KILLARA HIGH SCHOOL

YEAR 12 TERM 1 EXAMINATION

MATHEMATICS

APRIL 2003

TOTAL TIME: 3 Hours

TOTAL MARKS: 120

Read each question thoroughly.

Attempt all questions.

Start each part on a new page.

Marks may be deducted for untidy or carelessly arranged working.

All necessary working must be shown for full marks.

Marks will be deducted if the answer is not written as directed by the question.

Non-Programmable Calculators may be used.

QUESTION 1: (12 marks)

(i) Factorise completely $3t^2 - 18t - 81$ **(2 marks)**

(ii) Evaluate to 5 significant figures : $\dfrac{(6.1 \times 5.2)^2}{28.36 - 17.24}$ **(2 marks)**

(iii) Simplify $\dfrac{1}{x-y} - \dfrac{1}{x+y}$ **(2 marks)**

(iv) Given $f(x) = \begin{cases} x-2 & \text{for } x \le 2 \\ (x-2)^2 & \text{for } x > 2 \end{cases}$. Find (a) $f(-1)$

(b) $f(a+2), a > 0$ **(2 marks)**

(v) Find the primitive of $3x^2 - 2x^{-2} + 4$ **(2 marks)**

(vi) Simplify and Integrate $\dfrac{3x^3 - 5x^{\frac{1}{2}}}{x^3}$ **(2 marks)**

QUESTION 2: (12 marks)

(i) (a) Find the roots of the equation $x^2 + 5x - 14 = 0$ (2 marks)

(b) For what values of x is $x^2 + 5x - 14 > 0$? (2 marks)

(ii) If $\tan\alpha = \frac{3}{5}$ and $\pi \le \alpha \le \frac{3\pi}{2}$ find the exact values of :

(a) $\cos\alpha$ (2 marks)

(b) $\operatorname{cosec}\alpha$ (2 marks)

(iii) Using the table of values below, for a continuous function $y = f(x)$,

x	-1	-0.5	0	0.5	1
f(x)	1.25	0.75	1	1.75	2

evaluate $\int_{-1}^{1} f(x)\,dx$ using

(a) The Trapezoidal Rule (b) Simpsons Rule (4 marks)

[Give your answers correct to two decimal places]

QUESTION 3: (12 marks)

(i) The sum to n terms of a certain arithmetic series is $-n^2 + 34n$. Find

(a) The first term. (1 mark)

(b) S_{n-1} (2 marks)

(c) T_n where $T_n = S_n - S_{n-1}$ (1 mark)

(d) Show the common difference of the series is -2. (1 mark)

(e) The first value of 'n' for which the n-th term is negative. (2 marks)

(ii) (a) Express $0.\dot{3}\dot{5}$ as an infinite series of fractions. (1 mark)

(b) Hence calculate its value as a rational number. (2 marks)

(iii) Evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$. (2 marks)

QUESTION 4: (12 marks)

(i) Convert 105^0 to a radian measure in terms of π . (1 mark)

(ii) Convert $\frac{2\pi}{9}$ to degrees . (1 mark)

(iii) Convert $132^0\ 58'$ to radians, leaving your answer to 4 decimal places. (1 mark)

(iv) Convert $2\frac{2}{3}$ radians to degrees and minutes. (1 mark)

(v) The chord AB of a circle of radius 8 cm is 10 cm long.

(a) Show that the angle subtended by the chord at the centre of the circle is $77^0\ 22'$ (2 marks)

(b) Find the length of the arc AB to 2 decimal places. (2 marks)

(c) Calculate the area of the sector AOB to 2 decimal places. (2 marks)

(d) calculate the area of the minor segment to 2 decimal places. (2 marks)

QUESTION 5: (12 marks)

(i) Differentiate the following

(a) $y = \sin(2x+1)$ (2 marks)

(b) $y = \sin x \cos x$ (2 marks)

(c) $y = \cot x$ (2 marks)

(ii) Find the following integrals

(a) $\int \sec^2 3x\, dx$ (2 marks)

(b) $\int_0^{\frac{\pi}{3}} 4 \sin 2x\, dx$ (2 marks)

(c) $\int_0^{\frac{1}{2}} \cos \pi x\, dx$ (2 marks)

QUESTION 6: (12 marks)

(i) Consider the curve $y = 3x^2 - x^3 + 9x - 2$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (2 marks)

(b) Find the co-ordinates of the stationary points and determine their nature. (2 marks)

(c) Find the co-ordinates of any points of inflection. (2 marks)

(d) Sketch the graph of the function for the domain $-2 \le x \le 5$. (2 marks)

(e) State the absolute minimum value of the function over this domain. (1 mark)

(f) Write down the values of x where the curve $y = 3x^2 - x^3 + 9x - 2$ is increasing. (1 mark)

(ii) The derivative of a curve $y = f(x)$ is given by $f'(x) = x^2 - 3x - 4$.
Find the equation of the curve if it is known the curve passes through $(-2,1)$ (2 marks)

QUESTION 7:

(i) Integrate the following

(a) $\int \sqrt{x^5}\, dx$ (2 marks)

(b) $\int \frac{2}{3x^5}\, dx$ (2 marks)

(c) $\int (2x+1)^3\, dx$ (2 marks)

(ii) (a) Sketch the curve $y = (x+1)(x-1)(x-4)$ showing the x - intercepts only. (1 mark)

(b) Show that $\int_{-1}^{4} (x+1)(x-1)(x-4)\, dx = \frac{-125}{12}$ (2 marks)

(c) Explain why the result in (b) does not equal the total area of the regions bounded by the curve and the x - axis. (1 mark)

(d) Find the total area of the regions bounded by the curve and the x - axis. (2 marks)

QUESTION 8: (12 marks)

(i) Find the area enclosed by the curve $y = \sqrt[3]{x}$, the y - axis and the line $y = 2$. (3 marks)

(ii) (a) For the functions $y = x^2$ and $y = x(4 - x)$, show the x-ordinates of the points of intersection of the graphs of these functions are $x = 0$ and $x = 2$ (2 marks).

(b) Hence find the area enclosed between the two graphs of $y = x^2$ and $y = x(4 - x)$. (2 marks)

The area enclosed by these two graphs is rotated about the x - axis.

(c) Show that the volume of the solid of revolution formed by this rotation. is given by $V_x = \pi\int_0^2 16x^2 - 8x^3\,dx$ (3 marks)

(d) Calculate the volume of the solid of revolution formed by this rotation, leaving your answer in exact form. (2 marks)

QUESTION 9: (12 marks)

(i) Alex borrows \$10000 from a bank at 9 % reducible, and is repaid in equal monthly repayments over 5 years. If 'R' represents the monthly repayment and A_n is the amount outstanding on the loan after 'n' repayments.

(a) Write an expression to represent the amount outstanding on the loan, A_1, after the first repayment and the amount outstanding on the loan, A_2 after the second repayment. (2 marks)

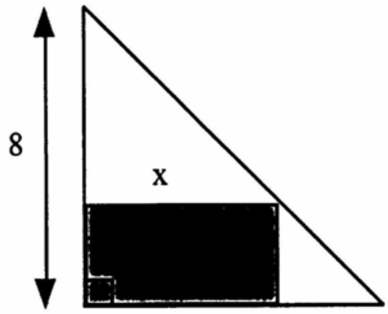
(b) Calculate the value of the monthly repayment. (2 marks)

(c) Using your results to parts (a) and (b), calculate the interest charged by the bank in the first three years. (2 marks)

(d) Calculate the total amount of interest paid to the bank over this ten year period. (1 mark)

QUESTION 9:(continued)

(ii)



A rectangle of side 'x' cm, as shown in the diagram is drawn inside a right angled isosceles triangle of height 8 cm.

(a) If the other side of the rectangle is labelled 'y', show, by using similar triangles, that $y = 8 - x$ (2 marks)

(b) Show that the area of the rectangle is given by $A = 8x - x^2$ (1 mark)

(c) Find the dimensions of the shaded part so that the area is a maximum. (2 marks)

QUESTION 10: (12 marks)

(i) (a) Given that $f(x) = \sin^2 x$, find $f'(x)$ and $f''(x)$ (2 marks)

(b) Show that $\dfrac{f''(x) + 2f(x)}{f'(x)} = \cot x$. (2 marks)

(ii) Draw the sketch of $y = \sin 2x, \quad 0 \le x \le 2\pi$ and on the same diagram draw the sketch of $y = |\operatorname{cosec} 2x|, \quad 0 \le x \le 2\pi$ (2 marks)

(iii) On the same set of axes, draw the sketch the graph of :

(a) $y = 2\sin x$ (2 marks)

(b) $y = \cos x$ (1 mark)

(c) $y = 2\sin x + \cos x$ for the values of x given by $0 \le x \le 2\pi$. (2 marks)

From your sketch, estimate the approximate maximum value of $y = 2\sin x + \cos x$. (1 mark)

End of Test

SOLUTIONS - 2003 TERM1 YR 12 ASSESSMENT.

Question 1:

(i) $3t^2 - 18t - 81$

$= 3(t^2 - 6t - 27)$ — ①

$= 3(t-9)(t+3)$ — ①

(ii) $90 \cdot 4818705$ — ①

$= 90 \cdot 482$ — ①

(iii) $\dfrac{1}{x-y} - \dfrac{1}{x+y}$

$= \dfrac{x+y-x+y}{(x-y)(x+y)}$ — ①

$= \dfrac{2y}{(x-y)(x+y)}$ — ①

(iv) (a) $f(-1) = (-1) - 2 = -3$ — ①

(b) $f(a+2) = ((a+2)-2)^2$

$= a^2$ — ①

(v) $y = 3x^2 - 2x^{-2} + 4$

Primitive $= \dfrac{3x^3}{3} - \dfrac{2x^{-1}}{-1} + 4x + c$

$= x^3 + 2x^{-1} + 4x + c$ → ②

① only for no 'c'

(vi) $\displaystyle\int \frac{3x^3 - 5x^{\frac{1}{2}}}{x^3}\,dx$

$= \displaystyle\int 3 - 5x^{-\frac{5}{2}}\,dx$ — ①

$= 3x - 5\dfrac{x^{-\frac{3}{2}}}{-\frac{3}{2}} + c$

$= 3x + \dfrac{10}{3}x^{-3/2} + c$ — ①.

[allow no 'C']

/12

Question 2

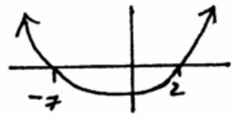
1(a) $x^2 + 5x - 14 = 0.$

$(x+7)(x-2) = 0$

$\therefore x = -7 \quad x = 2$ — ②

(b) $x^2 + 5x - 14 = 0.$

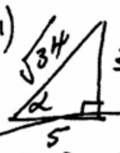
$(x+7)(x-2) > 0$



$x < -7$ — ①

$x > 2$ — ①

(ii)



$\cos\alpha = \dfrac{-5}{\sqrt{34}}$ — ②

3RD QUADRANT

$\operatorname{cosec}\alpha = \dfrac{-\sqrt{34}}{3}$ — ②

(iii)

a) $\displaystyle\int_{-1}^{1} f(x)\,dx = \frac{h}{2}\left[y_1 + 2(y_2+y_3+y_4) + y_5\right]$

$= \dfrac{0.5}{2}\left[1.25 + 2(0.75+1+1.75) + 2\right]$ → ①

$= 2.5625$

$= 2.56$ (2dp) — ①

b) $\displaystyle\int_{-1}^{1} f(x)\,dx = \frac{h}{3}\left[y_1 + 4(y_2+y_4) + 2(y_3) + y_5\right]$

$= \dfrac{0.5}{3}\left[1.25 + 4(0.75+1.75) + 2(1) + 2\right]$ → ①

$=$

$= 2.54$ (2dp) — ①

Question 3:

(i) $S_n = -n^2 + 34n.$

(a) $T_1 = S_1 = -(1)^2 + 34(1) = 33.$ — ①

(b) $S_{n-1} = -(n-1)^2 + 34(n-1)$ — ①

$= -n^2 + 2n - 1 + 34n - 34$

$= -n^2 + 36n - 35$ — ①

(c) $T_n = S_n - S_{n-1}$

$= -n^2 + 34n + n^2 - 36n + 35$

$= 35 - 2n$ — ①

(d) $T_1 = 33 \quad T_2 = 31 \quad T_3 = 29.$

$T_2 - T_1 = -2$

$T_3 - T_2 = -2$

$\therefore d = -2$ — ①

SOLUTIONS - 2003 TEAM 1 YR12 ASSESSMENT

Question 3 (cont)

e) $T_n = 35 - 2n < 0$ — (1)

$2n > 35$

$n > 17\frac{1}{2}$

$\therefore n = 18$ — (1)

T_{18} is negative.

1)
a) $0.\dot{3}\dot{5} = \frac{35}{100} + \frac{35}{10000} + \frac{35}{1000000} + \cdots$ — (1)

b) $S_\infty = \frac{a}{1-r}$ $a = \frac{35}{100}$, $r = \frac{1}{100}$

$= \dfrac{35/100}{99/100}$ — (1)

$= \frac{35}{99}$ — (1)

(c) $\lim\limits_{x \to 3} \frac{x^2 - 9}{x - 3}$

$= \lim\limits_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$ — (1)

$= \lim\limits_{x \to 3} x + 3 = 6$ — (1)

12

Question 4:

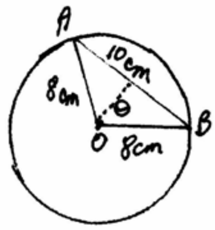
i) $105 \times \frac{\pi}{180} = \frac{7\pi}{12}^c$ — (1)

ii) $\frac{2\cancel{\pi}}{9} \times \frac{180}{\cancel{\pi}} = 40°$ — (1)

iii) $132°58' \times \frac{\pi}{180} = 2.3207$ — (1)

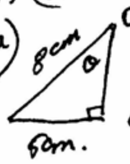
iv) $2\frac{2}{3} \times \frac{180}{\pi} = 152°47'$ — (1)

v)



(v) (continued)

a)



$\sin\theta = \frac{5}{8}$. $\theta = \sin^{-1}\frac{5}{8}$ — (1)

angle at centre of circle $= 2\theta$

$= 2 \times \sin^{-1}\frac{5}{8}$ — (1)

$= 77° 21' 51.75''$

$= 77° 22'$

b) arc $AB = r\theta^c$

$= 8 \times 77°22' \times \frac{\pi}{180}$ — (1)

$= 10.80$ (2dp) — (1)

c) Area $AOB = \frac{1}{2} r^2 \theta^c$

$= \frac{1}{2} \times (8)^2 \times 77°22' \times \frac{\pi}{180}$ (1)

$= 43.21 \text{ cm}^2$ (2dp) — (1)

d) Minor Segment Area

$= \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin\theta$ — (1)

$= \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2}(8)^2 \left(\frac{77°22' \times \pi}{180} - \sin 77°22'\right) \frac{\pi}{180}$

$= 11.98 \text{ cm}^2$ (2dp) — (1)

Question 5.

(1) a) $y = \sin(2x+1)$.

$\frac{dy}{dx} = 2\cos(2x+1)$ — (2)

b) $y = \sin x \cos x$
$= u \cdot v$.

$\frac{dy}{dx} = vu' + uv'$.

$= (\cos x)(\cos x) + (\sin x)(-\sin x)$

$= \cos^2 x - \sin^2 x$. — (2)

i) $y = \cot x = \frac{\cos x}{\sin x} = \frac{u}{v}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$

$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$ — 1

$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= \frac{-1}{\sin^2 x}$

$= -\operatorname{cosec}^2 x$ — 1

(ii) a) $\int \sec^2 3x \, dx$

$= \frac{1}{3} \tan 3x + C$ — 2

b) $\int_0^{\frac{\pi}{3}} 4 \sin 2x \, dx$

$= -2\left[\cos 2x\right]_0^{\frac{\pi}{3}}$ — 1

$= -2\left[\cos \frac{2\pi}{3} - \cos 0\right]$.

$= -2\left[-\frac{1}{2} - 1\right]$ — 1

$= 3.$

(c) $\int_0^{1/2} \cos \pi x \, dx$

$= \frac{1}{\pi}\left[\sin \pi x\right]_0^{\frac{1}{2}}$ — 1

$= \frac{1}{\pi}\left[\sin \frac{\pi}{2} - \sin 0\right]$

$= \frac{1}{\pi}\left[1 - 0\right]$ — 1

$= \frac{1}{\pi}$

12

Question 6:

(i) (a) $y = 3x^2 - x^3 + 9x - 2$

$y' = 6x - 3x^2 + 9.$ — 1

$y'' = 6 - 6x$ — 1

(b) Stationary points $y' = 0$.

$y = 6x - 3x^2 + 9$

$= 9 + 6x - 3x^2$

$= 3(3 + 2x - x^2)$

$= 3(3 - x)(1 + x) = 0$

$x = 3$, $x = -1$

$y = 25$, $y = -7$

$y'' < 0$, $y'' > 0$

$\therefore (3, 25)$ is a max. t.p. — 1

$(-1, -7)$ is a min t.p. — 1

c) Pts. of inflexion. $y'' = 0$

$6 - 6x = 0$

$6(1 - x) = 0$, $y = 9$ — 1

$x = 1$

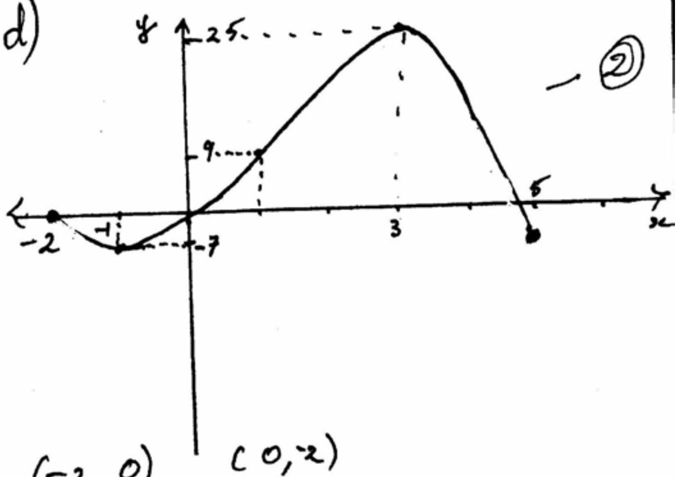
Test

x	1^-	1	1^+
y''	+ve	0	-ve

change in sign — 1

$(1, 9)$ is a pt of inflection

d) — 2



$(-2, 0)$ $(0, -2)$

$(5, -7)$

e) absolute minimum $= -7$ — 1

f) $-$[illegible] $< x < 3$ — 1

(ii) $f'(x) = x^2 - 3x - 4$

$f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x + C$ — 1

$f(-2) = -\frac{8}{3} - \frac{12}{2} + 8 + C = 1$

$-\frac{2}{3} + C = 1$ — 1

$f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x + 1\frac{2}{3}$

12

Question 7

(i) a) $\int x^{5/2}\,dx = \frac{2}{7}x^{\frac{7}{2}} + c$ (2)

b) $\int \frac{2}{3x^5}\,dx = \int \frac{2}{3}x^{-5}\,dx$

$= \frac{2}{3}\left(\frac{x^{-4}}{-4}\right) + c$

$= -\frac{1}{6}x^{-4} + c$ (2)

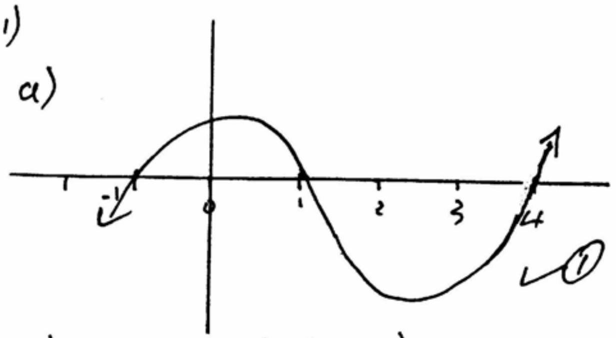
c) $\int (2x+1)^3\,dx$

$= \frac{(2x+1)^4}{4 \times 2} + c$

$= \frac{1}{8}(2x+1)^4 + c$ (2)

(ii)

a)



(1)

b) $(x+1)(x-1)(x-4)$

$= (x^2-1)(x-4)$

$= x^3 - 4x^2 - x + 4$

$\int_{-1}^{4} x^3 - 4x^2 - x + 4\,dx$ (1)

$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 - \frac{1}{2}x^2 + 4x\right]_{-1}^{4}$

$= \left[\frac{1}{4}(4)^4 - \frac{4}{3}(4)^3 - \frac{1}{2}(4)^2 + 4(4)\right] - \left[\left(\frac{1}{4}\right)(-1)^4 - \frac{4}{3}(-1)^3 - \frac{1}{2}(-1)^2 + 4(-1)\right]$

$= \frac{-125}{12}$, $\left[\frac{-40}{3} + \frac{35}{12}\right]$. (1)

c) Reasonable explanation areas above & below axis +ve & -ve regions.

(1)

d) $A = \int_{-1}^{1} x^3 - 4x^2 - x + 4\,dx + \left|\int_{1}^{4} x^3 - 4x^2 - x + 4\,dx\right|$ (1)

$= \left[\left(2\frac{5}{12}\right) - \left(-2\frac{11}{12}\right)\right] + \left|\left[\left(-13\frac{1}{3}\right) - \left(2\frac{5}{12}\right)\right]\right|$

$= 5\frac{1}{3} + \left|-15\frac{3}{4}\right| = \frac{253}{12}$.

$= 21\frac{1}{12}\,u^2$. (1)

Question 8

(i) $A = \int_0^2 y^3\,dy$ (1) Function

$= \left[\frac{1}{4}y^4\right]_0^2$ (1) Integral

$= 4\,u^2$. (1) answer.

(ii)(a) $y = x^2 \quad y = x(4-x)$

$x^2 = x(4-x)$

$x^2 = 4x - x^2$

$2x^2 - 4x = 0$

$2x(x-2) = 0$ (2)

$x = 0,\ x = 2$

(b) $A = \left|\int_0^2 2x^2 - 4x\,dx\right|$. (1) integral

$= \left|\left[\frac{2}{3}x^3 - 2x^2\right]_0^2\right|$

$= \left|-2\frac{2}{3}\right|$ (1) answer

$= 2\frac{2}{3}\,u^2$

c) $V_x = \pi \int_0^2 (4x - x^2)^2\,dx - \pi \int_0^2 (x^2)^2\,dx$ ①

$= \pi \int_0^2 16x^2 - 8x^3 + x^4 - x^4\,dx$ ①

$= \pi \int_0^2 16x^2 - 8x^3\,dx$ ①

d) $V = \pi \left[\frac{16}{3}x^3 - 2x^4 \right]_0^2$ ①

$= \pi \left[\frac{16}{3}(8) - 2(16) - 0 \right]$

$= \frac{32\pi}{3}$ units^3. ①

Question 9.

a) $A_1 = 10000(1.0075) - R.$ ①

$A_2 = 10000(1.0075)^2 - R(1.0075) - R.$ ①

b) $A_{60} = 10000(1.0075)^{60} - R\left(1 + (1.0075) + \ldots + (1.0075)^{59}\right)$

$0 = 10000(1.0075)^{60} - R\left(\frac{1.0075^{60} - 1}{0.0075}\right).$

$R = \frac{10000(1.0075)^{60}(0.0075)}{1.0075^{60} - 1}.$

$= \$207.58.$ ①

OR $A_{36} = 4543.97$

AMOUNT PAID ON LOAN = 5456.03.

INTEREST = $207.58 \times 3 - 400.73$

$= \$2016.85.$ ②

(d) TOTAL INTEREST.

$= 207.58 \times 60 - 10000$

$= \$2454.80.$ ②

(11)(a) $\frac{8-y}{x} = \frac{8}{8}$

$8 - y = x$

$y = 8 - x$ ②

b) $A = x(8 - x)$

$= 8x - x^2$ ①

c) $\frac{dA}{dx} = 8 - 2x$

$\frac{dA}{dx} = 0 \quad x = 4.$

$\frac{d^2A}{dx^2} = -2 < 0$ always

MAX occurs at $x = 4$. ①

Dimensions are 4×4 cm ①

Question 10

(i) (a) $f(x) = \sin^2 x$

$= (\sin x)^2$

$f'(x) = 2\sin x . \cos x$ 1

$f''(x) = vu' + uv'$

$u = 2\sin x \quad u' = 2\cos x$

$v = \cos x \quad v' = -\sin x$

$f''(x) = \cos x . 2\cos x + 2\sin x(-\sin x)$

$= 2\cos^2 x - 2\sin^2 x$ 1

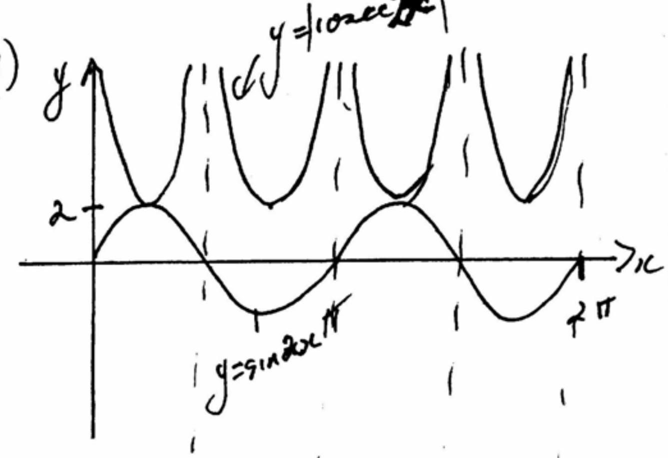
(b) $\dfrac{2\cos^2 x - 2\sin^2 x + 2\sin^2 x}{2\sin x \cos x}$ 1

$= \dfrac{2\cos^2 x}{2\sin x \cos x}$

$= \dfrac{\cos x}{\sin x}$ 1

$= \cot x$

(ii)



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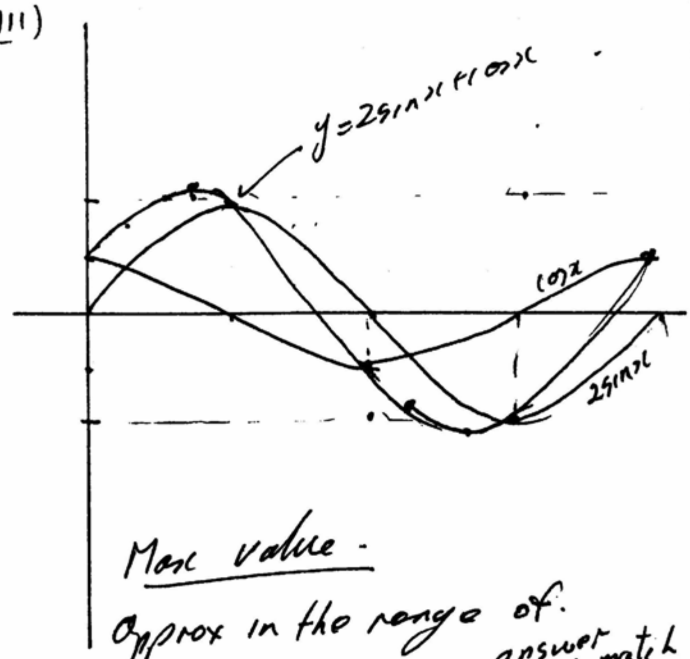
(iii)



Max value.

Approx in the range of:
2.13 to 2.26. - answer must match graph/sketch

correct max: 2.236.