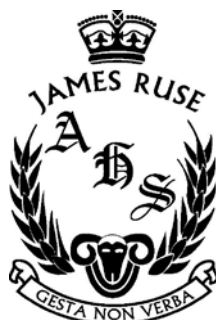


Name:	
Class:	



HIGHER SCHOOL CERTIFICATE EXAMINATION ASSESSMENT TASK 3 2019

MATHEMATICS

**Time Allowed – 90 Minutes
(Plus 5 minutes reading time)**

Total marks - 60

- All three questions may be attempted
- Department of Education approved calculators are permitted
- All necessary working should be shown in all questions
- Marks may not be awarded for careless or untidy work
- Write your student number neatly at the top of every page of your answer sheets

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 1, Question 2, etc. Each question must show your Candidate Number.

Question 1 (20 marks)

- a) Sketch the curve $y = 3e^{\frac{x}{2}}$ showing all important information. (Include at least 2 distinct points). 2
- b) Given a certain population N after time t is given by $N = Ae^{kt}$:
- i. Solve for k when $N = 20, A = 4, t = 3$ 2
 - ii. Solve for A in simplest form when $N = -9, k = \ln 3, t = 4$ 1
- c) The population of a particular town decreases at a rate that is proportional to its population at the time. If the population is denoted by P and time is denoted by t , write a differential equation that represents the sentence above. You must state the conditions of any constants that are introduced. 1
- d) Bensen walks/runs along a straight road such that his displacement x metres from a fixed tower on the road after t seconds is given by $x = t^2 - 7t - 18$.
- i. What is the distance between Bensen and the tower after 5 seconds? 1
 - ii. After how many seconds will Bensen be at the tower? 1
 - iii. What is Bensen's initial velocity? 1
 - iv. If Bensen is only capable of sustaining a speed of 8ms^{-1} , What is Bensen's displacement from the tower before this model is no longer valid? 2
 - v. What was Bensen's average velocity for the first 6 seconds? 1
- e) Find the velocity functions in terms of time for the following displacement functions:
- i. $x = t^3 - 3t^2 + 4t - 2$ 1
 - ii. $x = 3\sin t - 4\cos t$ 1
 - iii. $x = \sqrt{t^2 + 1}$ 1

- f) In a class of 16 girls and 14 boys, 2 people were chosen at random to complete a question on the board. Find the probability that:
- i. The students are of different genders 1
 - ii. Both are girls 1
- g) 4 coins are to be distributed randomly to 4 people. Each person may receive 0, 1, 2, 3 or all 4 coins. What is the probability that Robert:
- i. Gets all 4 coins? 1
 - ii. Gets no coins? 1
 - iii. Get at least one coin? 1

Question 2 (20 marks) Start a new sheet of paper

- a) Consider two rabbit colonies A and B with the same initial population. Their populations after t hours are given by

$$A(t) = P_0 e^{at} \text{ and } B(t) = P_0 e^{bt} \text{ respectively.}$$

When the rate at which B is growing is k times the rate at which A is growing, the population of B is twice the population of A .

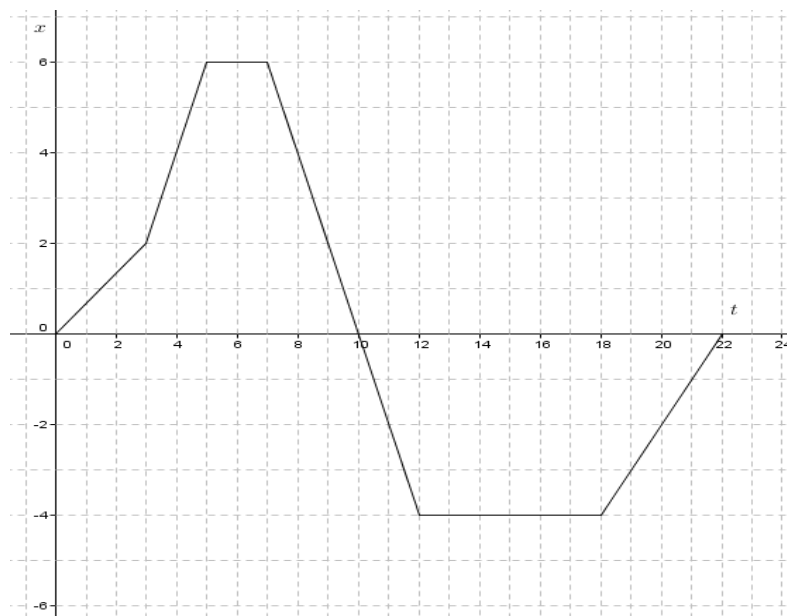
- i. Find the value of k in terms of a and b . 2
- ii. Hence, show that if $b = 2a$, then $t = \frac{\ln(2)}{a}$ when the population of B is twice the population of A . 1

Please turn over to continue Question 2

- b) The population of a certain insect is growing exponentially according to $N = 200e^{kt}$, where t is the time in weeks after the insects are first counted.

At the end of 3 weeks the insect population has doubled.

- i. Find the value of k in exact and simplified form. 2
 - ii. How many insects will there be after 12 weeks? 1
 - iii. At what rate is the population increasing after 6 weeks? (nearest whole number) 2
- c) The displacement-time graph below describes the hike of Randal from an origin, where x is in kilometres and t is in hours. Randal started his hike at 12am and finished at 10pm the same day.



- i. How long did Randal rest for throughout the trip? 1
- ii. Between what times did Randal travel the fastest? 1
- iii. What was Randal's average speed throughout the entire trip? Give your answer in km/h. 1
- iv. For safety reasons, there is a tower at Randal's starting position that can spot up to a radius of 2km. For how long was Randal not sighted? 1

Please turn over to continue Question 2

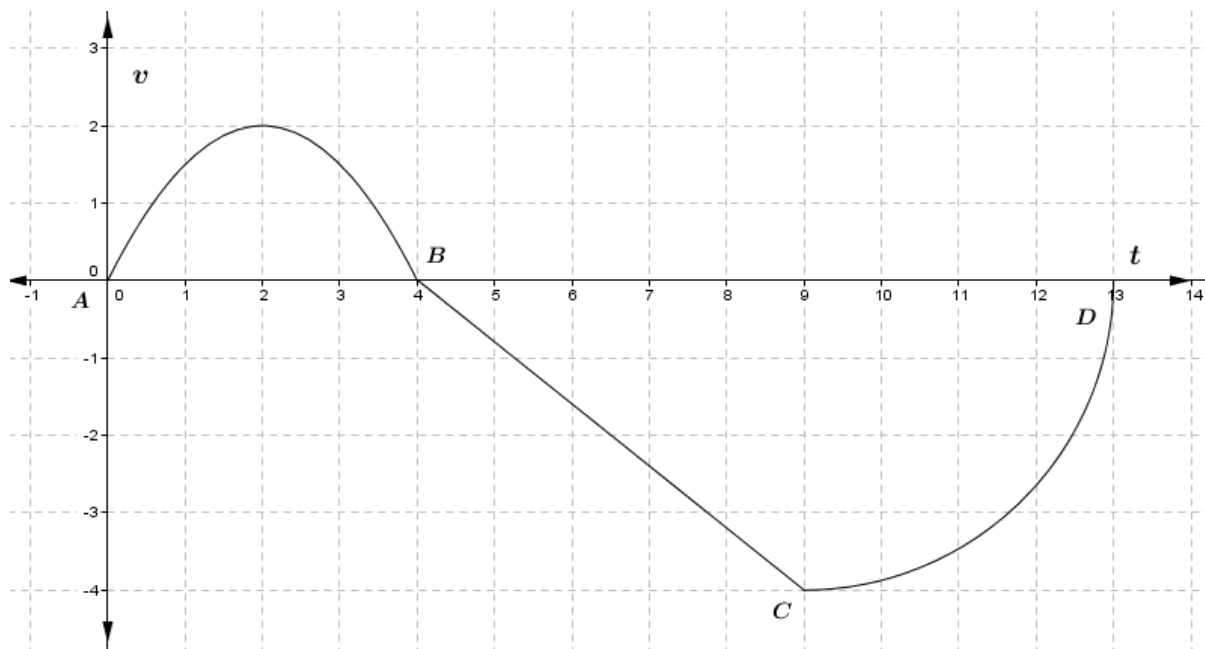
- d) A body is launched horizontally from an origin with an initial velocity of 15ms^{-1} and experiences an acceleration given by $\ddot{x} = -24e^{-2t}$ throughout its journey.
- Show that the velocity is given by $\dot{x} = 12e^{-2t} + 3$ 2
 - Find the displacement of the body after 5 seconds. (exact value) 2
 - Describe the motion of the body as $t \rightarrow \infty$. 2
- e) An experiment is conducted where an unbiased coin is tossed continually, until for the first time the same result appears twice in a row.
- Find the probability that the experiment ends on the 3rd toss. 1
 - Find the probability that the experiment finishes before the 4th toss. 1

Question 3 (20 marks) Start a new sheet of paper

- a) An air mattress deflates and decreases its air pressure P in millibars at a rate that is proportional to its air pressure at the time according to $\frac{dP}{dt} = -\omega P$ where t is in minutes. The mattress was initially pumped to an air pressure of 40 millibars.
- Show that $P = P_0 e^{-\omega t}$ is a solution to that differential equation. 1
 - After 5 minutes the air mattress' air pressure has decreased by 40%. Find the value of P_0 and ω . 2
 - The air mattress can be packed away once it reaches an air pressure of 5 millibars. However, when the air pressure of the mattress reaches below 10 millibars it decreases at a rate given by $\frac{dP}{dt} = -0.1P$. What is the total time required for the air mattress to deflate from 40 millibars down to 5 millibars? Give your answer to the nearest minutes and seconds. 3

Please turn over to continue Question 3

- b) Below is a velocity time graph for a particle travelling in a straight line from an origin, where t is in hours and v is in ms^{-1} .



From A to B, the graph is given by $y = -\frac{x(x-4)}{2}$, from B to C the graph is a straight line, and from C to D the graph is an arc of a circle with radius 4.

- i. After how long was the particle at rest? 1
- ii. After how long was the particle back at the origin? (Hours and Minutes) 2
- iii. What is the displacement of the particle at B, C and D respectively? (1 decimal place) 2
- iv. Between what time(s) did the particle have a positive acceleration? 1
- v. Between what time(s) was the particle 'slowing down'? 1
- vi. **(Start a new page for this part)** Copy the velocity time graph onto your answer sheet. Draw the corresponding displacement time graph directly below your diagram so that your horizontal axis are aligned. Show all the key information found in the previous parts. (Both of your graphs should be half a page in size). 3

Please turn over to continue Question 3

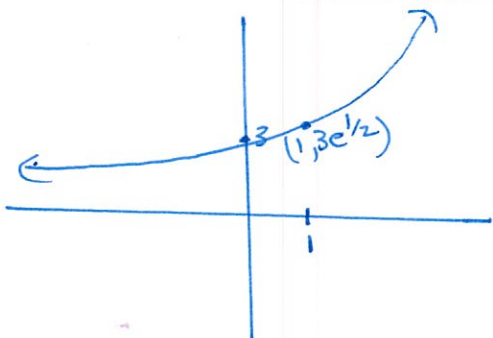
- c) A game involves throwing 2 dice and tossing 2 coins. The scoring system works as described below:

Result	Points
2 heads and double 6	5
2 tails and double 6	4
2 heads and a double other than 6	3
2 tails and double other than 6	2
Anything else	0

- i. Find the probability that after 1 turn, I will get no points. 2
- ii. Assuming that I was able to score from both my turns, what is the most likely score I will get after 2 turns? Show working. 2

End of Exam

MATHEMATICS : Question 1

Suggested Solutions	Marks Awarded	Marker's Comments
(a) 	1 for correct shape and domain all real x. 1 for points	student stopped at $x=0$ or had incorrect concavity
(b) (i) $N = Ae^{kt}$ $20 = 40e^{3k}$ $e^{3k} = 5$ $3k = \ln 5$ $k = \frac{\ln 5}{3}$ (ii) $-9 = Ae^{4\ln 3}$ $A = \frac{-9}{e^{4\ln 3}}$ $= \frac{-9}{81}$ $= -\frac{1}{9}$	1 1 1	well done student had difficulty dealing with e to a ln power
c) $\frac{dP}{dt} = -kP$ where $k > 0$	1	Significant number of students failed to state $k > 0$

MATHEMATICS : Question 1

Suggested Solutions

Marks Awarded

Marker's Comments

d) (i) $x = 5^2 - 35 - 18$
 $= -28$

Distance is 28m

(ii) $x=0 \rightarrow t^2 - 7t - 18 = 0$
 $(t-9)(t+2) = 0$

$t=9$ since $t \geq 0$

(iii) $v = 2t - 7$

$t=0 \quad v = -7 \text{ m/s}$

(iv) $2t - 7 > 8 \quad 2t - 7 < -8$

$t > \frac{15}{2}$

$t < -\frac{1}{2}$

since
 $t > 0$
 no soln.

$t = \frac{15}{2}$

$x = \frac{225}{4} - \frac{105}{2} - 18$

$x = -14 \frac{1}{4}$

(v) $t=6 \quad x = -24$

$t=0 \quad x = -18$

qu velocity = $-\frac{6}{6}$

$= -1 \text{ m/s}$

1

needed statement to gain mark.

1

well done

1

Had to have units

1

Some students gave full solutions but full marks were awarded for considering positive case only.

1

many students did - it consider $t=0$.

1

MATHEMATICS : Question.....1..

e) (i) $v = \frac{3t^2 - 6t + 4}{1}$

(ii) $v = \frac{3\cos t + 4\sin t}{1}$

(iii) $v = \frac{1}{2} (t^2 + 1)^{-\frac{1}{2}} \times 2t$
 $= \frac{t}{\sqrt{t^2 + 1}}$

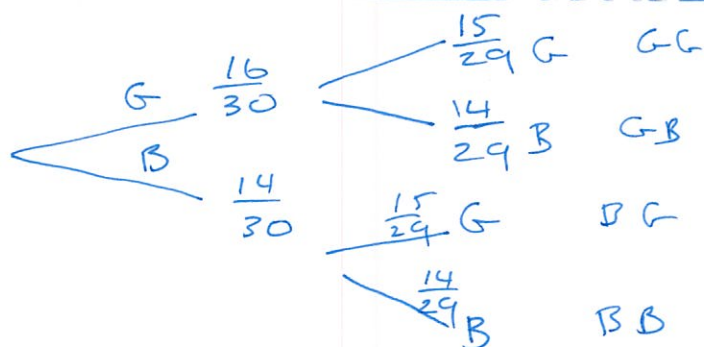
1

1

1

some issues
with dealing
with fractional
power.

f)



(i) $P(GB) + P(BG) = \frac{16}{30} \times \frac{14}{29} + \frac{14}{30} \times \frac{15}{29}$
 $= \frac{224}{435}$

(ii) $\frac{16 \times 15}{30 \times 29} = \frac{8}{29}$

students did
it with replacement

students did
not get
both occasions

carry forward
error given

(g) (i) $\left(\frac{1}{4}\right)^4 = \frac{1}{256}$

(ii) $\left(\frac{3}{4}\right)^4 = \frac{81}{256}$

(iii) $1 - P(\text{no aces})$
 $1 - \frac{81}{256} = \frac{175}{256}$

1

1

1

students who
clearly indicated
 $1 - P(\text{no aces})$ gained
CFE.

T2 24 2019

MATHEMATICS: Question... 2

1/4

Suggested Solutions

Marks

Marker's Comments

$$a) i) A(t) = P_0 e^{at}$$

$$A'(t) = a P_0 e^{at}$$

$$= a A(t)$$

$$B(t) = P_0 e^{bt}$$

$$B'(t) = b P_0 e^{bt}$$

$$= b B(t)$$

$$\frac{dB}{dt} = k \frac{dA}{dt} \Rightarrow bB = kaA$$

$$\text{but } B = 2A \quad 2AB = kaA$$

$$2b = ka$$

$$k = \frac{2b}{a}$$

METHOD 1

$$ii) \text{ If } b = 2a \text{ then } t = \frac{\ln 2}{a} \text{ (show)}$$

and B is twice A.

$$P_0 e^{bt} = 2 P_0 e^{at}$$

$$e^{bt} = 2 e^{at}$$

$$bt = \ln 2 e^{at}$$

$$= \ln 2 + at$$

$$t(b-a) = \ln 2$$

$$t(2a-a) = \ln 2$$

$$at = \ln 2$$

$$t = \frac{\ln 2}{a}$$

$$\text{since } b = 2a$$

(2)

Differentiating
and equating
1 mark.

writing k in
terms of a & b
1 mark.

mark was
awarded if
steps stating
 $B = 2A$ and $b = 2a$
was stated
where appropriate

(1)

Suggested Solutions

Marks

Marker's Comments

METHOD 2.

$$e^{bt} = 2e^{at}$$

$$\frac{e^{bt}}{e^{at}} = 2$$

$$e^{bt-at} = 2$$

$$e^{t(b-a)} = 2$$

$$\ln 2 = t(b-a)$$

$$t(3a-a) = \ln 2$$

$$t = \frac{\ln 2}{a}$$

bi) $[N = 200 e^{kt} \quad t = 3, N = 400]$ — 1 mark

$$400 = 200 e^{3k}$$

$$3k = \ln 2$$

$$k = \frac{\ln 2}{3}$$

$$N = 200 e^{\frac{\ln 2}{3} t}$$

ii) $t = 12, N = 200 e^{\frac{\ln 2}{3}(12)}$
 $= 3200$ — 1

correct substitution
and answer.

iii) $t = 6 \quad N = 200 e^{kt} \quad \frac{dN}{dt} = 200k e^{kt}$ — 1 mark for understanding rate means differentiate

$$\therefore \text{Rate} = \frac{dN}{dt} = 200 e^{\frac{\ln 2}{3}(6)} \cdot \frac{\ln 2}{3}$$

$$= 184.84$$

$$= 184 \text{ (nearest whole number)}$$

1 mark for nearest
whole number

No marks awarded if didn't
differentiate.

Suggested Solutions	Marks	Marker's Comments
c) i) 8 hours. — 1 mark ii) Fastest. 3 am to 5 am and 7 am - 12 pm 1 mark awarded if both time period mentioned. iii) Ave speed = $\frac{\text{dist travelled}}{\text{time taken}}$ $= \frac{20}{2} = \frac{10}{1} \text{ km/hr}$ — ① iv) Not signed from 3-5, 6-7, 7-9 11-12, 12-6 & 6-8 $\therefore 15 \text{ hrs.}$ — ① d) i) $\ddot{x} = -24e^{-2t}$ $\dot{x} = 15 \text{ m/s}$ $\dot{x} = \frac{-24e^{-2t}}{-2} + C_1$ $t = 0$ $15 = \frac{-24e^{-2(0)}}{-2} + C_1$ $C_1 = 3.$ $\therefore \dot{x} = 12e^{-2t} + 3$ ii) $x = \frac{12e^{-2t}}{-2} + 3t + C_2$ $x = 0, t = 0$ $= -6e^{-2t} + 3t + C_2$ $\therefore C_2 = 6$ when $t = 5$; $x = -6e^{-2(5)} + 3(5) + 6$ $= -6e^{-10} + 21 \text{ m}$		no mark awarded if mentioned between what time to what time. no marks awarded if units incorrect. pls mention units!! pls mark the dots clearly 1 mark for integrating 1 mark for finding C_1 1 mark for integrating and finding C_2 — 1 mark displacement in exact value.

Suggested Solutions	Marks	Marker's Comments
diii) as $t \rightarrow \infty$, $\dot{x} \rightarrow 3$, $\ddot{x} \rightarrow 0$, $x \rightarrow \infty$ The particle will continue to travel with a true velocity by continually slowing down towards a velocity of 3m/s and head away from the origin OR. as $t \rightarrow \infty$, the particle slows down until a limiting velocity of 3m/s & acceleration approaches 0 and the particle travels infinitely to the right of the origin.		— 1 mark } 1 mark to describe the motion. — no marks awarded if say slowing down and not mention 3m/s. — no mark awarded if stated speeding up. — have to mention moving away from origin.
e i) Let n be the number of tosses. Game ending on 3 tosses. THH or HTT $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ ii) game finishes before the 4th toss HH or TT or HTT or THH $\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{4}$ $= \frac{3}{4}$		① ①

MATHEMATICS: Question 3

Suggested Solutions

Marks Awarded

Marker's Comments

- a) Show that $P = P_0 e^{-\omega t}$ is a
(i) solution to the differential equation.

$$\frac{dP}{dt} = -\omega P_0 e^{-\omega t}$$

$$\text{Since } P = P_0 e^{-\omega t}$$

$$= -\omega P$$

1 mark

①

- (ii) Find the value of P_0 and ω after 5 minutes when pressure has decreased by 40%.

2 marks

when $t=0$, $P=40$

$$\therefore 40 = P_0 e^0$$

$$\therefore P_0 = 40$$

$$\text{Now } P = 40 e^{-\omega t}$$

$$\text{when } t=5 \quad P = \frac{3}{5} \times 40$$

$$= 24$$

sub in $P=24$ and $t=5$ in eq

$$24 = 40 e^{-5\omega}$$

$$\frac{3}{5} = e^{-5\omega}$$

$$-5\omega = \ln \frac{3}{5}$$

$$\omega = \frac{\ln \frac{3}{5}}{-5} \text{ or } \frac{\ln \frac{5}{3}}{5}$$

$$\text{or } 0.1012$$

①

①

Suggested Solutions

Marks Awarded

Marker's Comments

(iii) Total time required for the mattress to deflate from 40 millibars to 5 millibars

3 marks

From previously $P = 40 e^{-\ln \frac{5}{3} t}$

when $P=10$
 $10 = 40 e^{-\ln \frac{5}{3} t}$

$$\frac{1}{4} = e^{-\ln \frac{5}{3} t}$$

$$\frac{-\ln \frac{5}{3} t}{5} = \ln \left(\frac{1}{4} \right)$$

$$\therefore t = \frac{-5 \ln \left(\frac{1}{4} \right)}{\ln \frac{5}{3}} \text{ or } \frac{5 \ln \left(\frac{1}{4} \right)}{\ln \frac{3}{5}} \approx 13.569 \rightarrow \textcircled{1}$$

Time taken from 10 to 5 millibars

$$\frac{dP}{dt} = -0.1 P$$

$$\therefore P = P_0 e^{-0.1 t}$$

$$5 = 10 e^{-0.1 t}$$

$$\frac{1}{2} = e^{-0.1 t}$$

$$\ln \frac{1}{2} = -0.1 t$$

$$\ln 2 = 0.1 t$$

$$\therefore t = 10 \ln 2 \text{ or } 6.93 \rightarrow \textcircled{1}$$

$\therefore$ Total time is

$$= \frac{-5 \ln \frac{1}{4}}{\ln \frac{5}{3}} + 10 \ln 2$$

$$= 20.50 \text{ min}$$

$$= 20 \text{ minutes and } 30 \text{ seconds} \rightarrow \textcircled{1}$$

MATHEMATICS: Question 3

Suggested Solutions

Marks Awarded

Marker's Comments

b(i) Particle is at rest after 4 sec.

1 mark.

accepted 4 hours

(ii) After how long was the particle back at the origin.

Distance travelled in first 4 seconds is

$$\int_0^4 v \, dx$$

$$= -\frac{1}{2} \int x(x-4) \, dx$$

$$= -\frac{1}{2} \int x^2 - 4x \, dx$$

$$= -\frac{1}{2} \left[\frac{x^3}{3} - 2x^2 \right]_0^4$$

$$= -\frac{1}{2} \left[\frac{64}{3} - 32 \right] = \frac{16}{3} \text{ m}$$

$$\text{Area under BC} = \text{Area under A-B} = \frac{16}{3} \text{ m}$$

$$A = \frac{1}{2} bh = \frac{1}{2} t \left(\frac{4}{5} t \right) = \frac{16}{3}$$

$$= \frac{2}{5} t^2 = \frac{16}{3}$$

$$t^2 = \frac{40}{3}$$

$$t = \sqrt{\frac{40}{3}}$$

$$t = 3.6514 \text{ s}$$

$$\text{Total time} = 4 + \sqrt{\frac{40}{3}} \text{ hrs}$$

$$= 7 \text{ hours } 39 \text{ minutes}$$

OR.

equation

$$y = -\frac{4}{3}(t-4)$$

$$= -\frac{4}{3}t + \frac{16}{3}$$

$$\text{Area} = \int -\frac{4}{3}t + \frac{16}{3} \, dt + C = \frac{16}{3}$$

$$6t^2 - 48t + 16 = 0$$

MATHEMATICS: Question 3

- Page 4 -

Suggested Solutions

Marks Awarded

Marker's Comments

b (iii)

$$B = \frac{16}{3} \text{ m}$$

$$C = \frac{16}{3} - \frac{1}{2}(5)(4) \\ = -4\frac{2}{3} \text{ m}$$

$$D = -4\frac{2}{3} - \frac{1}{4}\pi(4^2) \\ = -4\frac{2}{3} - 4\pi \\ \approx -17.2 \text{ m.}$$

①

①

looked at
 $\frac{1}{4}\pi(4^2)$
to get a
mark in calculation

(iv) Between $t=0$ and $t=2$ } and
 $t=9$ and $t=13$ }

1 mark

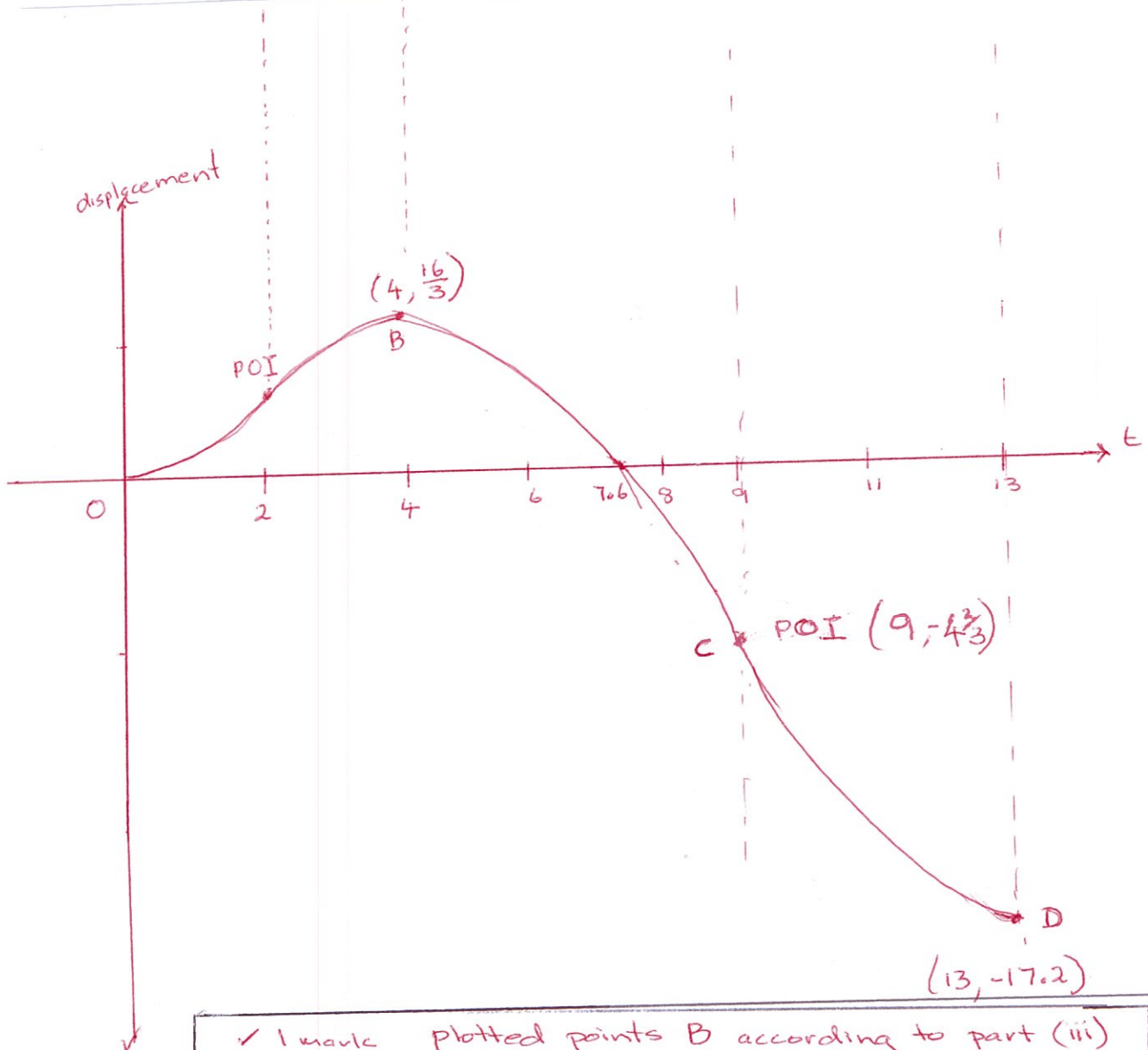
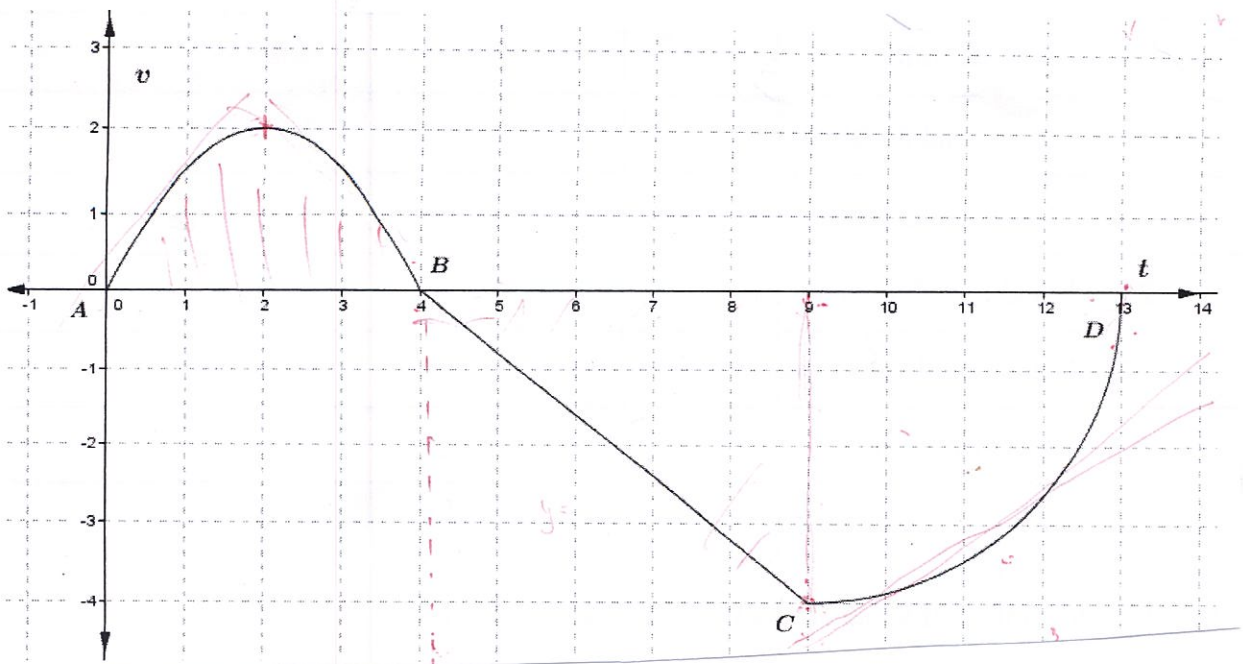
①

(v) Between $t=2$ and $t=4$ } and
between $t=9$ and $t=13$ }

1 mark

①

Out of 4
intervals if
one got
2 intervals or
3 intervals, you
got 1 mark.



- ✓ I mark plotted points B according to part (iii) and x-intercept of 7.6
- ✓ There is point of inflexion at $x=2$ and $x=9$
- ✓ Turning points at $(4, \frac{16}{3})$ and $(13, -17.2)$

MATHEMATICS: Question 3

- Page 6 -

Suggested Solutions	Marks Awarded	Marker's Comments
C(i) Probability that after 1 turn will get no points $P(5) = P(2H, 26) = \frac{1}{4} \times \frac{1}{36} = \frac{1}{144}$ $P(4) = P(2T, 26) = \frac{1}{4} \times \frac{1}{36} = \frac{1}{144}$ $P(3) = P(2H, 2 \neq 6) = \frac{1}{4} \times \frac{5}{36} = \frac{5}{144}$ $P(2) = P(2T, 2 \neq 6) = \frac{1}{4} \times \frac{5}{36} = \frac{5}{144}$ $P(\text{no points}) = 1 - \left(\frac{1}{144} + \frac{1}{144} + \frac{5}{144} + \frac{5}{144} \right)$ $= 1 - \frac{12}{144}$ $= \frac{11}{12}$	2 marks	Alternate Method Total no of outcomes $8 \times 36 = 288$ ① $1 - \frac{4 \times 6}{288}$
(ii) Able to score from both turns $P(10) = P(5) \times P(5) = \frac{1}{144} \times \frac{1}{144} = \frac{1}{20736}$ $P(9) = P(5) \times P(4) \text{ or } P(4) \times P(5) = \frac{1}{144} \times \frac{5}{144} + \frac{5}{144} \times \frac{1}{144} = \frac{10}{20736}$ $P(8) = P(4) \times P(4) \text{ or } P(5) \times P(3) \text{ or } P(3) \times P(5) = \frac{1}{20736} + 2 \times \frac{5}{20736} = \frac{11}{20736}$ $P(7) = P(5) \times P(2) \text{ or } P(2) \times P(5) = 2 \times \left(2 \times \frac{5}{144} \right) = \frac{20}{20736}$ $P(6) = P(3) \times P(3) + P(4) \times P(2) + P(2) \times P(4) = \frac{25}{20736} + 2 \times \left(\frac{5}{20736} \right) = \frac{30}{20736}$ $P(5) = P(3) \times P(2) + P(2) \times P(3) = 2 \left(\frac{25}{20736} \right) = \frac{50}{20736}$ $P(4) = P(2) \times P(2) = \frac{25}{20736}$ From the probabilities calculated P(5) is most likely	2 marks	1 mark for calculating probabilities 1 mark concluding that P(5) has the highest probability Note P(5) is likeliest Not only because P(2) and P(3) is most likely Incorrect working or no working will not be awarded any marks.