Killara High School

Year 11 Term 4 HSC Mathematics Assessment

December 2008

Time Allowed : 60 minutes — Total marks: 46 marks

Instructions: **Read Thoroughly**

All questions should be attempted.

Write your answers neatly, only using **one** side of your answer sheet.

Only **non-programmable** calculators may be used.

Show **all** necessary working.

Marks will be awarded for necessary working.

At the discretion of the examiners, marks may not be awarded for answers without adequate working.

At the discretion of the examiners, marks may be deducted for untidy or poorly arranged working.

PART A	/23
PART B	/23
TOTAL	/46

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x + C, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2} \right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2} \right) + C$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section A: Geometrical applications of Calculus (Total marks: 23)

1. A function is defined as $f(x)$. The first derivative of $f(x)$ is defined as $f'(x)$. The second derivative of the function is defined as $f''(x)$.

   (a) Briefly explain a purpose of finding $f'(x)$. (1)

   (b) Rewrite the following, completing the sentence.
   $f'(x)$ is often referred to as the ________________ Function. (1)

   (c) If $f(x)$ is concave up at a point when x=a, briefly explain what process is used to determine whether the concavity is up or down. (1)

2. A function is defined as $f(x) = x^3 - 6x^2 + 9x - 5$.

   (a) Find the Stationary points of the function. (3)

   (b) Find where any points of inflection of the function will lie. (2)

   (c) Determine the nature of each stationary point. (2)

   (d) Draw a neat pair of coordinate axes showing the scale and labeling each axis. (1)

   (e) Plot the stationary points and any points of inflection showing the important features and y-axis intercept. (3)

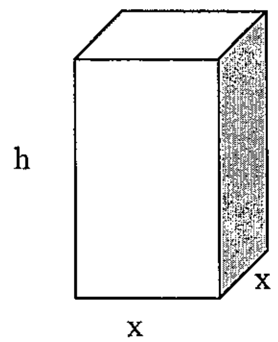
   (f) The curve will cut the x-axis between two values.
   Find which two values these are. (1)

3. A rectangular block with a square base has the total area of it's surface area equal to 150 cm^2 and the sides of the base are each x cm long.

   (a) Find an equation for h, the height of the block in terms of x. (2)

   (b) Hence show that the volume is $V = \frac{1}{2}(75x - x^3)$. (2)

   (c) Hence find the maximum volume of the box. (4)



Section B: The Quadratic Polynomial. (Total marks: 23)

1. If α and β are the roots of the quadratic equation $5x^2 - x + 15 = 0$, find without solving the equation the values of:

   a. $\alpha + \beta$ (1)

   b. $\alpha\beta$ (1)

   c. $\alpha^2\beta^3 + \alpha^3\beta^2$ (1)

   d. $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$ (2)

   e. $\alpha\beta - 2\alpha - 2\beta + 1$ (2)

2. A parabola is defined by the function $y = x^2 - 2x - 8$.

   a. Find the equation of the axis of symmetry. (1)

   b. Find the co-ordinates of the vertex. (1)

   c. Sketch the graph of $y = x^2 - 2x - 8$ showing all the intercepts with the x and y axes. (2)

   d. Hence solve the inequality: $x^2 - 2x - 8 < 0$. (1)

3. Find the value of m for which the quadratic equation $mx^2 - (2m+1)x + m = 0$ has two real unequal roots. (2)

4. If the equation $3x^2 + Mx + 8$ has one root which is 6 times the other, find the value of M. (3)

5. For what values of k is the expression $x^2 - 2kx + k + 2$ positive definite. (3)

6. Find all real roots of the equation $9^x - 28.3^x + 27 = 0$. (3)

Q1 a) • To find the gradient of the tangent to a curve
• to determine stationary PTS. (1)

b) Gradient
or Rabbit (1)

c) consider $f''(a)$ (1)
if $f''(a) > 0$ then concave up.
if $f''(a) < 0$ then concave down

Q2. a) $f(x) = x^3 - 6x^2 + 9x - 5$

$f'(x) = 3x^2 - 12x + 9$
$= 3(x^2 - 4x + 3)$
$= 3(x-1)(x-3)$

for st. pts $f'(x) = 0$
$(x-1)(x-3) = 0$
$x = 1, 3$ ✓✓
$(1, -1)$ and $(3, -5)$ ✓ (3)

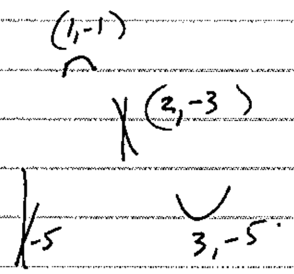
b) $f''(x) = 6x - 12$
$= 6(x-2)$

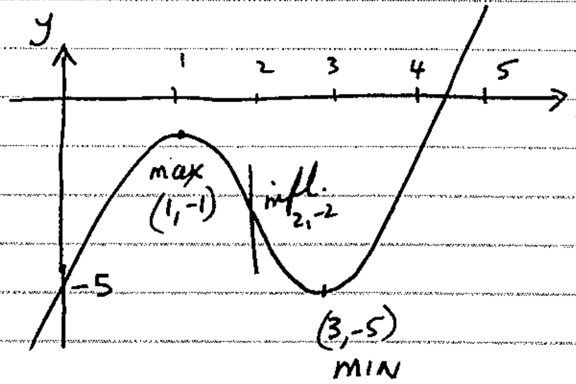
x	$1\frac{1}{2}$	2	$2\frac{1}{2}$
$f''(x)$	$-$	0	$+$

change of concavity ✓
$(x-2) = 0 \Rightarrow x = 2$ ✓

(2)

c) $f''(1) = 6 - 12 = -6 < 0$ ✓
$\therefore$ max at $(1, -1)$
$f''(3) = 18 - 12 = 6 > 0$ ✓
$\therefore$ min at $(3, -5)$ (2)





d) • — neat axes / labelled ✓
• — (scale) + units ✓ (1)

e)
$f(4) = -1$
$f(5) = 15$

graph as above
plot + label
• inflection ✓
• max ✓
• min ✓
• y intercept ✓
• approx x intercept ✓

(3)
−1 for each missing

(f) Between 4 and 5 (1)

Q3. a)

Surface area $= 2x^2 + 4xh$ ✓

$\therefore 150 = 2x^2 + 4xh$

$h = \dfrac{150 - 2x^2}{4x}$ ✓ (2)

b) $V = Ah$

$= x^2 h$

$= \left(\dfrac{150 - 2x^2}{4x}\right) \cdot x^2$ ✓CE

$= \dfrac{x}{4}(150 - 2x^2)$

$V = \dfrac{x}{2}(75 - x^2)$ ✓ (2)

c) $V = \dfrac{75x}{2} - \dfrac{x^3}{2}$

$\dfrac{dV}{dx} = \dfrac{75}{2} - \dfrac{3x^2}{2}$ ✓

for max/min $\dfrac{dV}{dx} = 0$

$\dfrac{3x^2}{\cancel{2}} = \dfrac{75}{\cancel{2}}$

$x^2 = 25$

$x = \pm 5$ but $x > 0$

$x = 5$ ✓

$\dfrac{d^2V}{dx^2} = \dfrac{-6x}{2} = -3x$

if $x = 5$ $\dfrac{d^2V}{dx^2} = -15 < 0$

$\therefore$ max vol when $x = 5$ (4)
✓

max vol $= \dfrac{5}{2}(75 - 25)$

$= \dfrac{5(50)}{2}$

$= \dfrac{250}{2}$

$= 125 \text{ cm}^3$ ✓

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Section B

Q4.

$5x^2 - x + 15 = 0$

a) $\alpha + \beta = -\dfrac{(-1)}{5} = \dfrac{1}{5}$ ✓ (1)

b) $\alpha\beta = \dfrac{15}{5} = 3$ ✓ (1)

c) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = \dfrac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$= \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2}$ ✓

$= \dfrac{\left(\frac{1}{5}\right)^2 - 2(3)}{(3)^2}$

$= \dfrac{\frac{1}{25} - 6}{9}$

$= \dfrac{-149}{225}$ ✓ (2)

d)

$\alpha\beta - 2\alpha - 2\beta + 1$

$= \alpha\beta - 2(\alpha + \beta) + 1$

$= 3 - 2\left(\frac{1}{5}\right) + 1$

$= 3 - \frac{2}{5} + 1$

$= 18/5$

e) $\alpha^2\beta^3 + \alpha^3\beta^2$

$= \alpha^2\beta^3(\beta + \alpha)$

$= 3^2\left(\frac{1}{5}\right) = \frac{9}{5}$ ✓ (1)

Q5

$y = x^2 - 2x - 8$

a) axis of symmetry $x = -\frac{b}{2a}$

$x = \frac{-(-2)}{2(1)} = 1$ ✓ (1)

b) at vertex $x = 1$

$y = 1 - 2 - 8 = -9$

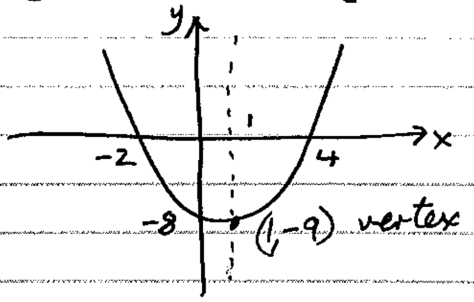
$(1, -9)$ ✓ (1)

c) if $y = 0$ $x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

x intercepts $x = 4, -2$ ✓

for y intercept $x = 0$, $y = -8$ ✓



(2)

d) $x^2 - 2x - 8 < 0$

$-2 < x < 4$ ✓ (1)

Q6. $mx^2 - (2m+1)x + m = 0$

for 2 real distinct roots $\Delta > 0$

$b^2 - 4ac = [-(2m+1)]^2 - 4(m)(m) > 0$ ✓$_{EX}$

$= (2m+1)^2 - 4m^2$

$= 4m^2 + 4m + 1 - 4m^2$

$= 4m + 1 > 0$

$4m > -1$

$m > -\frac{1}{4}$ ✓ (2)

Q7. Let roots be

$p, 6p.$

$p + 6p = -\frac{M}{3}$

$7p = -\frac{M}{3}$ $M = -21p$ — (A) ✓

$p.6p = \frac{8}{3}$

$6p^2 = \frac{8}{3}$

$p^2 = \frac{8}{18} = \frac{4}{9}$

$p = \pm\frac{2}{3}$ — (B) ✓

sub (B) into (A)

$M = -21.(\pm\frac{2}{3})$

$M = \pm 14$ ✓ (3)

Q8

positive definite

$\Delta < 0$ $a > 0$

$a = 1 > 0$

$\Delta = b^2 - 4ac$

$= (-2k)^2 - 4(1)(k+2)$

$= 4k^2 - 4k - 8$

$= 4(k^2 - k - 2)$

$= 4(k-2)(k+1)$

$4(k-2)(k+1) < 0$ ✓

$-1 < x < 2$ ✓✓ (3)

Question 9.

$$9^x - 28.3^x + 27 = 0$$

Let $m = 3^x$ $\therefore m^2 = (3^x)^2 = (3^2)^x = 9^x$

$$m^2 - 28m + 27 = 0$$
$$(m-27)(m-1) = 0$$

$m = 27$ or 1. ✓

$3^x = 27$ or $3^x = 1$

$x = 3$ or $x = 1$ ✓✓

(3)

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