Student Name:________________________________

Teacher:____

HURLSTONE AGRICULTURAL HIGH SCHOOL

2021

Year 12

Task 2

Mathematics Advanced

Examiners Ms L Yuen, Mr S Faulds, Ms T Tarannum, Mr G Huxley

General Instructions

- Reading time – 3 minutes
- Working time – 60 minutes
- Write using black pen.
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- A hand written double sided A4 summary sheet on the area of Work
- In Questions 9-12, show relevant mathematical reasoning and/or calculations

Total marks: 52

Section 1 – 8 marks

- Attempt Questions 1-8
- Allow about 5 minutes for this section.

Section 2 – 44 marks

- Attempt Questions 9-12
- Allow about 50 minutes for this section

Written by 2021 HAHS Maths Faculty, rewritten by Ariq Abdullah

Section 1

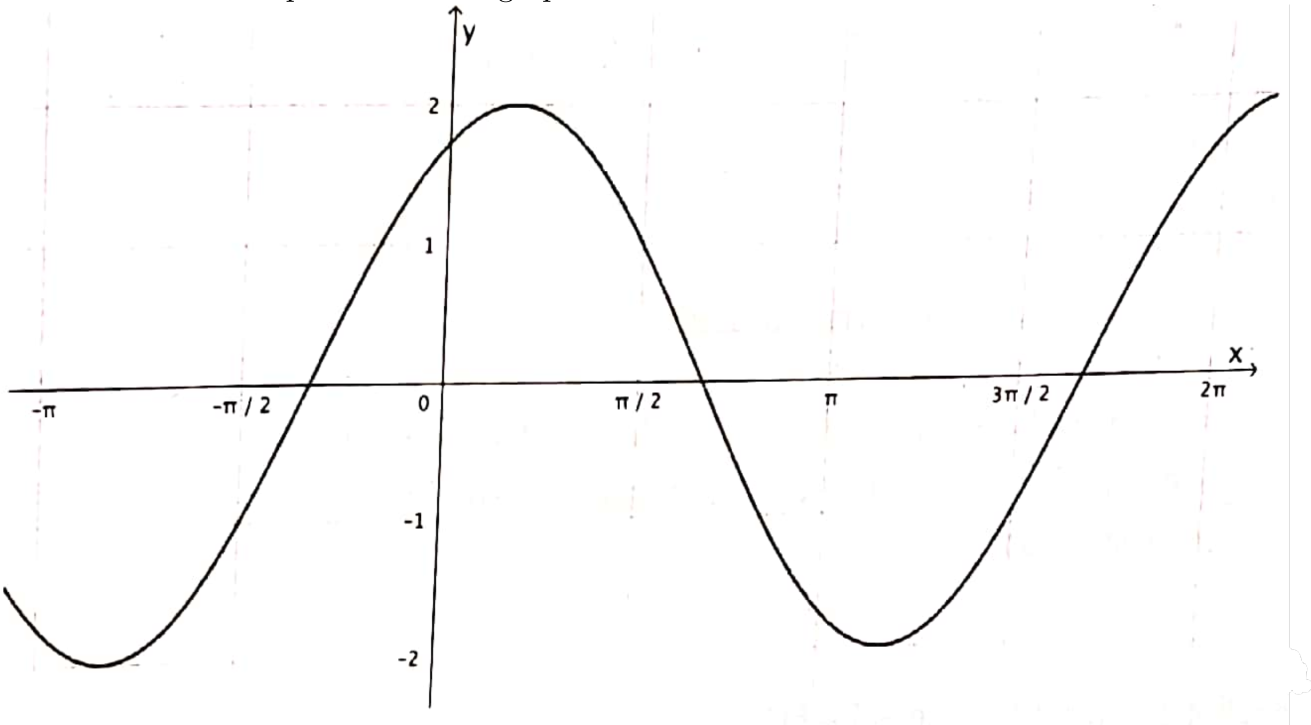
8 marks

Attempt Questions 1-8

Allow about 10 minutes from this section.

Question 1

What is the correct equation for the graph shown below?



Not to scale.

a. $y = 2\sin\left(x + \frac{\pi}{3}\right)$

b. $y = \sin\left(2x + \frac{\pi}{3}\right)$

c. $y = 2\cos\left(x + \frac{\pi}{6}\right)$

d. $y = \cos\left(2x + \frac{\pi}{6}\right)$

Question 2

What are the solutions for the equation $4\sin^2 x = 3$ over the domain $[-\pi, \pi]$?

a. $x = \frac{\pi}{6}$

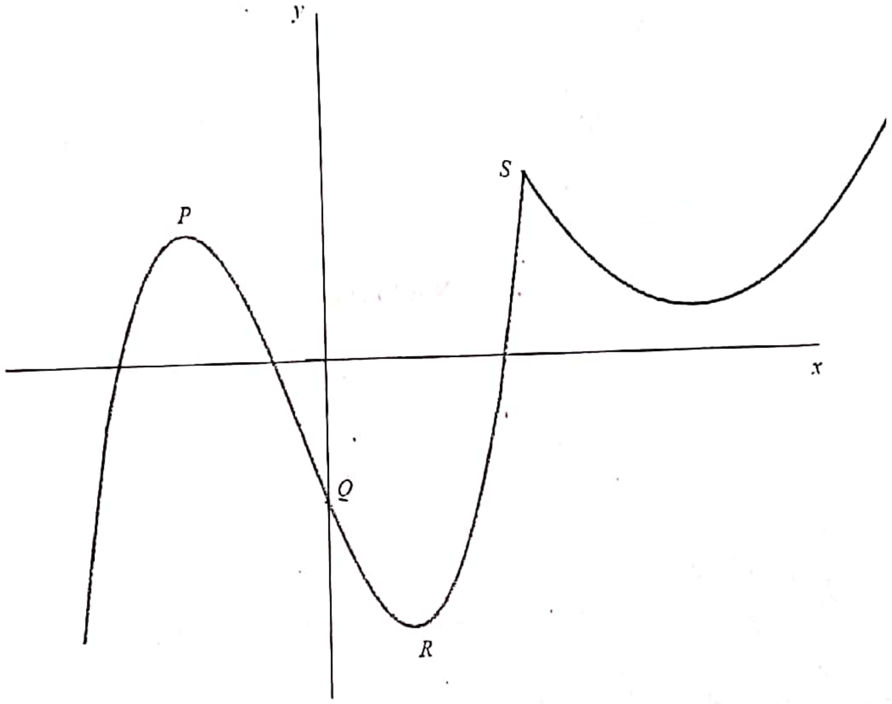
b. $x = -\frac{\pi}{3}, \frac{\pi}{3}$

c. $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

d. $x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

Question 3

At which point is the following function $y = f(x)$ not differentiable?



a. P
b. Q
c. R
d. S

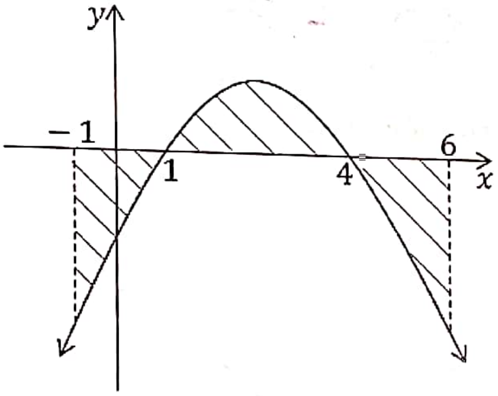
Question 4

Given $f(x) = x^3 - 3x$, what values of x is the function f(x) increasing.

a. $-1 \le x \le 1$
b. $x \le -1 \text{ } or \text{ } x \ge 1$
c. $x \le -1$
d. $x \ge 1$

Question 5

Which of the following expression gives the total area of the shaded region in the diagram?



Not to scale.

a. $\int_{-1}^{6} f(x).\,dx$

b. $-\int_{-1}^{0} f(x).\,dx + \int_{0}^{6} f(x).\,dx$

c. $-\int_{-1}^{1} f(x).\,dx + \int_{1}^{4} f(x).\,dx - \int_{4}^{6} f(x).\,dx$

d. $\int_{1}^{4} f(x).\,dx + 2\int_{4}^{6} f(x).\,dx$

Question 6

What is the primitive of $x^{-2} + 6x$?

a. $-\frac{1}{x} + 3x^2$

b. $\frac{1}{x} + 3x^2$

c. $-\frac{2}{x^3} + 6$

d. $\frac{2}{x^3} + 6$

Question 7

Which of the following expression gives $\frac{d}{dx}\cos^2 x$?

a. $2cosx$

b. $2cosxsinx$

c. $-2cosxsinx$

d. $2sinx$

Question 8

What is the equation of the tangent of the curve $y = 2sinx$ where $x = \pi$?

a. $y = 2(x - \pi)$

b. $y = -2(x + \pi)$

c. $y = -2(x - \pi)$

d. $y = 2x + \pi$

Section 2

44 marks

Attempt Questions 9-12

Allow about 50 minutes for this section

Question 9 (11 Marks) **Marks**

(a) Graphically show that the equation $cos x = x$ has only one solution for all real x. **2**

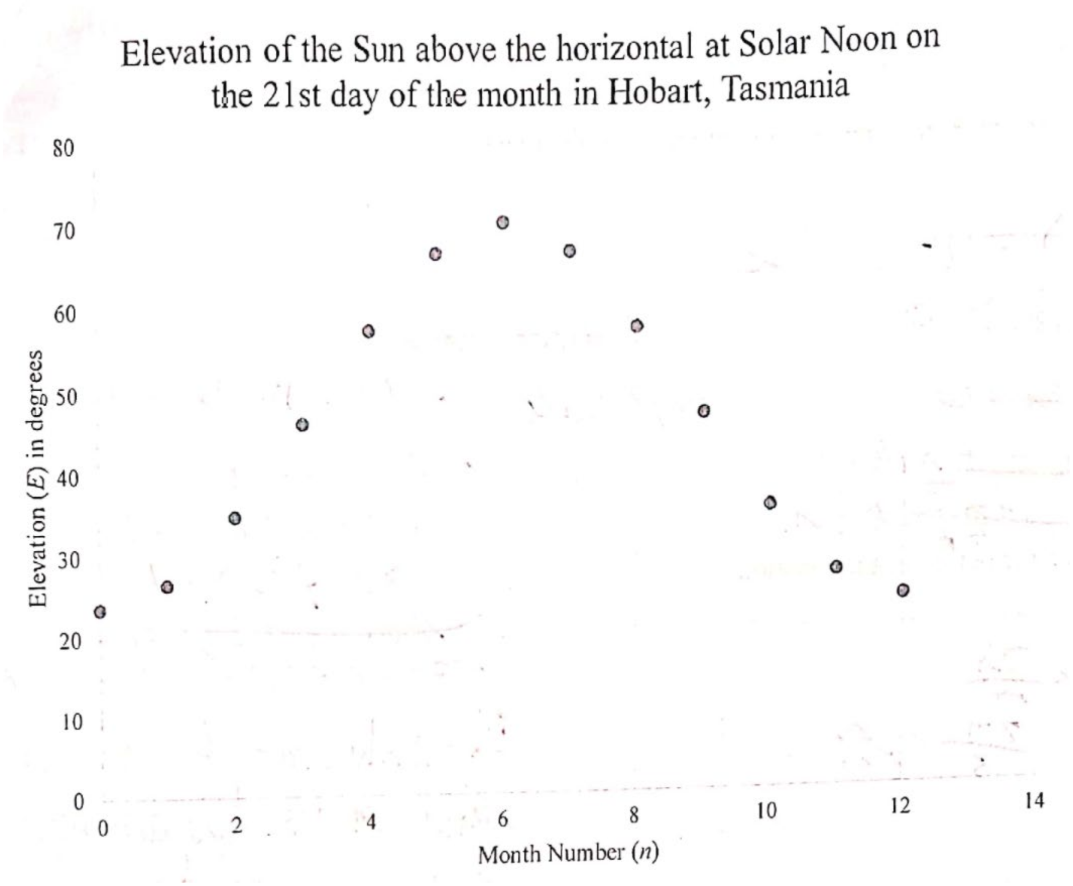
(b) Solve the equation $\sqrt{2} tan x = -1$, for $0 \leq x \leq 2\pi$ **3**

(c) A scientist living in Hobart, Tasmania collected data on the elevation of the Sun above the horizontal at solar noon on the 21st day of each month over the period 21st June, 2019 (month zero) to 21st June, 2020 (month 12). **3**

The scientist's data is presented below in tabular and graphical form.

Month number (June '19 – June '20)	0	1	2	3	4	5	6	7	8	9	10	11	12
Elevation of Sun (degrees above the horizontal	23.7	26.6	35	46.3	57.6	66.8	70.6	66.9	67.6	46.9	35.3	27	23.7

Question 9 continued on next page



Upon inspection of the graph, the scientist decides that trigonometric graph could be fitted to the data,

By considering the appropriate characteristics of the graph, find the trigonometric equation "$E = k\cos(an) + c$" to model the scientist's graph in terms of the month number, n, and elevation of the Sun, E.

(d) For the function $f(x) = 2\sin 3x - 1$:

(i) State the maximum and minimum values of the function. **1**

(ii) State the period of the function. **1**

(iii) Draw a neat sketch of the function for the domain $0 \leq x \leq 2\pi$, clearly identifying the characteristics of the graph found above. It is NOT necessary to show x intercepts. **1**

Question 10 (11 marks) **Marks**

(a) Find the equation of the tangent to the curve $y = x^2 - 2x$ at $(-1,3)$. **2**

(b) Consider the curve $y = x^3 - 6x^2 + 9x$

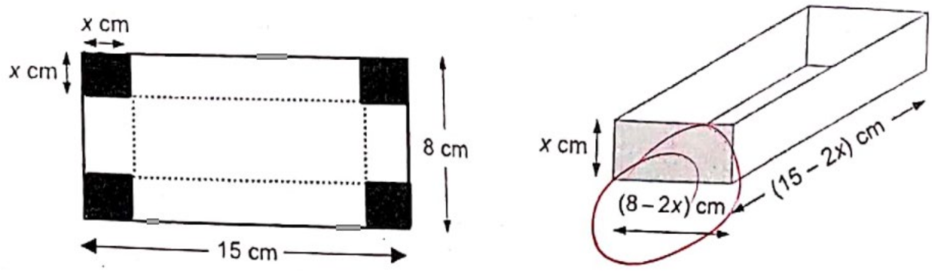
(i) Find the stationary points and determine their nature. **3**

Question 10 continued on next page

(ii) Find the points of inflexion. **1**

(iii) Sketch the curve, labelling the intercepts, stationary points, and point of inflexion. **1**

(c) A rectangular sheet of cardboard measures 15cm by 8cm. Four equal squares of sides x cm are cut out of the corners and the sides are folded up to form an open rectangular box as shown in the diagram below.



(i) Show that the volume of the box is $v = 2(60x - 23x^2 + 2x^3)$ **1**

(ii) Find the value of x so that the vox will have a maximum volume. Justify your answer. **3**

Question 11 (11 marks) **Marks**

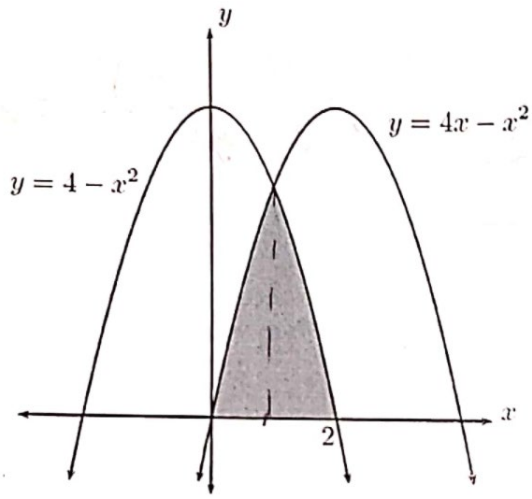
(a)

(i) Differentiate $y = \sqrt{9 - x^2}$ with respect to x. **2**

(ii) Hence, find $\int \left(\frac{6x}{\sqrt{9-x^2}}\right).dx$ **2**

(b) Evaluate $\int (4x+3)^6.dx$, expressing your answer in its simplest form **1**

(c) Find the area of the region shaded in the region below **3**



Not to scale

Question 11 continued on next page

(d) Using the trapezoidal rule with 4 subintervals, evaluate the approximate area under the curve $y = \frac{1}{x^2+1}$ between $x = 1$ and $x = 3$. **3**

Question 12 (11 marks) **Marks**

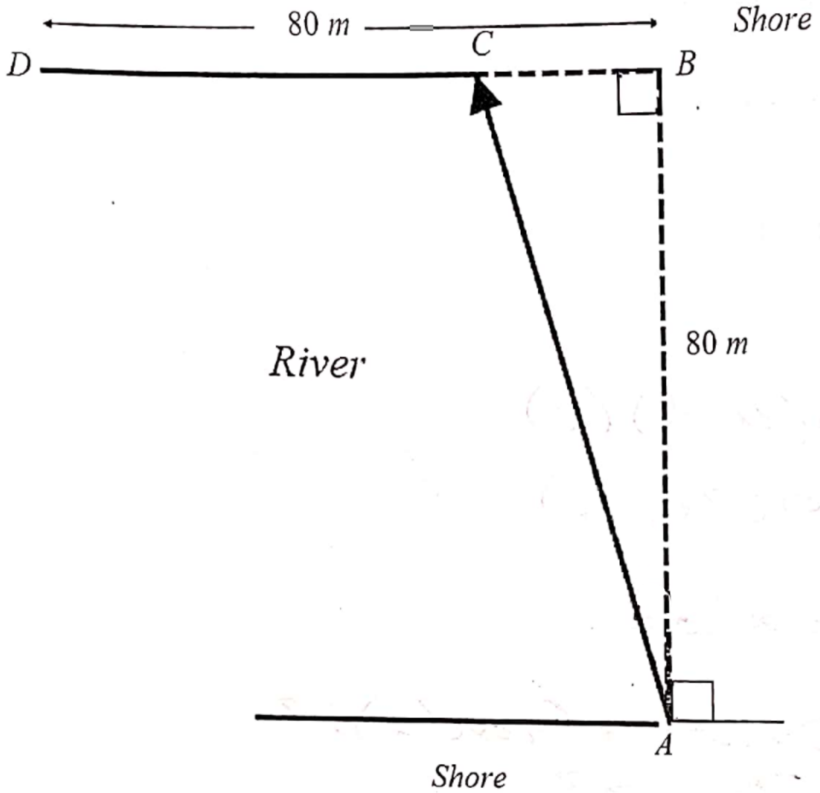
(a) Find $\frac{dy}{dx}$ of the following:

(i) $y = 3\sin\left(2x - \frac{\pi}{2}\right)$ **1**

(ii) $y = \dfrac{x^2}{\tan x}$ **3**

(iii) Find the first and second derivatives of the function $y = \sin^2 x - \sqrt{3}cosx$ **2**

(b) In the diagram, a man swims across a river from point A to point C. He then runs along the shore to D. It is given that $\angle ABD = \frac{\pi}{2}$, $AB = BD = 80m$. $\angle BAC = \theta$, where $0 \leq \theta \leq \frac{\pi}{4}$. Suppose the man swims at 40m/min and runs at 80m/min, and he can finish the whole journey in T minutes.



(i) Show that $T = \frac{2}{cos\theta} - tan\theta + 1$ **2**

(ii) Find the least value of T. (give your answer as an exact value.) **4**

End of Exam

Year 12	Mathematics Advanced	2021 HSC Task 2

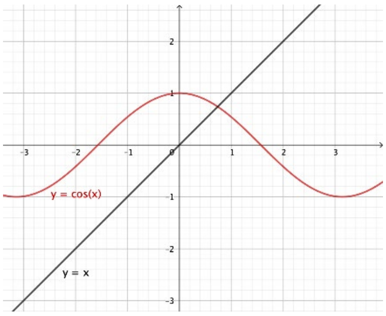
Answers for MC: (1) A, (2) C, (3) D, (4) B , (5) C, (6) A, (7) C, (8) C

Question No. 9 Solutions and Marking Guidelines

Outcomes Addressed in this Question

MA12-5 - applies the concepts and techniques of periodic functions in the solution of problems involving trigonometric graphs

Outcome	**Solutions**	**Marking Guidelines**
MA12-5	(a) Considering the graphs of the equations $y = x$ and $y = \cos(x)$, shown below:	Part (a) 2 marks for correct solution 1 mark – both graphs sketched correctly 1 mark – correct explanation for the reason for only one solution



There is only one point of intersection, hence, only one value of x will satisfy the equation $\cos(x) = x$.

Part (b)
3 marks for correct solution

1 mark – obtain first two correct solutions for $2x$ in 2nd and 4th quadrants

1 mark – obtain additional two solutions for $2x$ in 6^{th} and eighth quadrants.

1 mark – final correct solution obtained by halving answers for $2x$.

(b)

$$\sqrt{2}\tan 2x = -1, \qquad 0 \le x \le 2\pi$$

$$\tan 2x = \frac{-1}{\sqrt{2}}, \qquad 0 \le 2x \le 4\pi$$

$$2x = 0.804\pi, 1.804\pi, \qquad 0 \le 2x \le 4\pi$$

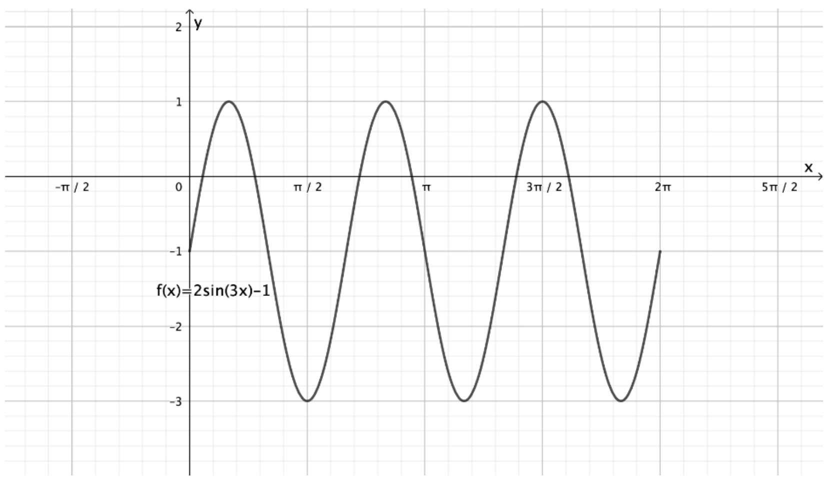
$$2.804\pi, 3.804\pi$$

OR $2.526, 5.667, 8.809, 11.951$

$$x = 0.402\pi, 0.902\pi, \qquad 0 \le x \le 2\pi$$

$$1.402\pi, 1.902\pi$$

OR $1.263, 2.833, 4.405, 5.975$

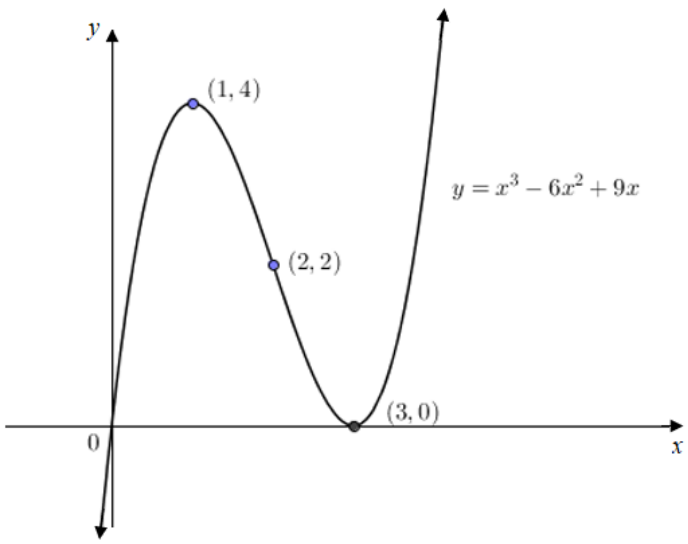
	(c) The graph is of the form $E = k\cos(an) + c$, where k is the amplitude, $a = \dfrac{2\pi}{\text{period}}$ (or $a = \dfrac{360}{\text{period}}$, if done in degrees) and c is the vertical shift of the graph. The graph is the negative version of the cosine graph since there is a minimum value where it crosses the E axis, and its amplitude will be $k = \dfrac{70.5 - 23.7}{2} = 23.4$. The period of the graph is 12, hence, $a = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$ (or $a = \dfrac{360}{12} = 30$, if done in degrees). There is a positive vertical shift of the graph, so $c = 70.5 - 23.4 = 47.1$. Hence, equation of the graph: $E = 47.1 - 23.4\cos\left(\dfrac{\pi n}{6}\right)$ or $E = 47.1 - 23.4\cos(30n)$, if done in degrees.	Part (c) 3 marks correct solution 1 mark for correct value of k 1 mark for correct value of a 1 mark for correct value of c and equation
	(d)(i) Max. = 1, Min. = –3 (ii) $T = \dfrac{2\pi}{3}$ (iii) 	Part (d) 1 mark – correct answer 1 mark – correct answer 1 mark- correct, neat graph

Year 12	Mathematics Advanced	2021 HSC Task 2
Question No. 10	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

MA12-6 - applies appropriate differentiation methods to solve problems

Outcome	Solutions	Marking Guidelines
MA12-6	a) $\frac{dy}{dx} = 2x - 2$ Slope of the tangent at (–1, 3) $\frac{dy}{dx} = 2(-1) - 2$ $= -4$ The equation of the tangent at (–1, 3) is: $\frac{y-3}{x+1} = -4$ $y - 3 = -4(x+1)$ $y = -4x - 1$	Part (a) 2 marks for correct solution 1 mark for finding the correct gradient. 1 mark for finding the equation of tangent correctly.
	b) $\frac{dy}{dx} = 3x^2 - 12x + 9$ $= 3(x^2 - 4x + 3)$ $= 3(x-1)(x-3)$ For stationary points, $\frac{dy}{dx} = 0$ $3(x-1)(x-3) = 0$ $x = 1, \quad x = 3$ $when\ x = 1,\ y = 4$ $x = 3,\ y = 0$ (1,4) and (3,0) are possible stationary points. $\frac{d^2y}{dx^2} = 6x - 12$ When $x = 1$ $\frac{d^2y}{dx^2} = 6(1) - 12$ $= -6$ $< 0 \Rightarrow$ max turning point at $(1, 4)$ When $x = 3$ $\frac{d^2y}{dx^2} = 6(3) - 12$ $= 6$ $> 0 \Rightarrow$ min turning point at $(3, 0)$	Part (b) 3 marks for correct solution 1 mark for correct derivative 2 marks for correct stationary points with indication for max and min

	$\therefore$ Minimum turning point at (3, 0) and maximum turning point at (1,4) and the curve pass through origin. For any points of inflection $\frac{d^2y}{dx^2}=0$ $6x-12=0$ $x=2$ x: 1.9, 2, 2.1 $\frac{d^2y}{dx^2}$: $\frac{-3}{5}$, 0, $\frac{3}{5}$ The concavity changes. (****Need to test concavity and must calculate the values of 2nd derivative) When $x = 2$ $y=(2)^3-6(2)^2+9(2)$ $=2$ $\therefore$ There is a point of inflexion at (2, 2).  c) Let V cm^3 be the volume of the open box formed. The dimensions of the box, in cm, are $x, (15-2x)$ and $(8-2x)$, where $0 \le x \le 4$. $V=x(15-2x)(8-2x)$ $=x(120-30x-16x+4x^2)$ $=120x-46x^2+4x^3$ $=2(60x-23x^2+2x^3)$	1 mark: showing points of inflexion correctly. 1 mark: All features correct and labelled, including max/min and inflexion point. Part (c) 1 mark for work shown correctly.

x	1.9	2	2.1
$\frac{d^2y}{dx^2}$	$\frac{-3}{5}$	0	$\frac{3}{5}$

	$\frac{dv}{dx}=2(60-46x+6x^2)$ $=4(30-23x+3x^2)$ $=4(30-23x+3x^2)$ $=4(x-6)(3x-5)$ put $\frac{dv}{dx}=0$ $(x-6)(3x-5)=0$ $x=\frac{5}{3}$ *or* 6 (*rejected as* $x\in[0,4]$ $\frac{d^2v}{dx^2}=4(-23+6x)$ *When* $x=\frac{5}{3}$ $\frac{d^2v}{dx^2}=4(-23+6\times\frac{5}{3})$ $=-52$ <0 $\therefore V$ is maximum when $x=\frac{5}{3}$.	3 marks for correct solution 1 mark for the correct derivative. 1 mark for correct value of x with reason 1 mark for x value calculated and maximum shown.

Outcome Addressed in this Question

MA12-3 - applies calculus techniques to model and solve problems

Outcome	**Solutions**	**Marking Guidelines**
MA12-3	(a)(i) $\frac{dy}{dx}=\frac{1}{2}(9-x^2)^{\frac{-1}{2}}(-2x)$ $=-x(9-x^2)^{\frac{-1}{2}}$ $=\frac{-x}{\sqrt{9-x^2}}$	Part (a) 2 marks Correct solution. 1 mark Substantial progress towards correct solution.
	(ii) $\int\frac{6x}{\sqrt{9-x^2}}dx=\int-6\frac{dy}{dx}dx$ $=-6y+\mathrm{C}$ $=-6\sqrt{9-x^2}+\mathrm{C}$ Alternate method $\int\frac{-x}{\sqrt{9-x^2}}dx=\sqrt{9-x^2}+\mathrm{C}$ $\frac{-1}{6}\int\frac{6x}{\sqrt{9-x^2}}dx=\sqrt{9-x^2}+\mathrm{C}$ $\therefore\int\frac{6x}{\sqrt{9-x^2}}dx=-6\sqrt{9-x^2}+\mathrm{C}$	2 marks Correct solution. 1 mark Substantial progress towards correct solution
	b) $\int(4x+3)^6dx=\frac{(4x+3)^7}{7\times4}+C$ $=\frac{1}{28}(4x+3)^7+C$	Part (b) 1 mark Correct solution

	c) Intersection of two graphs for x is $4-x^2=4x-x^2$ $4=4x$ $x=1$ The shaded area is: $A=\int_0^1(4x-x^2)dx+\int_1^2(4-x^2)dx$ $=\left[2x^2-\frac{x^3}{3}\right]_0^1+\left[4x-\frac{x^3}{3}\right]_1^2$ $=\left[(2(1)^2-\frac{(1)^3}{3})-(2(0)^2-\frac{(0)^3}{3})\right]_0^1+\left[(4(2)-\frac{(2)^3}{3})-(4(1)-\frac{(1)^3}{3})\right]_1^2$ $=\frac{5}{3}+\frac{16}{3}-\frac{11}{3}$ $=\frac{10}{3}$ units^2	Part (c) 3 marks Correct solution 1 mark Correct value of x 1 mark Correct expression of the shaded area 1 mark Correct answer
	(d) For four subintervals, $h=\frac{3-1}{4}=0.5$ <table><tr><td>x</td><td>1</td><td>1.5</td><td>2</td><td>2.5</td><td>3</td></tr><tr><td>$y=\frac{1}{x^2+1}$</td><td>$\frac{1}{2}$</td><td>$\frac{4}{13}$</td><td>$\frac{1}{5}$</td><td>$\frac{4}{29}$</td><td>$\frac{1}{10}$</td></tr></table> $A\simeq\frac{0.5}{2}\left[\frac{1}{2}+2\left(\frac{4}{13}+\frac{1}{5}+\frac{4}{29}\right)+\frac{1}{10}\right]$ $\simeq\frac{0.5}{2}\left[\frac{713}{377}\right]$ $\simeq\frac{713}{1508}$ $\simeq 0\cdot4728$ units^2	Part (d) 3 marks: Correct solution 1 mark: Correct value of interval 2 marks: Correct area 1 mark Substantial progress towards correct solution

Outcome Addressed in this Question

MA12-3 - applies calculus techniques to model and solve problems in trigonometric functions

	Solutions	Marking Guidelines
(a)	(i) $\frac{dy}{dx} = 3\cos(2x - \frac{\pi}{2}) \times 2$ $= 6\cos(2x - \frac{\pi}{2})$	**(i) 1 mark:** Correct
	(ii) $y = \frac{x^2}{\tan x}$ $U = x^2$ $U' = 2x$ $V = \tan x$ $V' = \sec^2 x$ $\frac{dy}{dx} = \frac{2x\tan x - x^2 \sec^2 x}{\tan^2 x}$	**(ii) 2 marks:** Correct **1 mark:** Relevant progress.
	(iii) $y = \sin^2 x - \sqrt{3}\cos x$ $\frac{dy}{dx} = 2\sin x\cos x + \sqrt{3}\sin x$ $\frac{d^2y}{dx^2} = 2(\sin x(-\sin x) + \cos x\cos x) + \sqrt{3}\cos x$ $= 2(-\sin^2 x + \cos^2 x) + \sqrt{3}\cos x$	**(iii)** **2 marks:** Both first and second derivatives correct. **1 mark:** One correct
(b)	(i) In ΔABC, $AC = \frac{80}{\cos\theta}$ $BC = 80\tan\theta$ $\therefore CD = 80 - 80\tan\theta$ $T = \frac{AC}{40} + \frac{CD}{80}$ $= \frac{80}{40\cos\theta} + \frac{80 - 80\tan\theta}{80}$ $= \frac{2}{\cos\theta} + 1 - \tan\theta$	**b)(i)** **2 marks:** Correct **1 mark:** Relevant progress.

(ii)

$$\frac{dT}{d\theta} = \frac{2\sin\theta}{\cos^2\theta} - \sec^2\theta$$
$$= 2\sec^2\theta\sin\theta - \sec^2\theta$$
$$= \sec^2\theta(2\sin\theta - 1)$$

$$\frac{dT}{d\theta} = 0$$

Then $\sec^2\theta(2\sin\theta - 1) = 0$

$\sec\theta = 0$ (rejected) or $\sin\theta = \frac{1}{2}$

$$\theta = \frac{\pi}{6}$$

So $\frac{dT}{d\theta} = 0 \ at \ \theta = \frac{\pi}{6}$

θ	0	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{4}$ or 45°
$\frac{dT}{d\theta}$	-1	0	0.828
	\	___	/

$\therefore T$ attain its minimum when $\theta = \frac{\pi}{6}$

The least value of

$$T = \frac{2}{\cos\frac{\pi}{6}} - \tan\frac{\pi}{6} + 1$$
$$= \frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}} + 1$$
$$= \frac{3}{\sqrt{3}} + 1$$
$$= \sqrt{3} + 1$$

(ii)

2 marks: Correct

1 mark: Relevant progress.

1 mark: for showing min value

1 mark for correct value of T

.