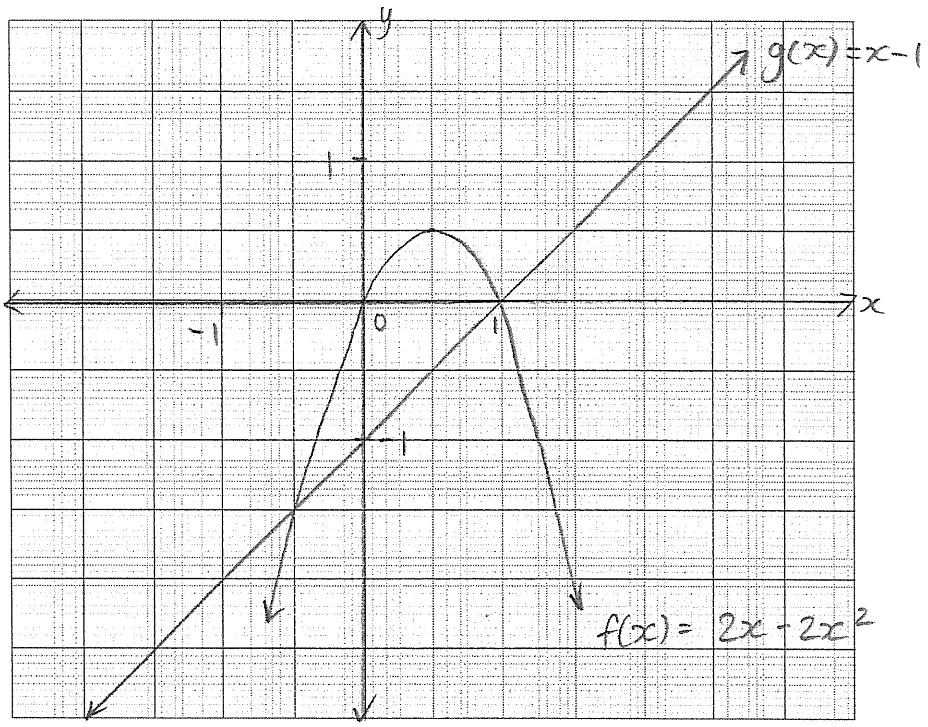
$7x-12 \geq 16$ or $7x-12 \leq -16$

$7x \geq 28$ $\quad$ $7x \leq -4$

$x \geq 4$ $\quad$ $x \leq -\frac{4}{7}$

(b)

(i) Sketch the graphs of $f(x) = 2x - 2x^2$ and $g(x) = x - 1$ on the same set of axes below. (2 marks)



(ii) Using your graphs from part (i), or otherwise solve the inequality: (2 marks)

$$x - 1 < 2x - 2x^2$$

$2x - 2x^2 = x - 1$

$2x^2 - x - 1 = 0$

$2x^2 - 2x + x - 1 = 0$

$2x(x-1) + (x-1) = 0$

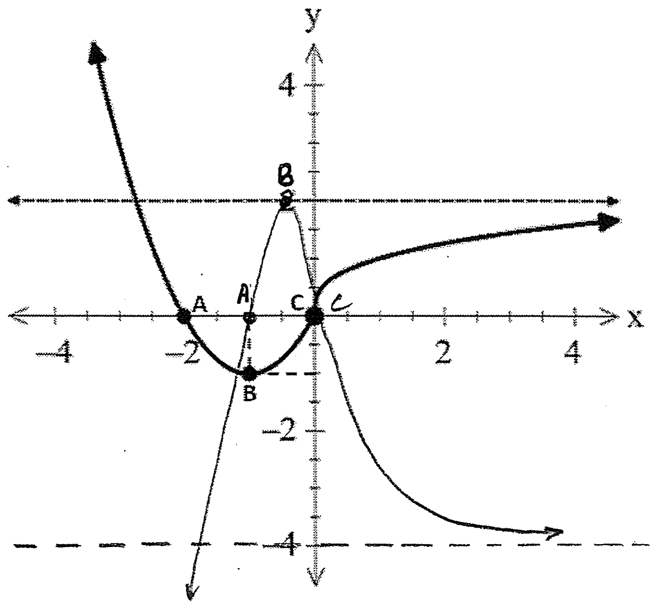
$(2x+1)(x-1) = 0$

$\therefore x = -\frac{1}{2}$ or $x = 1$

$-\frac{1}{2} < x < 1$

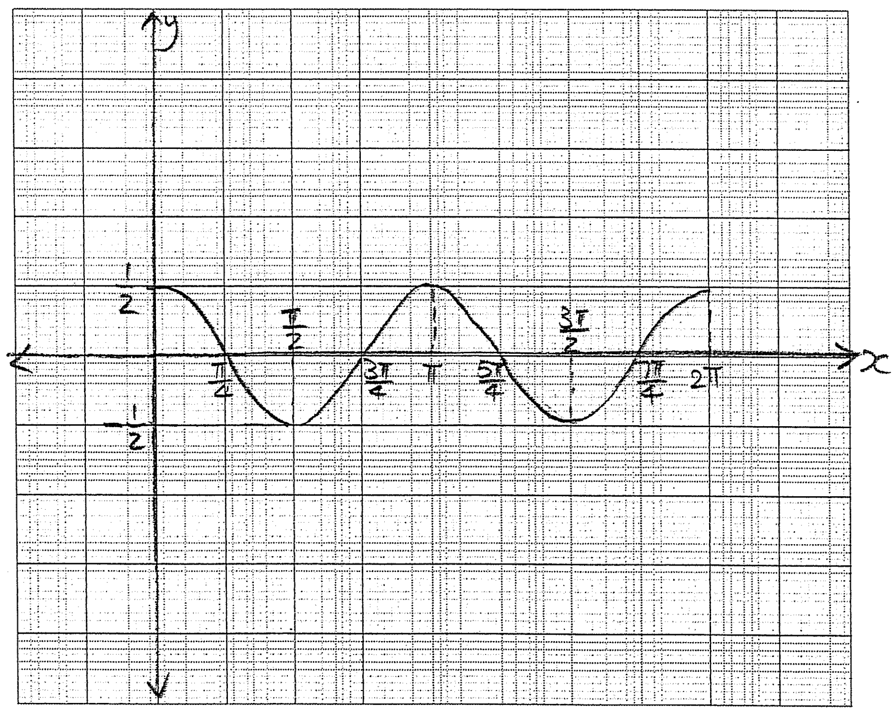
(c) The curve of $y = f(x)$ is given below. There is an asymptote at $y = 2$.

Sketch $y = -2f(2x)$, indicating the equivalent points to A, B and C on the new graph. (3 marks)



(d) Sketch the graph of $y = \frac{1}{2}\cos 2x$ for $0 \leq x \leq 2\pi$ showing the zeroes, and the amplitude. (2 marks)

$A = \frac{1}{2}$ $\quad P = \pi$



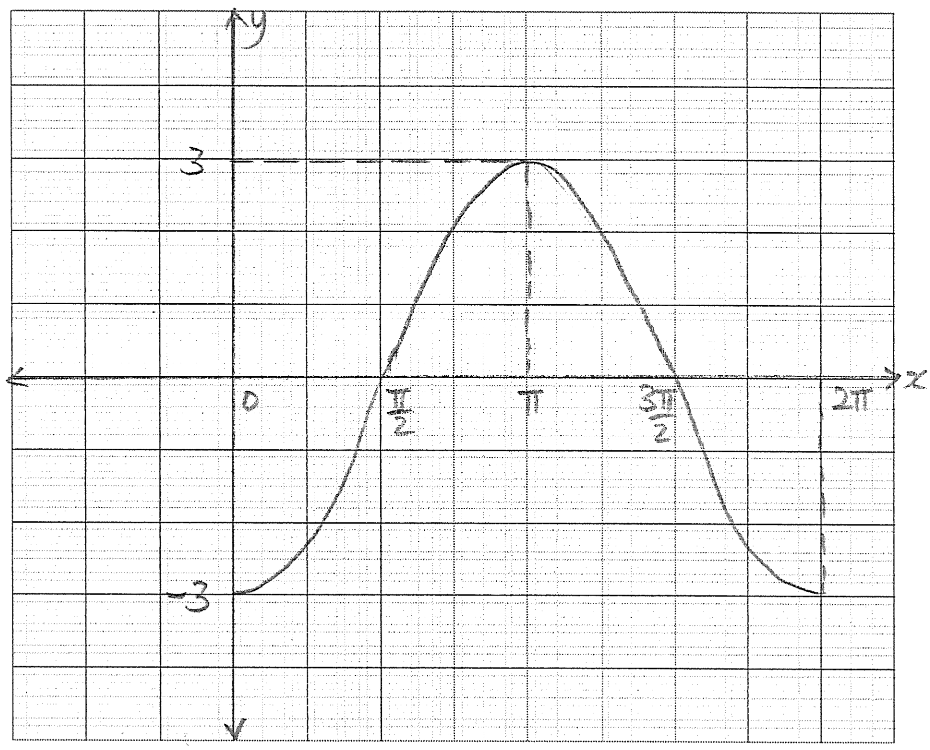
(e) Consider $y = 3\sin\left(x - \frac{\pi}{2}\right)$.

(i) State the amplitude and period. (2 marks)

Amplitude = 3

period = 2π

(ii) Hence sketch the graph of $y = 3\sin\left(x - \frac{\pi}{2}\right)$ for $0 \leq x \leq 2\pi$ showing the zeroes, and the amplitude. (2 marks)



QUESTION 9 (15 Marks)

(a) Consider $y = \frac{x+1}{x-4}$.

(i) Determine if the function is even or odd or neither. (2 marks)

$f(x) = \frac{x+1}{x-4}$ $f(x) \neq f(-x)$

$f(-x) = \frac{-x+1}{-x-4}$ $f(-x) \neq -f(x)$

$\therefore$ the function is neither even or odd

$-f(x) = -\frac{x-1}{x-4}$

(ii) Find any vertical asymptote(s). (1 mark)

Since $x \neq 4$, $x = 4$ is the vertical asymptote

(iii) Find the x and y intercepts. (2 marks)

x-int when $y = 0$

$\frac{x+1}{x-4} = 0$

$x + 1 = 0$ $\therefore (-1, 0)$

$\therefore x = -1$

y-int when $x = 0$

$y = -\frac{1}{4}$

$\therefore (0, -\frac{1}{4})$

(iv) Find any horizontal asymptote(s) and describe the behaviour of y for very large and small values of x. (2 marks)

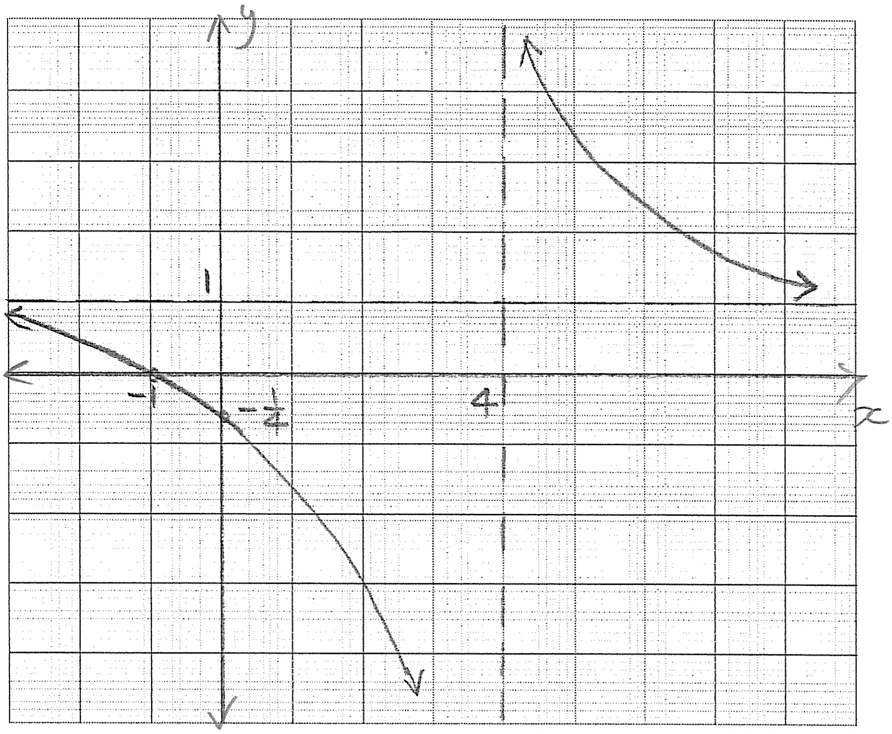
$y = \frac{\frac{x}{x} + \frac{1}{x}}{\frac{x}{x} - \frac{4}{x}} = \frac{1 + \frac{1}{x}}{1 - \frac{4}{x}}$

As $x \to \infty$, $y \to 1^{+}$

As $x \to -\infty$, $y \to 1^{-}$

$\therefore y = 1$ is the horizontal asymptote.

(v) Hence, sketch the graph of showing the features found in parts (i) to (iv). (2 marks)



(b) The first term of a geometric series is e^x and the fifth term is $81e^{9x}$.

(i) Show that the common ratio can be expressed by $r = 3e^{2x}$. (2 marks)

G.P with

$a = e^x$

$T_5 = 81e^{9x}$

$T_n = ar^{n-1}$

$81e^{9x} = e^x(r)^4$

$r^4 = 81e^{8x}$

$r = (81e^{8x})^{1/4}$

$\therefore r = 3e^{2x}$ as required.

(ii) Find an expression for the 8th term of the series. (1 mark)

$T_8 = e^x(3e^{2x})^7$

$= (3^7e^{14x})(e^x)$

$= 3^7e^{15x}$

(iii) Find the exact value of x if the limiting sum is 3. (3 marks)

$S_\infty = 3$

$$3 = \frac{e^x}{1-3e^{2x}}$$

$$3 - 9e^{2x} = e^x$$

$$9e^{2x} + e^x - 3 = 0$$

Let $u = e^x$, then

$$9u^2 + u - 3 = 0$$

$$u = \frac{-1 \pm \sqrt{1+108}}{18}$$

$$u = \frac{-1 \pm \sqrt{109}}{18}$$

Hence, $e^x = \frac{-1+\sqrt{109}}{18}$

$\left(e^x = \frac{-1-\sqrt{109}}{18} \text{ is not a solution since } e^x > 0 \right)$

$$\therefore x = \ln\left(\frac{\sqrt{109}-1}{18}\right)$$

End of examination