KILLARA HIGH SCHOOL

HSC ASSESSMENT 1 - 2006

YEAR 12 - MATHEMATICS

Time allowed: 1 hour

Instructions:

Attempt all questions
Approved calculators may be used
Start your answer to each question on a new page.
Total marks: 50

Question 1 (31 marks): Calculus

Marks

a. Differentiate the following functions.

i. $y = 2x - 5x^2 + 9$ **1**

ii. $y = 3\sqrt{x} - x^4 + \dfrac{2}{x^3} + 7$ **2**

iii. $y = (2x^5 - 1)(6 + x)$ **2**

iv. $y = \dfrac{3x^5 - 10}{1 + x^3}$ **2**

v. $y = (3x^4 - 5x)^6$ **1**

vi. $y = \dfrac{8}{\sqrt{3x^5 + 1}}$ **1**

b. Find $\lim\limits_{x \to \infty} \dfrac{x^2 - 4x + 1}{x^2 - 3}$ **2**

c. i. Find the coordinates of the point P where the tangent to the curve $y = x^2 - 4x$ is parallel to the line $y = 2x + 1$. **2**

ii. Find the equation of the normal at the point P. **2**

iii. Find the coordinates of the points M and N where the normal cuts the X and Y axes respectively. **2**

iv. Find the area of ΔOMN, where O is the origin. **1**

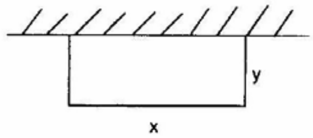
c. Given that $y = 3x^2 - x^3$,

i. Find the stationary points and distinguish between them. **3**

ii. Find the point of inflexion. **2**

iii. Sketch the curve. **3**

d. Farmer Brown has 500 m of wire with which to build a rectangular paddock. To save wire he decides to use an existing fence as one side of the paddock.



i. Show that the area, A m^2, of the paddock is given by $A = \dfrac{(500 - x)x}{2}$ **2**

ii. Find the dimensions of the paddock which would enclose the greatest area. **3**

Question 2 (19 marks): Sequence and series

a. The first term of an arithmetic progression is -25. The common difference is 4.

i. Write down the next two terms. **1**

ii. Find the 25^{th} term. **1**

b. Find $\sum_{n=9}^{20} 2n-1$ **2**

c. What is the least number of terms of the series $1+3+9+\ldots.$ that need to be taken to give a sum greater than 300. **3**

d. The 3^{rd} term of a geometric series is 24 and the 6^{th} term is 1536. Find

i. Find the common ratio. **2**

ii. Find the first term. **1**

e. By expressing the recurring decimal $0.12\dot{3}$ as the sum of an infinite number of terms of a series, find its value as a fraction in its simplest form. **3**

f. A woman borrows \$350000 in order to buy a house. Interest of 8.4% p.a. on the loan is calculated monthly on the balance still owing. The equal repayments, of \$M each, are made monthly and the loan is to be repaid over 25 years.

i. Show that the amount A_2 still owing after making two repayments of \$M each is given by $A_2 = 350000(1.007)^2 - M(1+1.007)$ **2**

ii. Find the amount, A, owing at the end of 25 years, in term of M. **1**

iii. Hence, show that $M = \dfrac{350000(1.007)^{300}(0.007)}{1.007^{300}-1}$ and find the value of M to the nearest dollar. **3**

End of paper

Solution

Question 1 : 21 marks

a) (i) $y' = 2 - 10x$ (1)

(ii) $y' = \frac{3}{2\sqrt{x}} - 4x^3 - \frac{6}{x^4}$ (2)

(iii) $y' = 10x^4(6+x) + 2x^5 - 1$ (1)

$= 60x^4 + 10x^5 + 2x^5 - 1$

$= 12x^5 + 60x^4 - 1$ (1)

iv) $y' = \frac{15x^4(1+x^3) - 3x^2(3x^5-10)}{(1+x^3)^2}$ (1)

$= \frac{15x^4 + 15x^7 - 9x^7 + 30x^2}{(1+x^3)^2}$

$= \frac{6x^7 + 15x^4 + 30x^2}{(1+x^3)^2}$ (1)

v) $y' = 6(3x^4 - 5x)^5 \times (12x^3 - 5)$ (1)

$= 6(12x^3-5)(3x^4-5x)^5$

vi) $y' = \frac{-4}{\sqrt{(3x^5+1)^3}} \times 15x^4$

$= \frac{-60x^4}{\sqrt{(3x^5+1)^3}}$ (1)

(b) $= \lim_{x \to \infty} \frac{\frac{x}{x^2} - \frac{4x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{3}{x^2}} = 1$ (2)

(c) (i) $y' = 2x - 4$ (Sub. $m = 2$.)

$2 = 2x - 4$ (1)

$\begin{cases} x = 3 \\ y = -3 \end{cases}$ (1)

(ii) $m_{normal} = -\frac{1}{2}$ (1)

$\frac{y+3}{x-3} = -\frac{1}{2} \Rightarrow 2y + 6 = -x + 3$

$2y + x + 3 = 0.$ (1)

(iii) $\therefore M\left(0, -\frac{3}{2}\right)$ (1) and $N(-3, 0)$ (1)

iv) $A = \frac{1}{2} \times 3 \times \frac{3}{2} = \frac{9}{4} = 2.25$ sq. units (1)

(d) $y' = 6x - 3x^2$

(i) Stationary points : $\left.\begin{aligned} 6x - 3x^2 &= 0 \\ 3x(2-x) &= 0 \end{aligned}\right\}$ $x = 0,\ y = 0$; $x = 2,\ y = 4$.

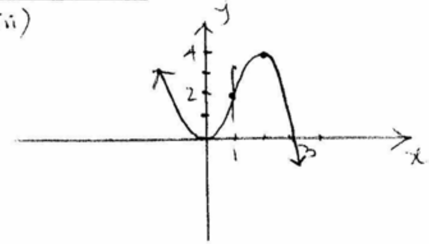
$y'' = 6 - 6x$ $f''(0) > 0$ min. pt. $(0,0)$ $f''(2) < 0$ max. point. $(2,4)$ (1)

ii) Pts. of inflexion : $6 - 6x = 0 \Rightarrow x = 1$

$f''(x<1) > 0$ $f''(x>1) < 0$ (1) $\therefore$ pt. of inflexion $(1,2)$ (1)

Q1 Continue

(c)(iii)



(1) Correct Shape

(2) showing all critical points

$(-\tfrac{1}{2})$ any missing or incorrect

(d)(i) $x + 2y = 500$

$y = \dfrac{500 - x}{2}$ (1)

$A = \dfrac{500 - x}{2} \times x = \dfrac{(500 - x)x}{2}$ (1)

(ii) $\dfrac{dA}{dx} = 250 - x = 0$

$\Rightarrow x = 250$ (1)

$A''(x) = -1 < 0$

$\therefore$ always concave down

$\therefore x = 250$ is a max (1)

$y = 125$ (1)

Quest. 2 · 17 marks

(a) (i) $-21, -17$ (1)

ii) $T_{25} = -25 + (25-4) \times 4$

$= 71$ (1)

(b) $17, 19, \ldots \ldots 39$ (1)

$S_{12} = \dfrac{12}{2}(17 + 39)$ (1)

$= 336$

(c) $\dfrac{1(3^n - 1)}{3-1} > 300$ (1) $\Rightarrow 3^n - 1 > 600$

$n > \dfrac{\log 601}{\log 3}$ (1)

$n > 5.822$

$\therefore n = 6$ (1)

(d)

(i) $ar^5 = 1536$

$ar^2 = 24$ $\div$ (1)

$r^3 = 64 \Rightarrow r = 4$ (1)

(ii) $24 = 16a \Rightarrow a = 1.5$ (1)

(e) $0.12\dot{3} = 0.12333\ldots = \dfrac{12}{10} + \dfrac{3}{1000} + \dfrac{3}{10000} + \cdots$ (1)

$S_\infty = \dfrac{\frac{3}{1000}}{1 - \frac{1}{10}} = \dfrac{1}{300}$ (1)

$\therefore 0.12\dot{3} = \dfrac{12}{10} + \dfrac{1}{300} = \dfrac{37}{300}$ (1)

(e)

i) $A_1 = 350000(1+.007)^1$

$= 350{,}000(1.007) - M$

$A_2 = [350{,}000(1.007) - M]\,1.007 - M$ ①

$= 350000(1.007)^2 - M(1 + 1.007)$ ①

$\vdots$

$r = 8.4\% \div 12$
$= .007$

$n = 12 \times 25$
$= 300$ months.

(ii) $A_{25} = 350000(1.007)^{300} - M(1 + 1.007 + \dots + 1.007^{299})$ ①

(iii) At end of 25 years $A_{25} = 0$

$$0 = 350\,000(1.007)^{300} - M\underbrace{(1 + 1.007 + \dots + 1.007^{299})}_{S_{300} = \frac{1(1.007^{300} - 1)}{.007}}$$

$\Rightarrow 0 = 350000(1.007)^{300} - \dfrac{M(1.007^{300} - 1)}{.007}$ ①

$\dfrac{M(1.007^{300} - 1)}{.007} = 350\,000(1.007)^{300}$

$\therefore M = \dfrac{350000(1.007)^{300}(.007)}{1.007^{300} - 1}$ ①

$\boxed{M = \$2795}$ ①