KILLARA HIGH SCHOOL

YEAR 11 MATHEMATICS EXAMINATION

TERM 4, 2005

Time Allowed: 70 minutes

Instructions: Answer on one side of your paper only.
Attempt all questions.
All necessary working should be shown in every question.
Marks may not be awarded for careless or badly arranged work.
Board-approved calculators may be used.
Begin each section on a new page.

Total Marks: 70 marks

SECTION 1 : INTRODUCTORY CALCULUS (41 marks)

Marks

1. Differentiate the following functions with respect to x:

   i) $y = 6 + 3x - 5x^2$ **1**

   ii) $f(x) = 6 - x$ **1**

   iii) $y = x^2\sqrt{x}$ **2**

   iv) $y = \dfrac{3x^3 - x}{x^2}$ **2**

2. Evaluate:

   i) $\lim\limits_{x \to 2} x^2$ **1**

   ii) $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x - 3}$ **2**

3. a) Differentiate the function $f(x) = x^2 - 3x - 2$ from first principles, using the compressed formula $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, or otherwise. **3**

b) Find the point P on the curve $y = x^2 - 3x - 2$ at which the tangent is parallel to the line $3x - y + 2 = 0$. **3**

4. Differentiate:

i) $y = \sqrt{x^3 - 2}$ **3**

ii) $y = (x^2 + 4)(2x + 3)^3$ **3**

5. Find the equation of the normal to the curve $y = \frac{x}{x-2}$ at (3,3). **4**

6. There are two points on the curve $y = x - \frac{3}{x}$ where the gradients of the tangents to the curve are equal. One of the points is (3,2). Find the other point. **4**

7. The sum of two numbers is 18. Find these numbers if their product is a maximum. **3**

8. For the function $y = x^3 - 3x$

a) find any stationary points in the domain $-3 \le x \le 1\frac{1}{2}$, and determine their nature. **4**

b) Neatly sketch the function $y = x^3 - 3x$ for the given domain. **3**

c) What are the maximum and minimum values of the function $y = x^3 - 3x$ in that domain? **2**

SECTION 2: SEQUENCES AND SERIES (29 Marks)

Marks

1. For each of the following sequences,
   i) determine if it is Arithmetic or Geometric, justify by giving the value of d or r as applicable, then
   ii) find the term indicated in brackets.

   a) 80, –20, 5, ... (T_9) — **1, 2**

   b) –22, –16, –10, ... (T_{53}) — **1, 2**

2. a) For an Arithmetic Series, find S_{30}, when the first term = 6 and the last term = –4. — **2**

   b) For 12, 16, 20, ... find S_{22}. — **3**

   c) For 8, –24, 72, ... find S_{12}. — **3**

   d) Evaluate $\sum_{n=5}^{17} 3n+1$ — **3**

3. What is the first term and common ratio of the Geometric sequence where $T_5 = -10$ and $T_9 = -\frac{5}{8}$? — **3**

4. A boy earns \$6 the first day, \$7 the second day, \$8 the third day. If his earnings continue to increase in the same manner, how long will it take him to earn \$105 ? — **4**

5. For the series $5 + 1 + \frac{1}{5} + \ldots$, find

i) S_∞ **2**

ii) S_5 in fraction form **2**

iii) the difference $S_\infty - S_5$ **1**

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Section 1.

① i) $y = 6 + 3x - 5x^2$

$\frac{dy}{dx} = 3 - 10x$. ✓

ii) $f(x) = 6 - x$

$f'(x) = -1$ ✓

iii) $y = x^2\sqrt{x}$

$= x^{\frac{5}{2}}$ ✓

$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}$

$= \frac{5\sqrt{x^3}}{2}$. ✓

iv) $y = \frac{3x^3 - x}{x^2}$

$= 3x - x^{-1}$ ✓

$\frac{dy}{dx} = 3 + x^{-2}$

$= 3 + \frac{1}{x^2}$ ✓

(6

② i) $\lim_{x \to 2} x^2 = 4$ ✓

ii) $\lim_{x \to 3} \frac{x^3 - 27}{x - 3}$

$= \lim_{x \to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)}$

$= \lim_{x \to 3} x^2 + 3x + 9$ ✓

$= 9 + 9 + 9$

$= 27$. ✓

(3.

③ $f(x) = x^2 - 3x - 2$.

a)

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. ✓

$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 3(x+\Delta x) - 2 - (x^2 - 3x - 2)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 3x - 3\Delta x - x^2 + 3x + 2}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 - 3\Delta x}{\Delta x}$ ✓

$= \lim_{\Delta x \to 0} 2x + \Delta x - 3$.

$= 2x - 3$. ✓

b) $y = x^2 - 3x - 2$

$\frac{dy}{dx} = 2x - 3$.

$\tan \theta = 2x - 3$.

Line $3x - y + 2 = 0$. $m = 3$. ✓

$\therefore 2x - 3 = 3$

$2x = 6$

$x = 3$. ✓

When $x = 3$.

$y = 3^2 - 3(3) - 2$ ✓

$= -2$. $\therefore P(3, -2)$

(6.

④ i) $y = \sqrt{x^3 - 2}$

$= (x^3 - 2)^{\frac{1}{2}}$. ✓

$\frac{dy}{dx} = \frac{1}{2}(x^3 - 2)^{-\frac{1}{2}}(3x^2)$. ✓

$= \frac{3x^2}{2\sqrt{x^3 - 2}}$. ✓

ii) $y = (x^2+4)(2x+3)^3$ ✓

$\frac{dy}{dx} = (2x+3)^3(2x) + (x^2+4) \times 3(2x+3)^2(2)$ ✓

$= 2x(2x+3)^3 + 6(2x+3)^2(x^2+4)$

$= 2(2x+3)^2\{x(2x+3) + 3(x^2+4)\}$

$= 2(2x+3)^2\{2x^2 + 3x + 3x^2 + 12\}$

$= 2(2x+3)^2(5x^2 + 3x + 12)$.

(6.

⑤ $y = \frac{x}{x-2}$ ✓.

$\frac{dy}{dx} = \frac{(x-2)(1) - x(1)}{(x-2)^2}$.

$= \frac{-2}{(x-2)^2}$. ✓

Tang. m.

at $(3,3) = \frac{-2}{1}$. ✓

$\therefore$ Grad. normal $= \frac{1}{2}$. ✓

Eqn. normal: $y - y_1 = m(x - x_1)$

$y - 3 = \frac{1}{2}(x - 3)$

$2(y - 3) = x - 3$.

$2y - 6 = x - 3$. ✓

$x - 2y + 3 = 0$

(4

⑥ $y = x - \frac{3}{x}$

$= x - 3x^{-1}$

$\frac{dy}{dx} = 1 + \frac{3}{x^2}$. ✓

at $(3,2)$ $m = 1 + \frac{3}{9}$

$= 1\frac{1}{3} = \frac{4}{3}$. ✓

When $m = \frac{4}{3}$, $\frac{4}{3} = 1 + \frac{3}{x^2}$. ✓

$4x^2 = 3x^2 + 9$

$x^2 = 9$

$x = \pm 3$.

$\therefore$ Other point $x = -3$.

$y = -3 - \frac{3}{-3}$.

$= -2$.

$\therefore (-3, -2)$. ✓

(4

⑦. Let numbers be x, $18 - x$.

$\therefore$ Product $P = x(18 - x)$

$P = 18x - x^2$. ✓

$\frac{dP}{dx} = 18 - 2x$.

When $\frac{dP}{dx} = 0$

$0 = 18 - 2x$. ✓

$2x = 18$

$x = 9$.

x	9^-	9	9^+
$\frac{dy}{dx}$	+	0	−

✓

$\therefore$ Max Product when numbers are 9, 9.

(3

(8)

a) $y = x^3 - 3x$

$\frac{dy}{dx} = 3x^2 - 3.$

Stat pts. $\frac{dy}{dx} = 0$

$3x^2 - 3 = 0$

$3(x+1)(x-1) = 0.$

$\therefore x = -1$, or $1.$

When $x = -1$

$y = (-1)^3 - 3(-1)$

$= 2.$ $(-1, 2)$ ✓

When $x = 1.$

$y = (1)^3 - 3$

$= -2.$ $(1, -2).$ ✓

x	-1^-	-1	-1^+		1^-	1	1^+
$\frac{dy}{dx}$	+	0	−		−	0	+
	⌒				⌣		

$\therefore$ $(-1, 2)$ Max T.P ✓ $(1, -2)$ min T.P. ✓ (4)

b) $y = x^3 - 3x$

Intercept on y axis $x = 0.$

$(0, 0)$

Intercept on x axis, $y = 0$

$0 = x(x^2 - 3)$

$\therefore x = 0, \pm\sqrt{3}.$

Extremities:

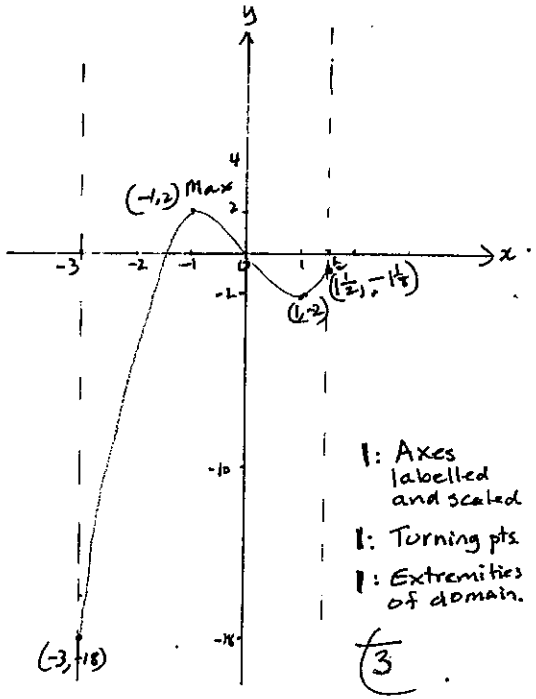
At $x = -3$, $y = (-3)^3 - 3(-3)$

$= -27 + 9$

$= -18.$

$x = 1\frac{1}{2}$, $y = (1\frac{1}{2})^3 - 3(\frac{3}{2})$

$= -\frac{9}{8}.$



1: Axes labelled and scaled

1: Turning pts

1: Extremities of domain.

(3)

c) Maximum value: 2 ✓

Minimum value: −18 ✓ (2)

(41)

SECTION 2.

(1). 80, −20, 5, ...

a) i) Geometric ✓

$r = \frac{-20}{80} = \frac{5}{-20} = -\frac{1}{4}$

ii) $T_9 = ar^8$

$= 80 \times (-\frac{1}{4})^8$ ✓

$= \frac{80}{65536}$

$= \frac{5}{4096}$ ✓

b) −22, −16, −10, ...

i) Arithmetic. $d = 6.$ ✓

ii) $T_n = a + (n-1)d$

$T_{53} = -22 + 52(6)$ ✓

$= 290.$ ✓ (6)

2. a) A.P. $a = 6$, $l = -4$

$S_n = \frac{n}{2}(a + l)$

$S_{30} = 15(6 - 4)$ ✓

$= 30.$ ✓

b) 12, 16, 20, ...

A.P. $a = 12$, $d = 4$ ✓

$S_n = \frac{n}{2}\{2a + (n-1)d\}.$

$S_{22} = 11\{24 + 21(4)\}$ ✓

$= 1188.$ ✓

c). 8, −24, 72, ...

G.P. $r = -3.$

$S_n = \frac{a(1 - r^n)}{1 - r}.$

$S_{12} = \frac{8 \cdot (1 - (-3)^{12})}{1 + 3}$ ✓

$= 2(1 - 531441)$ ✓

$= -1062880.$ ✓

d) $\sum_{n=5}^{17} 3n + 1$

$= 16 + 19 + 22 + \cdots + 52$ ✓

A.P. $a = 16$, $l = 52$, $n = 17 - 5 + 1$

$= 13.$

$S_n = \frac{n}{2}(a + l)$

$= \frac{13}{2}(16 + 52)$ ✓

$= 442.$ ✓ (11)

(3). T_5: $ar^4 = -10$ —①

T_9 $ar^8 = -\frac{5}{8}.$ —②

②÷①.

$r^4 = \frac{1}{16}$ ✓

$r = \pm\frac{1}{2}.$ ✓

Sub in ①. $a = \frac{-10}{r^4}$

$= \frac{-10}{\frac{1}{16}}$

$= -160.$ ✓

$\therefore$ $a = -160$ $r = \frac{1}{2}$

or $a = -160$ $r = -\frac{1}{2}$ (3)

(4). 6, 7, 8, ...

A.P. $a = 6$, $d = 1.$

$S_n = \frac{n}{2}\{2a + (n-1)d\}.$

$105 = \frac{n}{2}\{12 + n - 1\}$ ✓

$210 = n\{11 + n\}$ ✓

$210 = 11n + n^2$

$0 = n^2 + 11n - 210$ ✓

$0 = (n - 10)(n + 21)$

$\therefore n = 10$ or $-21.$

$\therefore$ 10 days. ✓ (4)

(5) $5 + 1 + \frac{1}{5} + \cdots$ G.P. $a = 5$, $r = \frac{1}{5}.$

i) $S_\infty = \frac{a}{1 - r}$ ✓

$= \frac{5}{1 - \frac{1}{5}}$

$= 6\frac{1}{4}$ ✓

ii) $S_n = \frac{a(1 - r^n)}{1 - r}$

$S_5 = \frac{5(1 - (\frac{1}{5})^5)}{1 - \frac{1}{5}}$ ✓

$= 5(\frac{1 - \frac{1}{3125}}{\frac{4}{5}})$

$= 6\frac{31}{125}$ ✓

iii) $S_\infty - S_5$

$= 6\frac{1}{4} - 6\frac{31}{125}$

$= \frac{1}{500}.$ ✓

(5)

(29)