Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS

TUESDAY 15th May 2012

General Instructions

- Reading Time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total Marks - 40

- Attempt questions 1 - 8
- Answer questions 1 – 4 on the multiple choice answer sheet provided
- For questions 5 – 8, start each question in a new booklet

QUESTION NO	MARK
1 - 4	/4
5	/9
6	/8
7	/9
8	/10
TOTAL	/40

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 15% of the Higher School Certificate Course Assessment.

Answer questions 1 – 4 on the multiple choice answer sheet provided **Mark**

Question 1 **1**

The range of the function $f(x) = 2\cos 3x$ is:

(A) $0 \le f(x) \le 2\pi$ (B) $-3 \le f(x) \le 3$

(C) $-2 \le f(x) \le 2$ (D) $0 \le f(x) \le \dfrac{3\pi}{2}$

Question 2 **1**

Evaluate $3\sin\dfrac{\pi}{7}$ correct to four significant figures

(A) 0.02349 (B) 0.02350

(C) 1.301 (D) 1.302

Question 3

For the series $a + ar + ar^2 + \ldots\ldots$, $|r| < 1$, the difference between the limiting sum and the sum to n terms is given by: **1**

(A) $\dfrac{-ar^n}{1-r}$ (B) $\dfrac{a(2-r^n)}{r-1}$

(C) $\dfrac{ar^n}{1-r}$ (D) $\dfrac{r^n}{r-1}$

Question 4

The sum of the first n terms of a series is given by $S_n = n(3n+1)$. **1**

The seventh term is?

(A) 154 (B) 40

(C) 46 (D) 148

Question 5 (9 marks) Use a SEPARATE writing booklet. **Marks**

(a) An infinite geometric series has a first term of eight and a limiting sum of twelve. Calculate the common ratio. **2**

(b) The third term of an arithmetic series is 32 and the sixth term is 17.

(i) Find the common difference **1**

(ii) Find the sum of the first twelve terms. **2**

(c) A ball is dropped from a height of 2 metres onto a hardwood floor and bounces. After each bounce, the maximum height reached by the ball is 75% of the previous maximum height. Thus, after it first hits the floor, it reaches a height of 1.5 metres before falling again, and after the second bounce, it reaches a height of 1.125 metres before falling again.

(i) What is the maximum height reached after the third bounce? **1**

(ii) Explain why the sequence formed by the successive maximum heights is geometric. **1**

(iii) What is the total distance travelled by the ball from the time it was first dropped until it eventually comes to rest on the floor. **2**

End of question 5

Question 6 (9 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\sum_{r=3}^{5} \frac{1}{r}$ **1**

(b) Explain why the geometric series $2+\frac{2}{\sqrt{2}-1}+\frac{2}{\left(\sqrt{2}-1\right)^2}+\ldots\ldots\ldots$ **1**

does NOT have a limiting sum.

(c) Mr Jon West, a salmon farmer, started his business on the 1st January 2001, with a stock of 1 000 000 salmon. He had a contract to supply 154 000 salmon at a price of \$10 per salmon to a retailer in December each year. In the period between January and the harvest at the end of December each year, the number of fish increase by 10%.

(i) Find the number of salmon immediately after the second harvest in December, 2002. **1**

(ii) Show that S_n, the number of salmon just after the nth harvest, is given by $S_n = 1\ 540\ 000 - 540\ 000(1\cdot1)^n$ **2**

(iii) When will Jon West have sold all his fish, and what will his total income be then? **3**

End of question 6

Question 7 (9 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the exact value of $\cos\dfrac{\pi}{4}+\cos\dfrac{2\pi}{3}$, **2**

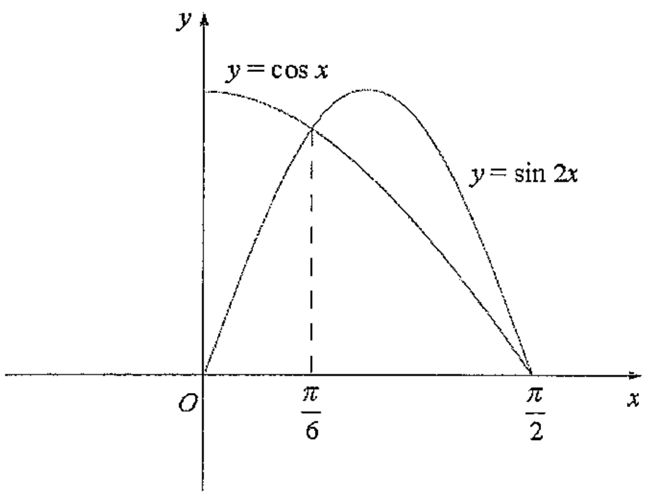
expressing your answer with a common denominator

(b) Differentiate:

(i) $x\cos(x+1)$ **2**

(ii) $\dfrac{\tan x}{x}$ **2**

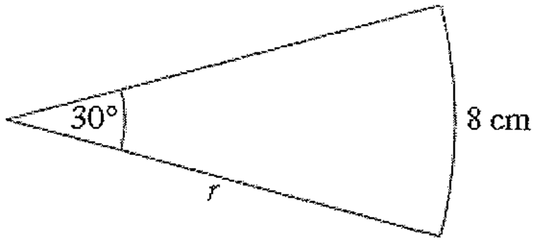
(c) The diagram below shows the graphs of the functions $y=\cos x$ and $y=\sin 2x$ between $x=0$ and $x=\dfrac{\pi}{2}$. The two graphs intersect at $x=\dfrac{\pi}{6}$ and $x=\dfrac{\pi}{2}$. Calculate the shaded area. **3**



End of Question 7

Question 8 (10 marks) Use a SEPARATE writing booklet. **Marks**

(a) 2



NOT TO SCALE

Find the length of the radius of the sector of the circle shown in the diagram above.

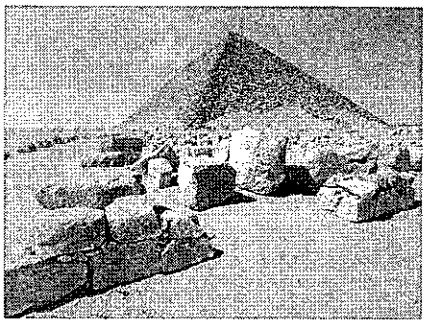
Answer correct to the nearest millimetre.

(b) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\sec^2 3x \, dx$ 2

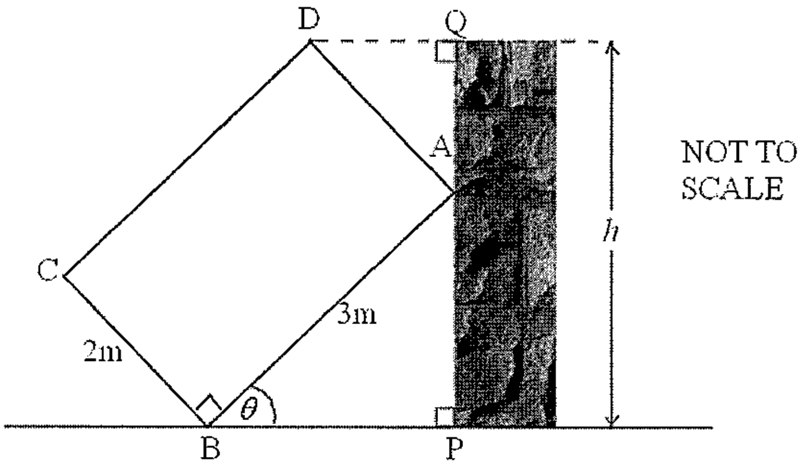
Question 8 is continued on the next page

Question 8 continued:

(c) Pharoah Djoser's Step Pyramid at Saqqara was built in 2630BC. In order to build the pyramids, huge rectangular granite blocks were floated from quarries right to the base of the Pyramids. The stones were then polished by hand and pushed up ramps to their intended positions.



In the earliest of pyramids, burial chambers were underground.
Imhotep, the architect of the Step Pyramid, was experimenting with the different ways in which he could place each block in order to ensure that the wall of the chamber had maximum strength. The block measured approximately 3m by 2m and is labelled ABCD in the diagram. It was placed against a vertical wall, labelled PQ, making an angle of θ with the horizontal. The height of D above the ground is h metres.



(i) Triangle DQA is right-angled at Q. Show that $\angle DAQ = \theta$. **1**

(ii) Hence show that $h = 3\sin\theta + 2\cos\theta$, where $0 \le \theta \le \dfrac{\pi}{2}$ **1**

(iii) Obtain an expression for $\dfrac{dh}{d\theta}$, and hence find the value of θ for which $\dfrac{dh}{d\theta} = 0$ **2**

(iv) Using calculus, explain why for this value of θ, the value of h is a maximum. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Multiple Choice Answer Sheet

Completely fill the response oval representing the most correct answer.

1. A ○ B ○ C ○ D ○

2. A ○ B ○ C ○ D ○

3. A ○ B ○ C ○ D ○

4. A ○ B ○ C ○ D ○

SOLUTIONS

MULTIPLE CHOICE 1 C 2 D 3 C 4 B

MATHEMATICS ASSESSMENT YR 12. 15th May, 2012

QUESTION 5

a) $a = 8$ $\quad S_\infty = \frac{a}{1-r}$

$S_\infty = 12$

$12 = \frac{8}{1-r}$

$12 - 12r = 8$

$-12r = -4$

$r = \frac{1}{3}$

Award zero for bald answer.

1 mark for correct substitution into correct formula

1 mark for correct value of r.

b)(i) $T_3 = a + 2d = 32$ —— ①

$T_6 = a + 5d = 17$ —— ②

① − ② ⇒ $-3d = 15$

$d = -5$

OR.

32, ___, ___, 17

$(32 - 17) \div 3$

difference = −5

1 mark for correct answer with working

(ii) $S_{12} = ?$ $\quad d = -5$ —— *

* → ① $\quad a - 10 = 32$

$a = 42$.

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_n = \frac{12}{2}(2 \times 42 + 11 \times -5)$

$S_n = 6(84 - 55)$

$S_n = 174$

OR $T_{12} = 42 + 11 \times -5$

$T_{12} = -13$.

$S_n = \frac{n}{2}[a + L]$.

$S_n = \frac{12}{2}[42 - 13]$

or $6[42 - 13]$.

$S_n = 174$

2 marks for writing out all 12 terms, adding & correct answer

1 mark for correct substitution into either Sn formula.

1 mark for correct evaluation for Sn.

c) Series is $2 + 1.5 + 1.125 + \ldots\ldots$

1st 2nd 3rd

(i) $T_3 = 0.75 \times 1.125$

$= 0.84375\text{m}$.

1 mark.

(ii) The series is geometric because each term is formed by multiply the previous term by 0.75 (the ratio)

1 mark.

(iii) Total distance =

$2 + 2 \times (1.5 + 1.125 + \ldots\ldots)$

$S_\infty = \frac{1.5}{1 - 0.75}$

$= 2 + 2 \times 6$

$= 14\text{m}$.

1 mark for $2 \times \left(\frac{1.5}{1 - 0.75}\right)$

1 mark for 12.

0 for $\frac{1.5}{1 - 0.75}$

QUESTION 6.

a) $\sum_{r=3}^{5} \frac{1}{r} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ $= \frac{47}{60}$ or $0.78\dot{3}$	1 mark for correct answer with correct working
b) $2 + \frac{2}{\sqrt{2}-1} + \frac{2}{(\sqrt{2}-1)^2}$ $r = \frac{1}{\sqrt{2}-1} \doteqdot 2.414 > 0$ $\therefore$ No limiting sum because $\lvert r\rvert \not< 1$ or $\lvert r\rvert > 1$	1 mark. Must show $r = \frac{1}{\sqrt{2}-1}$ & have correct explanation.
c) At end of 2001 (1st), no of fish $= 1000\,000(1.1) - 154000$	
(i) At end of 2002, no of fish $= 1000\,000(1.1)^2 - 154000(1.1) - 154000$ $= \$886,600$	1 mark for correct answer with working
(ii) At end of 2003 $(n=3)$, no of fish $= 1000000 \times 1.1^3 - 154000(1 + 1.1 + 1.1^2)$ $\vdots$ At end of nth year, no of fish $= 1000\,000(1.1)^n - 154000\underbrace{(1 + 1.1 + \dots + 1.1^n)}$ GP $a=1$, $r=1.1$ $S_n = \frac{1(1.1^n - 1)}{1.1 - 1} = \frac{1.1^n - 1}{0.1}$	All working must be there
$\therefore S_n = 1000\,000(1.1)^n - \frac{154000(1.1^n - 1)}{0.1}$	1 mark for this line
$= 1000\,000(1.1)^n - 1540\,000(1.1^n - 1)$ $= 1000\,000(1.1)^n - 1540\,000(1.1^n) + 1540\,000$	all working must be there
$= 1540\,000 - 540000(1.1)^n$	1 mark for getting to this line
(iv) $1540\,000 - 540\,000(1.1)^n = 0$ when all fish sold. $\therefore 540\,000(1.1)^n = 1540\,000$ $1.1^n = 1540\,000 \div 540\,000$	1 mark for this line
$n = \ln\left(\frac{1540000}{540000}\right) \div \ln 1.1$ $n = 10.99\dots$ $\therefore$ It will take 11 years	1 mark for correct soln of equation
Income $= 154000 \times 11 \times \$10$ $= \$16\,940\,000$	1 mark for number of years $\times 154000 \times 10$.

QUESTION 7

Solution	Marking
a) $\cos\frac{\pi}{4} + \cos\frac{2\pi}{3}$ $= \frac{1}{\sqrt{2}} + \cos\left(\pi - \frac{\pi}{3}\right)$ $= \frac{1}{\sqrt{2}} - \frac{1}{2}$ $= \frac{2 - \sqrt{2}}{2\sqrt{2}}$ OR $\frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{\sqrt{2}-1}{2}$	1 mark for correct exact value of $\cos\frac{\pi}{4}$ or $\cos\frac{2\pi}{3}$ 1 mark for final answer
b)(i) $\frac{d}{dx}\left(x\cos(x+1)\right)$ $= x \cdot -\sin(x+1) + \cos(x+1)\cdot 1$ $= \cos(x+1) - x\sin(x+1)$	* subtract 1 for no $y=$ or $\frac{d}{dx}$ 1 mark for correct use of product rule 1 mark for final answer
(ii) $\frac{d}{dx}\left(\frac{\tan x}{x}\right) = \frac{x \cdot \sec^2 x - \tan x \cdot 1}{x^2}$	1 mark for correct use of quotient rule 1 mark for $\frac{d}{dx}\tan x = \sec^2$
c) Area $= \int_0^{\frac{\pi}{6}} \sin 2x\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx$ $= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}} + \left[\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $= \left[-\frac{1}{2}\cos\frac{2\pi}{6} + \frac{1}{2}\cos 0\right] + \left[\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right]$ $= \left[-\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\right] + \left[1 - \frac{1}{2}\right]$ $= \frac{1}{4} + \frac{1}{2}$ $= \frac{3}{4}$ unit^2.	1 mark. for correct expression. 1 mark for correct sol'n of $\int_0^{\frac{\pi}{6}} \sin 2x\,dx$ $(\frac{1}{2})$ OR $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx$ $(\frac{1}{2})$ 1 mark for correct final answer. If no other marks, 1 mark for a correct answer from an incorrect definite integral.

COCB

QUESTION 8.

a) $l = r\theta$. $8 = r \times \frac{\pi}{6}$ $r = \frac{48}{\pi}$ or $15 \cdot 228$ or $15 \cdot 3$ cm.	1 mk correct sub. into formula 1 mark correct answer [ignore rounding].
b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\sec^2 3x\,dx = \left[\frac{2}{3}\tan 3x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= \frac{2}{3}\tan 3.\frac{\pi}{3} - \frac{2}{3}\tan\frac{3\pi}{4}$ $= \frac{2}{3}.\ 0 - \frac{2}{3}.-1$ $= \frac{2}{3}$.	1 mark for correct expression 1 mark for correct answer. <u>Must</u> be from correct working
c)(i) $\angle BAP = 90^\circ - \theta$ ($\angle$ sum of $\triangle BAP$) $\angle DAB = 90^\circ$ (angle in rectangle - given) $\angle DAQ = 180^\circ - 90^\circ - (90^\circ - \theta)$ (angle sum of $\triangle QAP$) $\angle BAQ = \theta$	1 mark. Must have all steps & reasons (SHOW θ!)
(ii) $h = PA + AQ$. In $\triangle BAP$, $\sin\theta = \frac{AP}{3}$ $\therefore AP = 3\sin\theta$. In $\triangle DQA$, $\cos\theta = \frac{AQ}{2}$ $\therefore AQ = 2\cos\theta$ $\therefore h = 3\sin\theta + 2\cos\theta$	<u>1 mark</u> Must have all working (SHOW θ)
(iii) $\frac{dh}{d\theta} = 3\cos\theta - 2\sin\theta$. When $\frac{dh}{d\theta} = 0$, $3\cos\theta - 2\sin\theta = 0$ $3\cos\theta = 2\sin\theta$ $\tan\theta = \frac{3}{2}$. $\therefore \theta = \tan^{-1}(1.5) \doteqdot 56^\circ 19'$.	1 mark. 1 mark. Must evaluate angle, ignore rounding
iv) $\frac{d^2h}{d\theta^2} = -3\sin\theta - 2\cos\theta$ When $\theta = 56^\circ 19$, $\frac{d^2h}{d\theta^2} = -3\sin 56^\circ 19' - 2\cos 56^\circ 19'$ $= -3 \cdot 6055 \ldots$ < 0 $\therefore$ Maximum value of h occurs when $\theta = 56^\circ 19'$ because $\frac{d^2h}{d\theta^2} < 0$	1 mark — <u>must</u> evaluate or explain each part 1 mark for <u>EXPLANATION</u> (must explain in full because its what is asked)