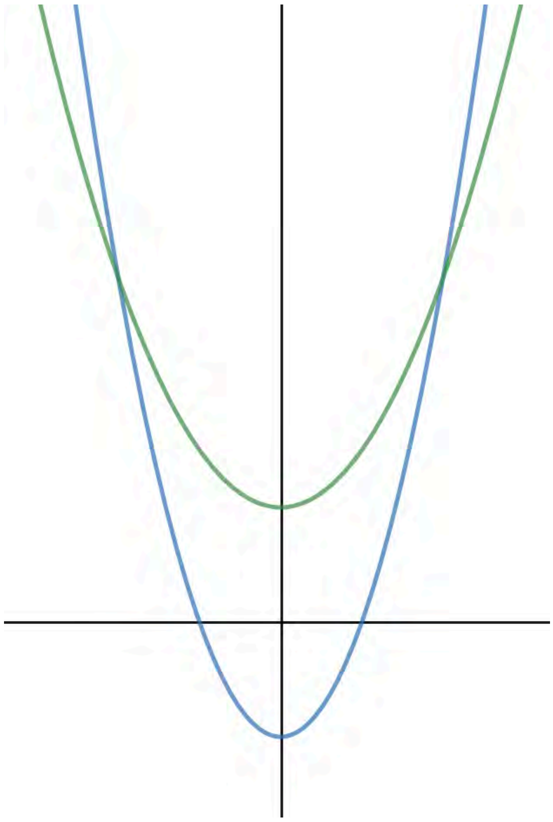
Question 1 (10 marks) Use a SEPARATE writing booklet

a. $$\int_{-2}^{1} x^3 - x + 3\, dx$$ **2**

b. $$\int \frac{x^4 + x^2 - 2}{x^2}\, dx$$ **2**

c. Find the exact area contained by the function $f(x) = \sqrt{3x - 6}$, the x axis and the line $x = 5$. **3**

d. i) Find the points of intersection of the curves $y = 2x^2 - 1$ and $y = x^2 + 1$ **1**

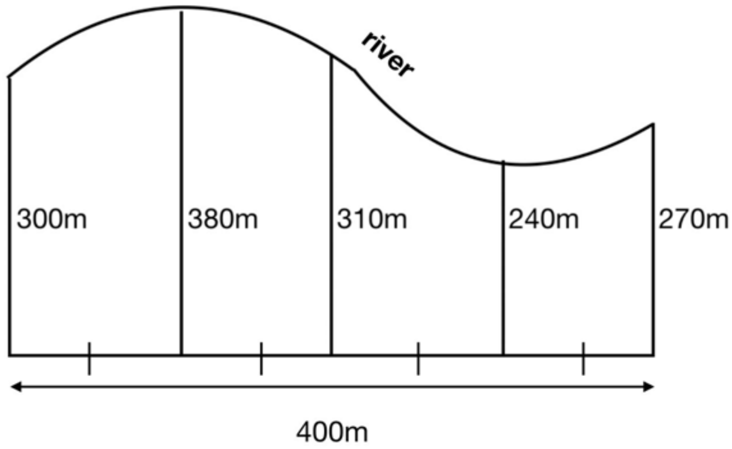


ii) Find the exact area contained between the curves $y = 2x^2 - 1$ and $y = x^2 + 1$ **2**

End of Question 1

Question 2 (10 marks) Use a SEPARATE writing booklet

a. A nature reserve borders a river as seen in the diagram below. An environment and landscape management company needs to know the area of this reserve to quote a price to maintain it. **2**



Use the trapezoidal rule to estimate the area of the nature reserve.

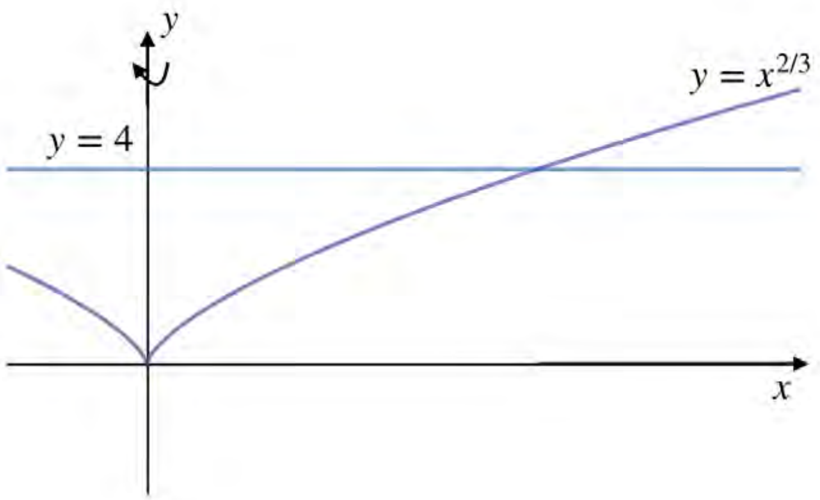
b. i) Differentiate $f(x) = (2x^2 - 3)^8$ **1**

ii) Hence, or otherwise, integrate $g(x) = x(2x^2 - 3)^7$ **2**

c. Use Simpson's rule with 5 function values to evaluate: **2**

$$\int_0^2 \frac{dx}{x+2}$$

d. Find the volume of the solid of revolution formed when the area bounded by the y axis, the curve $y = x^{\frac{2}{3}}$ and the line $y = 4$ is rotated about the y axis. **3**



End of Question 2

Question 3 (9 marks) Use a SEPARATE writing booklet

a. Use the first derivative to find values of x for which $f(x) = x(12 - x^2)$ is increasing. **3**

b. A function $f(x) = Ax^3 + Bx^2 + Cx + 1$ has a local maximum at point $(-1,3)$ and a point of inflection at point $\left(\frac{-1}{3}, \frac{49}{27}\right)$. Find the values of A, B and C. **3**

c. Find the equation of the normal to the curve $y = \frac{x^2}{3-x^2}$ at the point $\left(1, \frac{1}{2}\right)$. **3**

End of Question 3

Question 4 (11 marks) Use a SEPARATE writing booklet

a. A function $y = f(x)$ has the following characteristics:

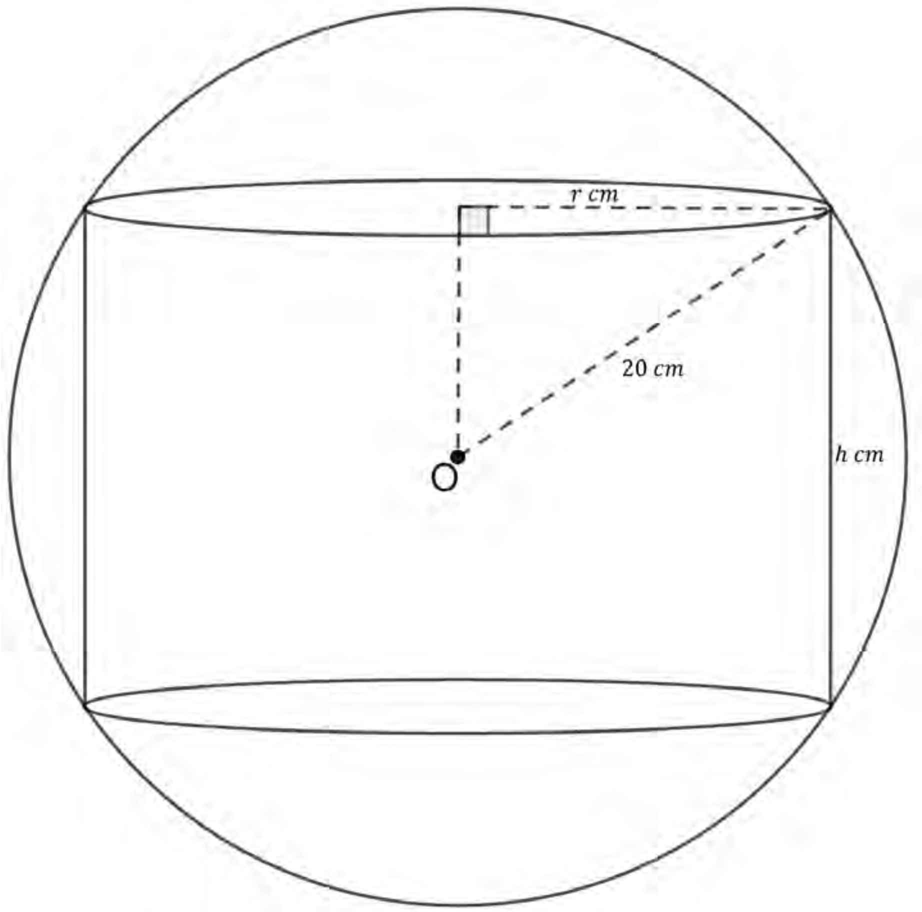
- The graph is a smooth, continuous curve and the function is differentiable for all real x
- $f'(x) = 0$ when $x = 2$ and -7
- $f''(x) < 0$ for $-4 < x < 2$
- $f''(x) > 0$ for $x < -4$ and $x > 2$

i) Identify the x values of any points of inflection. **1**

ii) Identify the x values of all stationary points and determine their nature. **2**

iii) Sketch a possible graph of the function, clearly showing the x value of each key point. **2**

b. A solid cylinder is to be cut from a sphere of radius 20cm. O is the centre of both the sphere and cylinder.



i) Find an expression for the radius r of the cylinder in terms of its height h. **1**

ii) Hence or otherwise, show that the volume of the cylinder is given by: **2**

$$v = 400\pi h - \frac{\pi h^3}{4}$$

iii) Find the height that would maximise the volume of the cylinder. **3**

End of Examination

Solutions

Tuesday, 26 February 2019 7:22 PM

Question One

a) $\int_{-2}^{1} x^3 - x + 3 \, dx$

$= \left[\frac{x^4}{4} - \frac{x^2}{2} + 3x\right]_{-2}^{1}$ | mark for integration

$= \left(\frac{1^4}{4} - \frac{1^2}{2} + 3\cdot 1\right) - \left(\frac{(-2)^4}{4} - \frac{(-2)^2}{2} + 3\cdot -2\right)$

$= \left(\frac{1}{4} - \frac{1}{2} + 3\right) - \left(\frac{16}{4} - \frac{4}{2} - 6\right)$

$= \frac{11}{4} - -4$

$= \frac{27}{4}$ | mark for answer

b) $\int \frac{x^4 + x^2 - 2}{x^2} \, dx$

$= \int x^2 + 1 - \frac{2}{x^2} \, dx$ | mark for integration

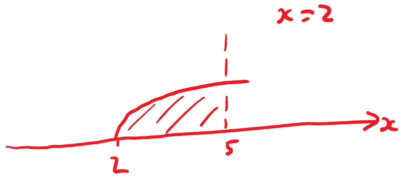
$= \frac{x^3}{3} + x + \frac{2}{x} + c$ | mark for constant '$+c$'

c) $f(x) = 0$ when $\sqrt{3x-6} = 0$

$3x - 6 = 0$

$3x = 6$

$x = 2$



Area $= \int_{2}^{5} \sqrt{3x-6} \, dx$ | mark for boundaries

$= \int_{2}^{5} (3x-6)^{\frac{1}{2}} \, dx$

$= \left[\frac{(3x-6)^{\frac{3}{2}}}{\frac{3}{2} \times 3}\right]_{2}^{5}$ | mark for integration

$= \left[\frac{2}{9}(3x-6)^{\frac{3}{2}}\right]_{2}^{5}$

$= \frac{2}{9}\left[(3x-6)^{\frac{3}{2}}\right]_{2}^{5}$

$= \frac{2}{9}\left((3\cdot 5 - 6)^{\frac{3}{2}} - (3\cdot 2 - 6)^{\frac{3}{2}}\right)$

$= \frac{2}{9}\left(9^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$

$= \frac{2}{9}\left(9^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$

$= \frac{2}{9} \times 27$

$= 6 \text{ units}^2$ 1 mark for answer

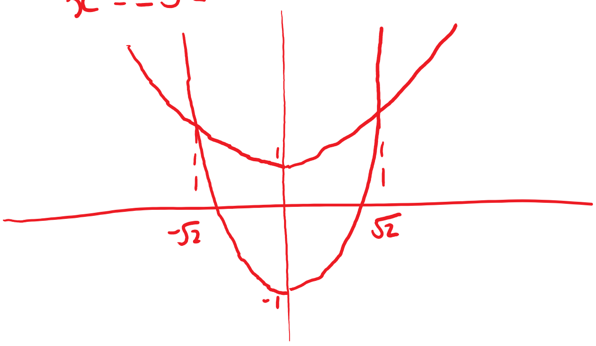
d) $y = 2x^2 - 1$ $y = x^2 + 1$

i) For points of intersection

$2x^2 - 1 = x^2 + 1$

$x^2 = 2$

$x = \pm\sqrt{2}$ 1 mark



ii) $A = \int \text{top} - \text{bottom}\ dx$

$= \int_{-\sqrt{2}}^{\sqrt{2}} (x^2+1) - (2x^2-1)\ dx$ 1 mark for correct integral

$= \int_{-\sqrt{2}}^{\sqrt{2}} -x^2 + 2\ dx$

$= \left[-\frac{x^3}{3} + 2x\right]_{-\sqrt{2}}^{\sqrt{2}}$

$= \left(\left(\frac{-\sqrt{2}^3}{3} + 2\sqrt{2}\right) - \left(\frac{-(-\sqrt{2})^3}{3} - 2\sqrt{2}\right)\right)$

$= \left(\frac{-2\sqrt{2}}{3} + \frac{6\sqrt{2}}{3}\right) - \left(\frac{2\sqrt{2}}{3} - \frac{6\sqrt{2}}{3}\right)$

$= \frac{8\sqrt{2}}{3} \text{ units}^2$ 1 mark for answer

Question Two

a) $h = \frac{400}{4}$ 1 mark for $h = 100$ or $b - a = 100$

a) $h = \frac{400}{4}$
$= 100$

1 mark for $h = 100$ or $b - a = 100$

$$A \approx \frac{100}{2}\left(300 + 2\times(380+310+240) + 270\right)$$
$$\approx 121500\,m^2$$

1 mark for answer

bi) $f(x) = (2x^2-3)^8$

$$f'(x) = 8\cdot 4x(2x^2-3)^7$$
$$= 32x(2x^2-3)^7$$

1 mark for derivative

ii $g(x) = x(2x^2-3)^7$

$$\int x(2x^2-3)^7\,dx = \frac{1}{32}\int 32x(2x^2-3)^7\,dx$$

1 mark for taking $\frac{1}{32}$

$$= \frac{1}{32}\left((2x^2-3)^8 + c\right)$$
$$= \frac{2x^2-3^8}{32} + c$$

1 mark for answer

c)

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$f(x)$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{1}{4}$
Weight	1	4	2	4	1

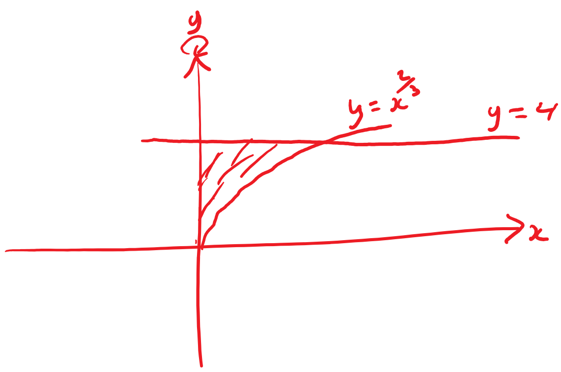
$h = \frac{1}{2}$

1 mark for $h = \frac{1}{2}$ or $b - a = \frac{1}{2}$

$$\int_0^2 \frac{dx}{x+2} \approx \frac{\frac{1}{2}}{3}\left(\frac{1}{2} + 4\times\left(\frac{2}{5}+\frac{2}{7}\right) + 2\times\frac{1}{3} + \frac{1}{4}\right)$$
$$\approx \frac{1747}{2520} \text{ or } 0.693 \text{ to 3 d.p.}$$

1 mark for answer

d)



$$V = \pi\int x^2\,dy$$

For $y = x^{\frac{2}{3}}$

$y^3 = x^2$

1 mark for $x^2 = y^3$

1 mark for boundaries

$V = \pi \int_0^4 y^3 \, dy$

$= \pi \left[\frac{y^4}{4} \right]_0^4$

$= \pi \left((4^3) - (0) \right)$

$= 64\pi$ units^3 1 mark for answer

Question Three

$f(x) = x(12 - x^2)$

$= 12x - x^3$

Increasing $\Rightarrow f'(x) > 0$ 1 mark for $f'(x) > 0$

$f'(x) = 12 - 3x^2$ 1 mark for derivative

$12 - 3x^2 > 0$

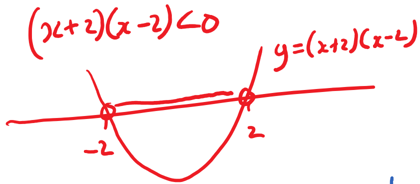
$12 > 3x^2$

$4 > x^2$

$x^2 < 4$

$x^2 - 4 < 0$

$(x+2)(x-2) < 0$



$\therefore -2 < x < 2$ 1 mark for answer

ie. $f(x)$ is increasing when $-2 < x < 2$

b) $f(x) = Ax^3 + Bx^2 + Cx + 1$

max TP at $(-1, 3) \Rightarrow f'(x) = 0,\ f''(x) < 0$

Point of inflection at $\left(-\frac{1}{3}, \frac{49}{27}\right) \Rightarrow f''(x) = 0$

Now $f'(x) = 3Ax^2 + 2Bx + C$

Now $f'(x) = 3Ax^2 + 2Bx + C$

and

$f''(x) = 6Ax + 2B$

When $x = -\frac{1}{3}$ $f''(x) = 0$

$f''(-\frac{1}{3}) = 6A(-\frac{1}{3}) + 2B = 0$ | mark for substituting $x = -\frac{1}{3}$ in $f''(x) = 0$

$-2A + 2B = 0$

$2B = 2A$

$\boxed{B = A}$ ①

When $x = -1$, $f'(x) = 0$ | mark for substituting $x = -1$ in $f'(x) = 0$

$f'(x) = 3Ax^2 + 2Bx + C$ sub ①

$= 3Ax^2 + 2Ax + C$

$f'(-1) = 3A(-1)^2 + 2A(-1) + C = 0$

$3A - 2A + C = 0$

$A + C = 0$

$\boxed{C = -A}$ ②

Subbing ① and ②

$f(x) = Ax^3 + Ax^2 - Ax + 1$

$f(-1) = 3$

$= A(-1)^3 + A(-1)^2 - A(-1) + 1$

$= -A + A + A + 1$

$= A + 1 = 3$

$A = 2$

$\therefore f(x) = 2x^3 + 2x^2 - 2x + 1$ | mark for answer

c) $y = \frac{x^2}{3 - x^2}$ $(1, \frac{1}{2})$

Let $u = x^2$ $u' = 2x$

$v = 3 - x^2$ $v' = -2x$

$\therefore y' = \frac{vu' - uv'}{v^2}$

$= \frac{(3 - x^2) \cdot 2x - x^2 \cdot (-2x)}{(3 - x^2)^2}$

$= 6x - 2x^3 + 2x^3$

$$(3-x^2)^2$$

$$= \frac{6x - 2x^3 + 2x^3}{(3-x^2)^2}$$

$$= \frac{6x}{(3-x^2)^2}$$ 1 mark for derivative

When $x=1$ $y' = \frac{6 \cdot 1}{(3-1^2)^2}$

$$= \frac{6}{4}$$

$$= \frac{3}{2}$$

$\therefore$ Gradient of tangent is $\frac{3}{2}$

and

gradient of normal is $-\frac{2}{3}$ 1 mark for gradient of normal

$$y - \frac{1}{2} = \frac{-2}{3}(x-1)$$

$$= \frac{-2x}{3} + \frac{2}{3}$$

$y = \frac{7}{6} - \frac{2x}{3}$ or $4x + 6y - 7 = 0$ 1 mark for answer

Question Four

ai) Points of inflection at $x = 2, -4$ (as concavity changes) 1 mark

ii) Stationary points $\Rightarrow f'(x) = 0$ at $x = 2, -7$ 1 mark for x-values

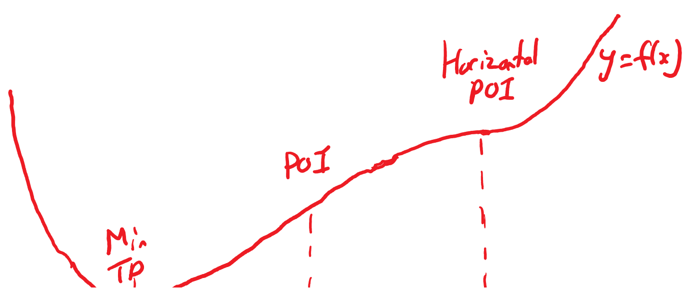
Also a point of inflection at $x=2$

$\therefore$ Horizontal point of inflection at $x=2$

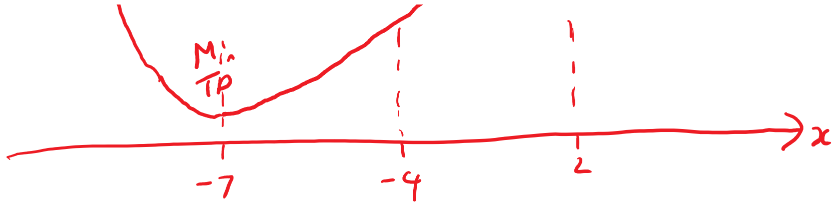
$f''(-7) > 0$ 1 mark for nature – Hor POI – Min TP

$\therefore$ Minimum turning point at $x = -7$

iii)



-1 mark for each incorrect/missing
* Shape
* Label of x-values
* Type of key point



* Label of x-values
* Type of key point

b) i) $r^2 = 20^2 - \left(\frac{h}{2}\right)^2$

$= 400 - \frac{h^2}{4}$

$r = \sqrt{400 - \frac{h^2}{4}}$ | 1 mark for answer

ii) $V = \pi r^2 h$

$= \pi\left(400 - \frac{h^2}{4}\right)h$ | 1 mark for substitution

$= 400\pi h - \frac{\pi h^3}{4}$ as required | 1 mark for answer

iii) $\frac{dV}{dh} = 400\pi - \frac{3\pi h^2}{4}$ | 1 mark for derivative

$\frac{dV}{dh} = 0$ when $400\pi - \frac{3\pi h^2}{4} = 0$

$\frac{3\pi h^2}{4} = 400\pi$

$h^2 = 400 \times \frac{4}{3}$

$= \frac{1600}{3}$

$h = \pm\sqrt{\frac{1600}{3}}$

$= \pm\frac{40}{\sqrt{3}}$

$= \pm\frac{40\sqrt{3}}{3}$

Ignore the negative case as $h > 0$

$h = \frac{40\sqrt{3}}{3}$ | 1 mark for height

d^2V

$\frac{}{4}\pi h$

$\therefore \frac{d^2V}{} < 0$ for any $h > 0$

dh^2

$\therefore$ Max turning point at $h = \frac{40\sqrt{3}}{3}$

1 mark for finding nature of stationary point

i.e. Maximum volume when height is $\frac{40\sqrt{3}}{3}$ m