Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

2021 **YEAR 12**

Mathematics Advanced

Task 3

Date: 24/05/2021

Q	Marks
MC	/5
6	/4
7	/2
8	/2
9	/2
10	/3
11	/4
12	/3
13	/4
14	/4
15	/7
16	/4
Total	/44

General Instructions:

- Reading time – 5 minutes
- Working time – 55 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks:
44

Section I - 5 marks

- Allow about 5 minutes for this section

Section II - 39 marks

- Allow about 50 minutes for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 25% of the course.

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Section I

5 marks
Allow about 5 minutes for this section

Use the multiple-choice sheet for Question 1–5

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1 Given the function $f(x) = -(x-3)^2 - 10$. Write the equation for $-f(-x)$.

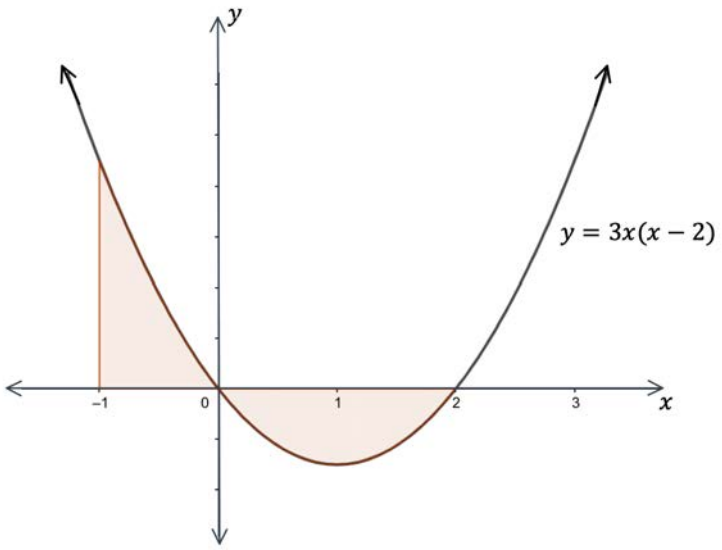
(A) $(-x+3)^2 + 10$

(B) $(x+3)^2 + 10$

(C) $(x+3)^2 - 10$

(D) $-(x+3)^2 + 10$

2 The area of the region shaded on the graph $y = 3x(x-2)$ is:



(A) 0 square units

(B) $\frac{8}{3}$ square units

(C) 8 square units

(D) 18 square units

Examination continues on next page

3 The period of the function $y = 4\sin\left(\frac{x}{3}\right)$ is:

(A) 6π

(B) 4

(C) $\frac{2\pi}{3}$

(D) $\frac{1}{4}$

4 The function $y = \ln x$ is transformed by being dilated vertically by a scale factor of $\frac{1}{3}$ and then translated horizontally 4 units to the left.
What is the transformed function?

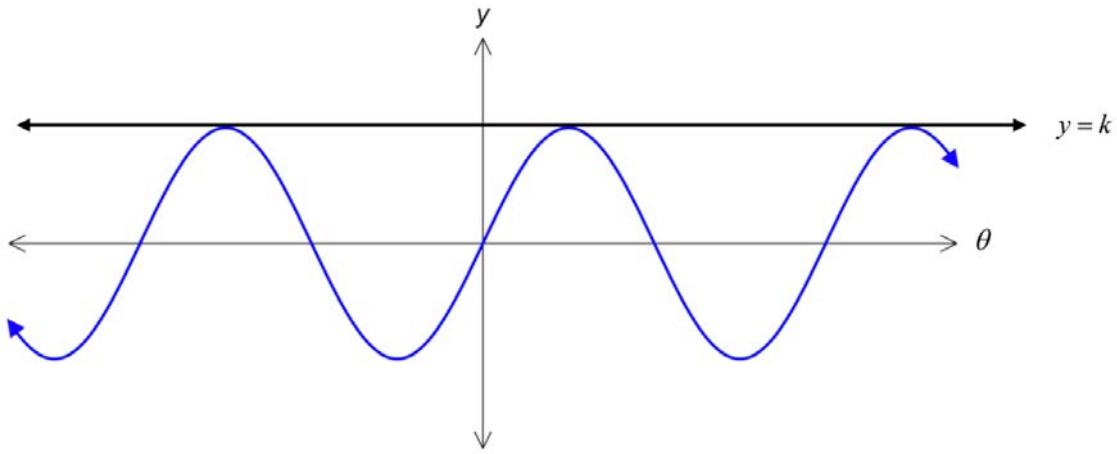
(A) $\frac{1}{3}\ln(x-4)$

(B) $\frac{1}{3}\ln(x+4)$

(C) $3\ln(x-4)$

(D) $3\ln(x+4)$

5 The curve $y = 2\sin 2\theta$ and the horizontal line $y = k$ are drawn below.



The solutions of the equation $2\sin 2\theta - k = 0$ as shown on this diagram are:

(A) $\theta = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}$

(B) $\theta = -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

(C) $\theta = -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}$

(D) $\theta = -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$

End of Section I

Examination continues on next page

Section II

In Questions 6 – 17, your response should include relevant mathematical reasoning and/or calculations.

Question 6 (4 marks)

Find:

(a) $\int (x^3 - 4x + 5)\ dx$ **2**

(b) $\int \frac{x^4 + 1}{x^2}\ dx$ **2**

Question 7 (2 marks)

Find: **2**

$$\int \left(\frac{x}{1-x^2}\right)\ dx$$

Question 8 (2 marks)

Evaluate: **2**

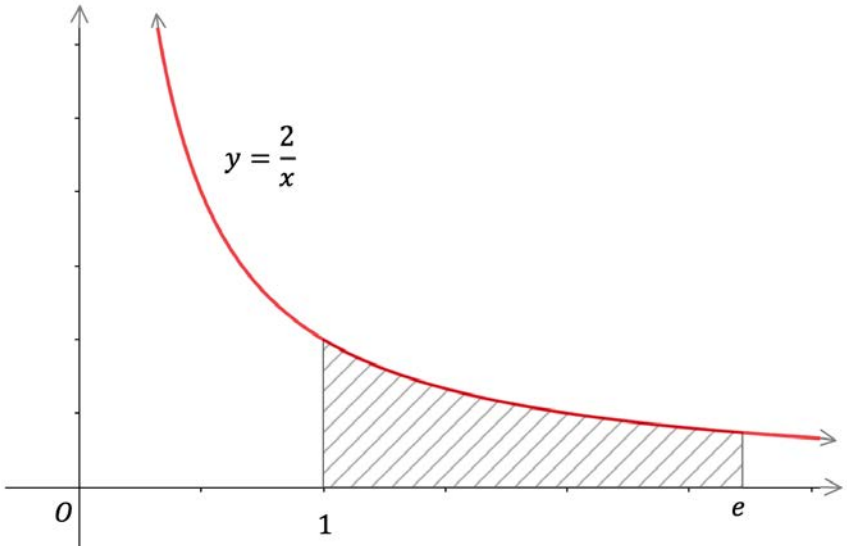
$$\int_0^{\frac{\pi}{6}} (1-\sec^2 2x)\ dx$$

Examination continues on next page

Question 9 (2 marks)

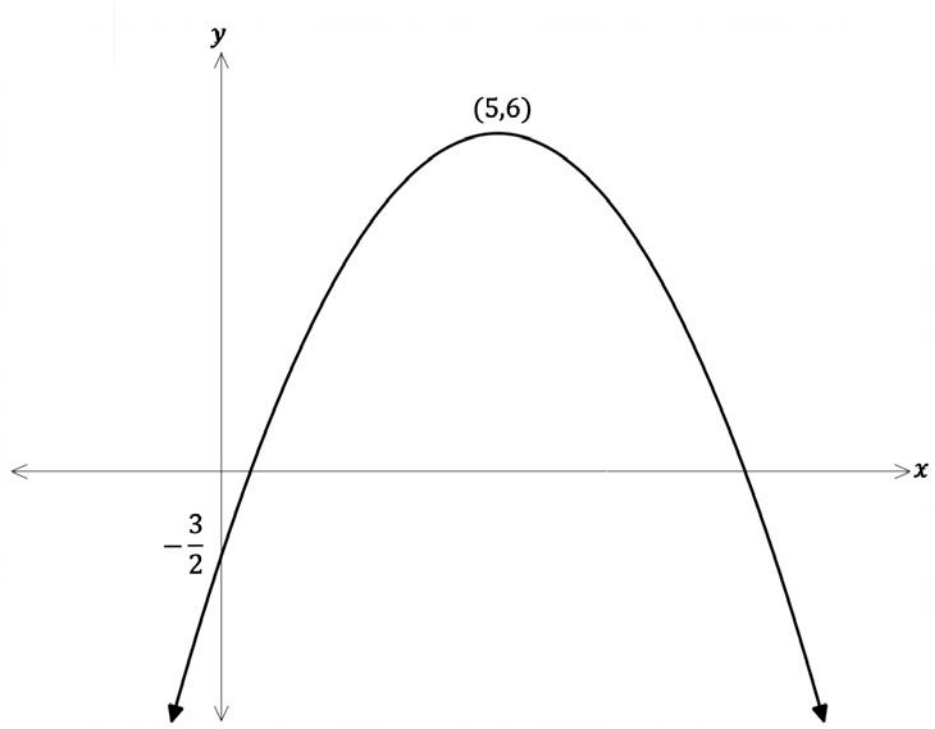
The diagram shows the area under the curve $y = \frac{2}{x}$ from $x = 1$ to $x = e$. **2**

Find the exact value of the shaded area.



Question 10 (3 marks)

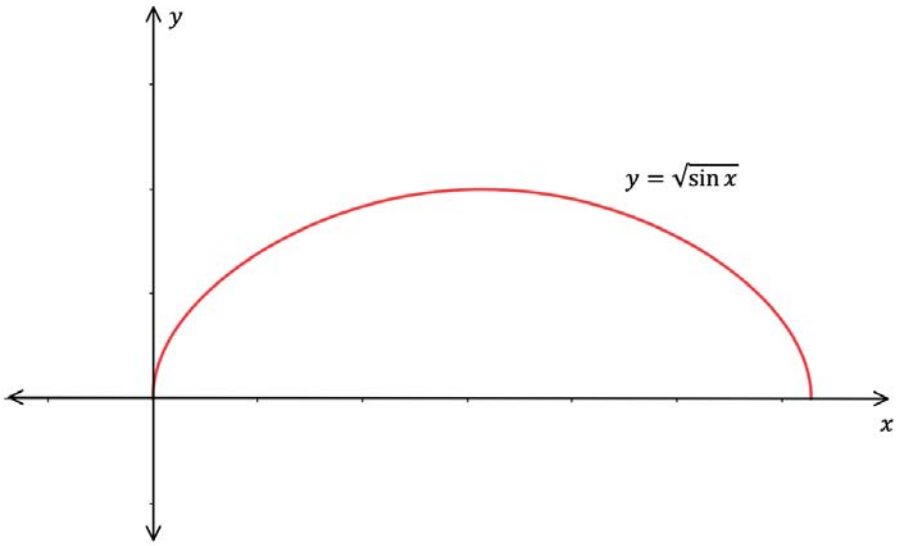
The function $f(x) = x^2$ has been transformed into a new function whose graph is shown below.



Find the equation of the new function in the form $g(x) = kf(x + b) + c$, where k, b, c are constants. **3**

Question 11 (4 marks)

The graph of the function $y = \sqrt{\sin x}$ is shown below. You may use this diagram to answer the following questions.



(a) Complete the table of values for the function $y = \sqrt{\sin x}$ in the table given below. (Round your answers to four decimal places) **1**

x	1	1.5	2
y			

(b) Use Trapezoidal rule, with three function values, to estimate the value of **2**

$$\int_1^2 \sqrt{\sin x}\, dx\ .$$

(Use the rounded values in (a)and give your answer correct to four decimal places)

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Question 11 (continued)

(c) Using the graph of $y = \sqrt{\sin x}$ and without further calculation, explain how the exact value of $\int_1^2 \sqrt{\sin x}\, dx$ compare with your result in (b). **1**

(You need to justify your answer giving reasoning and accurate mathematical terminology)

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Examination continues on next page

Question 12 (3 marks)

A circular metal plate of area A cm^2 is being heated. It is given that the area of the plate increases at the rate of **3**

$$\frac{dA}{dt} = \frac{\pi t}{32} \text{ cm}^2/h,$$

where t is the time in hours. If initially the plate had a radius of 6 cm, what is the exact area of the plate after 8 hours of heating?

Question 13 (4 marks)

The velocity of an object is given by the equation $v = 6t - 8 - t^2$ where the time (t) is in seconds and the velocity (v) is in m/s.

Initially, the object is 5m right of the origin.

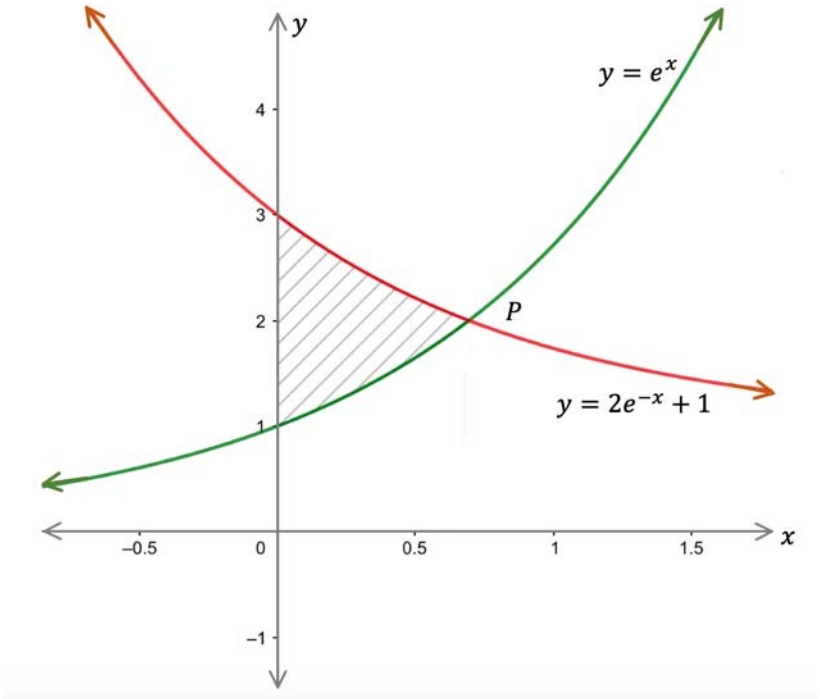
(a) When is the object at rest? **1**

(b) Find the distance travelled by the object in the first four seconds. **3**

Examination continues on next page

Question 14 (4 marks)

The curves $y = e^x$ and $y = 2e^{-x} + 1$ are shown below.



(a) By substitution, or otherwise, show that the x −value of P is $x = \ln 2$. **1**

Question 14 (continued)

(b) Evaluate $\int_0^{ln2} (2e^{-x} + 1)dx$, and hence show that the shaded area in the diagram is $\ln 2$ square units. **3**

Examination continues on next page

Question 15 (7 marks)

A tsunami is an ocean wave caused by an earthquake. The water first goes down from its normal level and then rises to an equal distance above the sea level.

A tsunami hits Morada harbour, where the water level normally has a depth of 10 meters. The meteorology station recorded that from the moment the sea started to go down, the tsunami wave hit a minimum depth (trough) after 5 minutes and rose to a maximum (crest) height of 22 metres after 15 minutes.

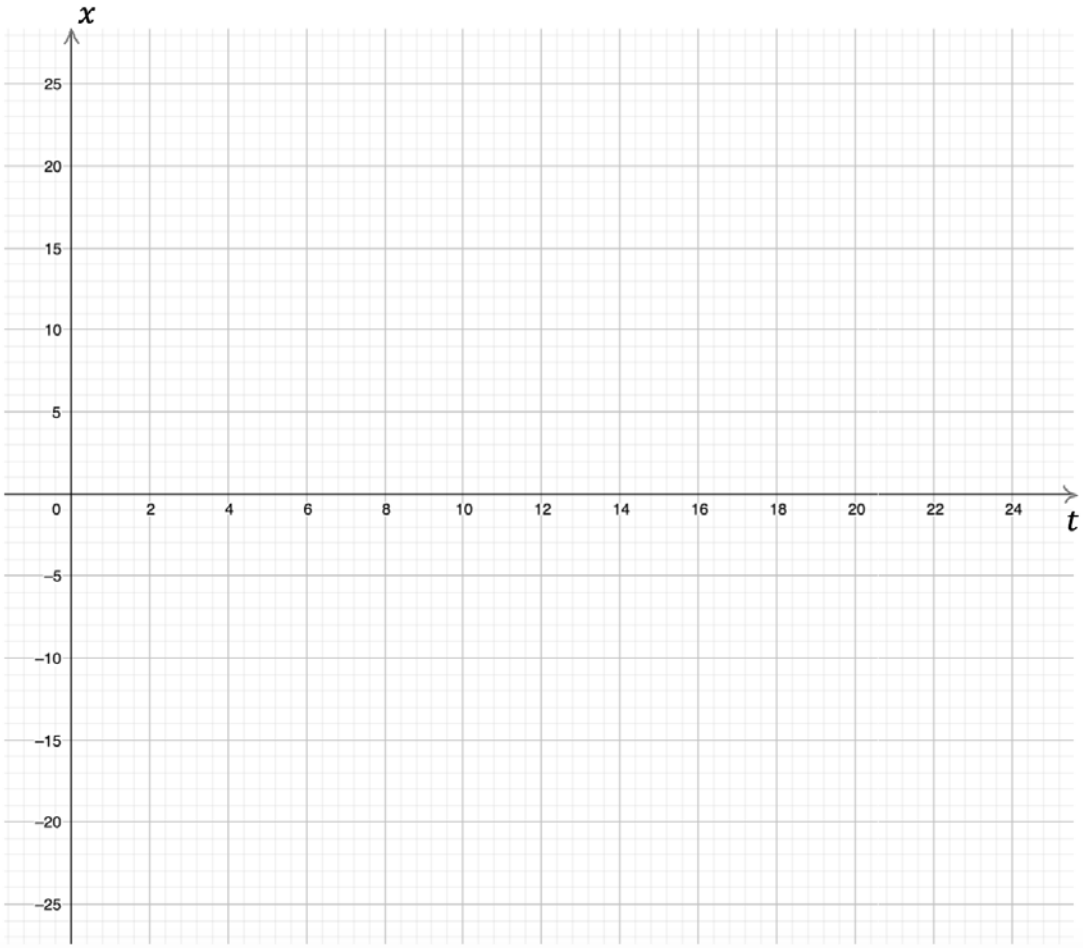
We may assume that the depth of water can be modelled as a sinusoidal function over time.

(a) Show that the equation of the tsunami wave can be modelled as **2**

$$x = 10 - 12 \sin \frac{\pi}{10} t.$$

Question 15 (continued)

(b) Sketch one cycle of the curve in (a) on the set of the axes given below: **3**

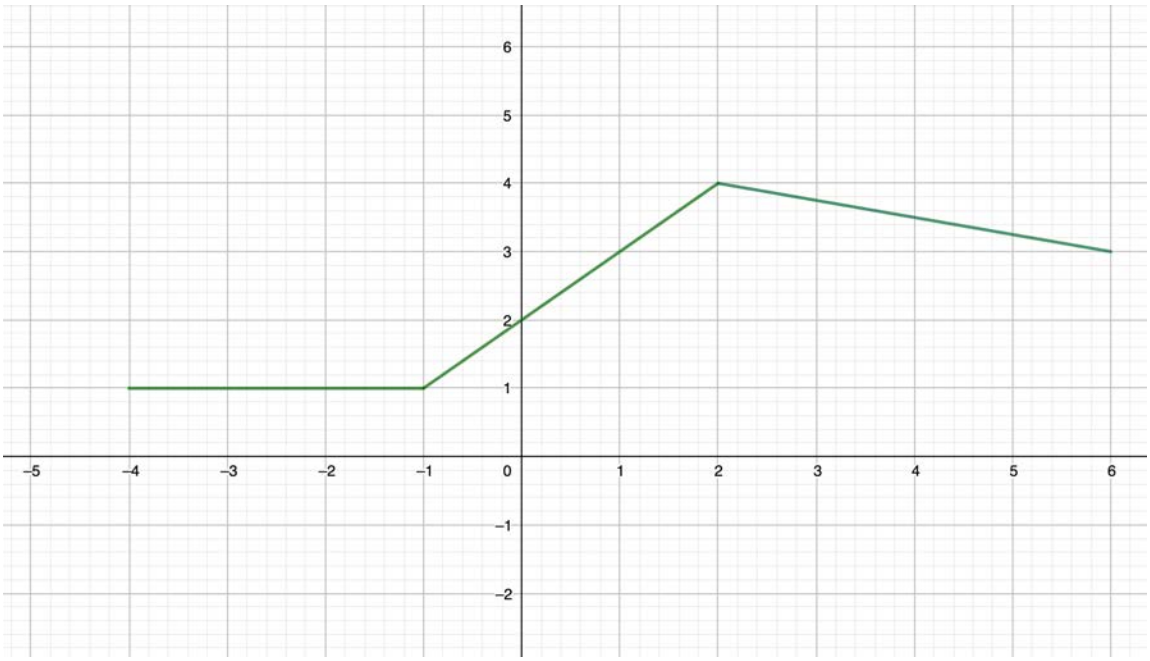


(c) After how many minutes will the wave hit the base of the observation tower if it is on the edge of a 20 metre cliff at the mouth of the harbour? **2**

Examination continues on next page

Question 16 (4 marks)

The graph of $y = f(x)$ is given below.



On the same axes given above, sketch the graph of $g(x) = 2f\big(2(x-1)\big) - 2$ **4**

(Use the space below for any calculations you may have)

End of Examination

Extra writing space

If you use this space, clearly indicate which question you are answering.

Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number								
Student Number								

HSC Mathematics Advanced Task 3

Section I – Multiple Choice Answer Sheet
Use this multiple-choice answer sheet for questions 1 – 5. Detach this sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your A ● B ✕ C ○ D ○ mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

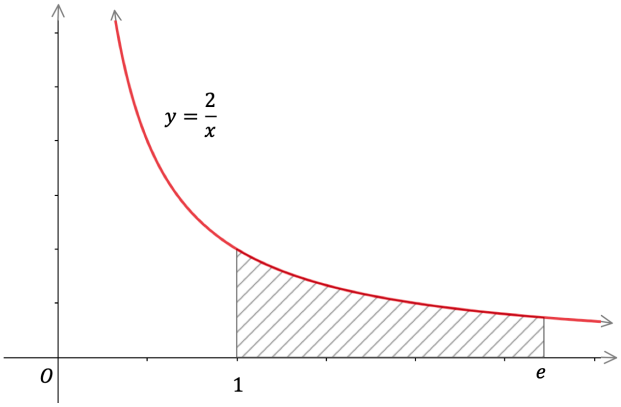
1.	A○	B○	C○	D○
2.	A○	B○	C○	D○
3.	A○	B○	C○	D○
4.	A○	B○	C○	D○
5.	A○	B○	C○	D○

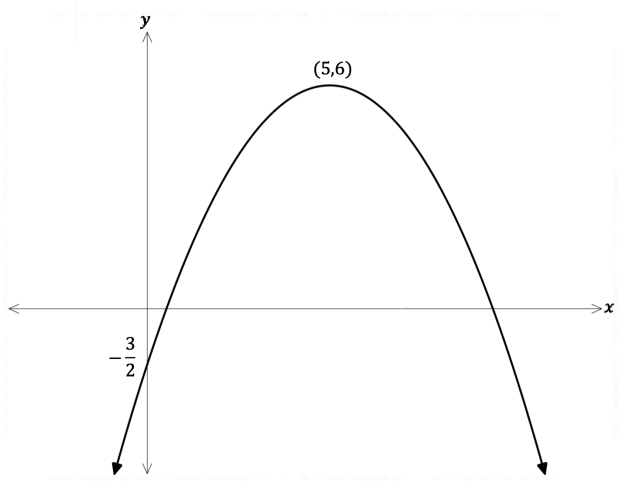
2021 Advanced Task 3 Marking scheme

1.	$f(x) = -(x-3)^2 - 10$ $f(-x) = -(-x-3)^2 - 10$ $f(-x) = -(x+3)^2 - 10$ $-f(-x) = (x+3)^2 + 10$ B		
2.	$x = -1,\ y = 3(-1)(-1-2) = 9$ Making approximations, the total area can be $1 \times 9 = 9$ So, the answer is C		
3.	$T = \frac{2\pi}{\frac{1}{3}} = 6\pi$ A		
4.	$y = \frac{1}{3}\ln(x+4)$ B		
5.	$sin2\theta = 1, \quad first\ solution\ is \frac{\pi}{4}.$ The period of $y = sin2\theta$ is π Then solutions are $\frac{\pi}{4}, \frac{5\pi}{4}, \ldots\ldots$ D		
Question 6			
6a.	$\int (x^3 - 4x + 5)\ dx$ $= \frac{x^4}{4} - 2x^2 + 5x + C$	1 mark: correctly integrates 1 mark: Constant of integration given	Done well
6b.	$\int \frac{x^4+1}{x^2}\ dx$ $\int x^2 + x^{-2}\ dx$ $= \frac{x^3}{3} + \frac{x^{-1}}{-1} + C$ $= \frac{x^3}{3} - \frac{1}{x} + C$	2 marks: separates the terms and integrates correctly 1 mark: separates the terms, but minor error in integration (ignore +C)	A few students gave $\frac{x^3}{3} - \ln x + c$ as their answers instead of $\frac{x^3}{3} - \frac{1}{x} + c$

Question 7		
$\int \left(\frac{x}{1-x^2}\right)\ dx$ $= \int -\frac{1}{2}\left(\frac{-2x}{1-x^2}\right)\ dx$	1 mark: Adjusts the integrand to $\frac{f'(x)}{f(x)}$ form and gives	Small number of students did not adjust the integrand to $\frac{f'(x)}{f(x)}$ form and therefore their answers did not

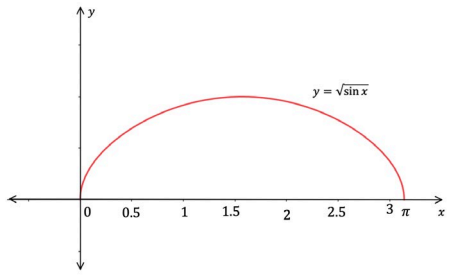
$= \frac{1}{2}\ln\|1-x^2\| + C$	$\frac{1}{2}\ln(1-x^2) + C$ 1 mark: correct integration, including the absolute sign	have a minus sign in front.
Question 8		
$\int_0^{\frac{\pi}{6}} (1-\sec^2 2x)\,dx$ $= \left[x - \frac{tan2x}{2}\right]_0^{\frac{\pi}{6}}$ $= \frac{\pi}{6} - \frac{\sqrt{3}}{2} - (0-0)$ $\frac{\pi - 3\sqrt{3}}{6}$	2 marks: correct integration and evaluation 1 mark: correct integration, minor error in evaluation	Small number of students gave $x - 2\tan 2x$ as their answers instead of $x - \frac{\tan 2x}{2}$ upon integration of $1-\sec^2 2x$

Question 9		
 Shaded area = $\int_1^e \frac{2}{x}\,dx$ $2[\ln(x)]_1^e = 2[\ln e - \ln 1]$ $= 2[1-0] = 2\ sq.units$	1 mark: correct expression for area and correctly integrates 1 mark: correct evaluation with area in sq. units	Done well

Question 10			
 Reflection on x-axis 5 units right 6 units up $g(x) = kf(x+b)+c$ $g(x) = k(x-5)^2+6$ Sub. $\left(0,-\frac{3}{2}\right)$, $-\frac{3}{2} = k \times 25 + 6$ $k = -\frac{3}{10}$ $g(x) = -\frac{3}{10}(x-5)^2+6$	1 mark for each of the values of a, b, k (be mindful of the error carried forward (Total 3 marks)	Most were fine with finding b and c from the vertex. Substituting to find k was poorly done. Careful if you change the function to be $g(x) = -k(x-5)^2+6$ that k will be $+\frac{3}{10}$

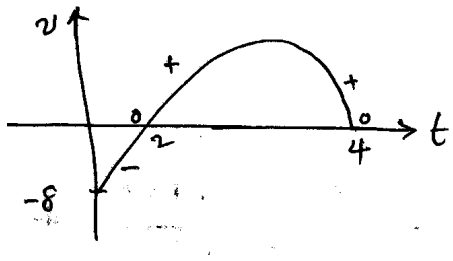
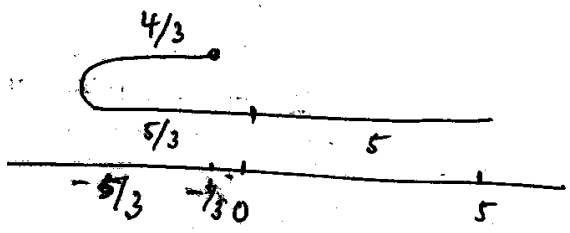
Question 11			
a.		1 mark: using radian measure to complete the table.	Poorly done. Use radians by default. Only use degrees if the question specifically tells you to,
b.	$\int_1^2 \sqrt{\sin x}\, dx$ $= \frac{0.5}{2}(0.9173 + 2 \times 0.9987 + 0.9536)$ $= 0.967075$ ≈ 0.9671	2 marks: correct answer from correct application of Trapezoidal rule 1 mark: correct application of trapezoidal rule (required)	Poorly done for a straightforward application of a formula. Careful if you are using the formula sheet that you know what a and b and n stand for. Otherwise, don't use the formula sheet and just use $A \approx \frac{h}{2}(first + last + 2(sum\ of\ the\ middle)$

x	1	1.5	2
y	0.9173	0.9987	0.9536

c.	 The area of the trapeziums is less than the exact area under the curve as the function is concave down. $\frac{d^2y}{dx^2} < 0$.	1 mark: Explains the result with appropriate diagram and reasoning. Must state the $y = \sqrt{\sin x}$ is concave down	You must state that the function is concave down. Some students do not have a conceptual understanding of how to draw trapeziums on this function.

Question 12			
$\frac{dA}{dt} = \frac{\pi t}{32}$ $A = \int \frac{\pi t}{32} dt$ $A = \frac{\pi t^2}{32 \times 2} + C \quad *$ when $t = 0, A = \pi \times 6^2 = 36\pi$ $\therefore 36\pi = 0 + C$ $\rightarrow C = 36\pi \quad **$ $\therefore A = \frac{\pi t^2}{64} + 36\pi$ When $t = 8$, $A = \frac{\pi(8)^2}{64} + 36\pi \quad **$ $\therefore A = 37\pi \text{ cm}^2$	3 marks: Correct solution with all the three processes clearly shown and gives the answer in sq.cm. 2 marks: Integrates to find the area expression. Steps 2 or 3 minor error 1 mark: Integrates to get the correct primitive function, and attempts to find C	Many did not take the time to understand the scenario of the question. If a plate of metal is heated it expands and the area increases. $A = \int \frac{dA}{dt} dt$ This is an indefinite integral, so it requires +C. For a circle: $A = \pi r^2$

Question 13			
a.	At rest, $\dot{x} = 0$ $6t - 8 - t^2 = 0$ $-(t^2 - 6t + 8) = 0$ $(4 - t)(t - 2) = 0$ $t = 2\ sec, \quad t = 4\ sec$	1 mark: sets $\dot{x} = 0$ and solves for t. Correctly gives $t = 2$, $t = 4$ with working	Fairly well answered

b.	 $v = 6t - 8 - t^2$ $\frac{dx}{dt} = 6t - 8 - t^2$ $x = \int 6t - 8 - t^2 \, dt$ $x = 3t^2 - 8t - \frac{t^3}{3} + C$ $t = 0, \ x = 5$ $5 = 0 - 0 - 0 + C \quad \Rightarrow C = 5$ $x = 3t^2 - 8t - \frac{t^3}{3} + 5$ $t = 0, \quad x = 5$ $t = 2, \ x = -\frac{5}{3}$ and $t = 4, \quad x = -\frac{1}{3}$ Distance travelled:  Distance = $5 + \frac{5}{3} + \frac{4}{3} = \ 8 \, m$	1 mark: evaluates the constant of integration, and correctly gives the expression for displacement 2 marks: correct answer from correct working 1 mark: calculates the displacements at various points or attempts to sketch the $x - t$ Function, and tries to calculate the distance travelled. Or Might try to integrate the v-t function, and attempts to find the distance travelled.	Many integrated between 0 and 4. Need to draw out the situation to know when the displacement is positive and negative.

Question 14			
a.	$y = e^x$ and $y = 2e^{-x} + 1$ At $x = ln2,\ y = e^{ln2} = 2$ At $x = ln2,\ y = 2e^{-ln2} + 1$ $= 2 \times \frac{1}{2} + 1 = 2$ $x = ln2$ satisfies both lines. Or $e^x = 2e^{-x} + 1$ $e^{2x} = 2 + e^x$ $e^{2x} - e^x - 2 = 0$ $(e^x + 1)(e^x - 2) = 0$ $e^x = -1 \quad (reject\ as\ e^x > 0$ $e^x = 2 \quad \therefore\ x = \ln 2$	1 mark: correct proof with full working. (It is a 'show that' question.	Very poorly answered. Many just wrote one side and did not show both sides as $y = 2$
b.	$\int_0^{ln2} (2e^{-x} + 1)dx$ $= [-2e^{-x} + x]_0^{ln2}$ $= -1 + \ln 2 + 2$ $= 1 + \ln 2$ $\int_0^{ln2} (2e^{-x} + 1 - e^x)dx$ $= \int_0^{ln2} (2e^{-x} + 1)dx - \int_0^{ln2} e^x dx$ $= 1 + \ln 2 - [e^x]_0^{ln2}$ from (b) $= 1 + \ln 2 - (2 - 1)$ $= \ln 2\ unit^2$	1 mark: correctly integrates 2 marks: correctly calculates the area 1 mark: Gives the expression, Area = $\int_0^{ln2} (2e^{-x} + 1 - e^x)dx$ and integrates correctly or Gives the expression, Area = $\int_0^{ln2} (2e^{-x} + 1 - e^x)dx$ and uses the result from (b).	Most achieved 1 mark but forgot to subtract area 2.

Question 15

a.

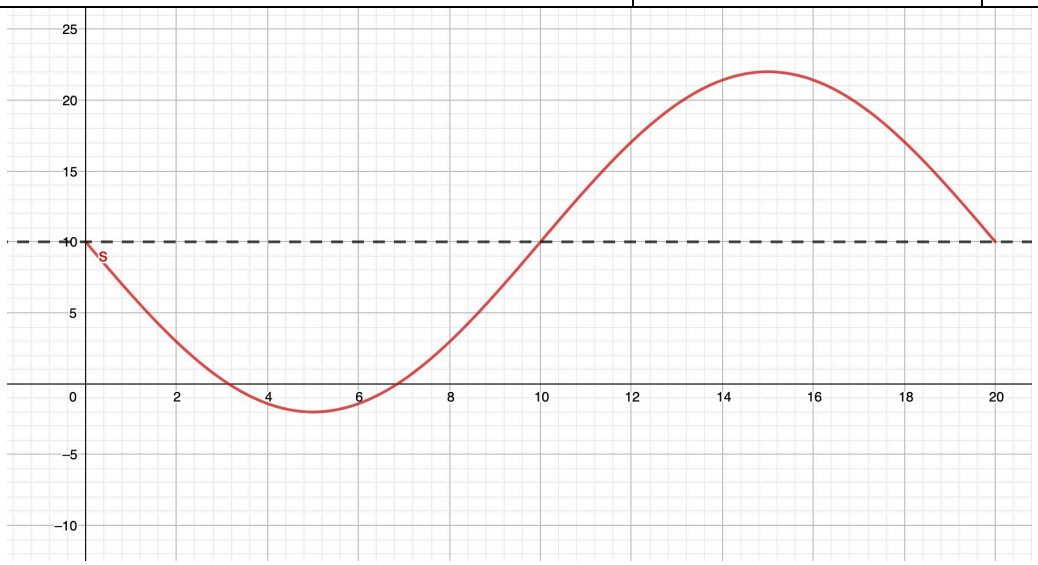
Amplitude = $22 - 10 = 12$,
Duration of a full cycle = $2 \times (15 - 5) = 20\ minutes$
Period, $T = \frac{2\pi}{20} = \frac{\pi}{10}$
At t = 0, $x = 10$ and minimum at t = 5.
Centre of motion $x = 10$
Hence, $x = 10 - 12\sin\frac{\pi}{10}t$

2 marks: calculates amplitude, period, centre of motion and explains why it is a negative sine curve

1 mark: correct amplitude, centre of motion and explains why it is a negative sine curve (Period calculation may be incorrect)

Note:
- Half period is the time between crest and trough
- Centre of motion is midway between the lowest/highest point

b.



1 mark: $y = -\sin x$ graph is drawn.
1 mark: Amplitude is 12 and period is 20 minutes
1 mark: centre of motion at $x = 10$ and minimum turning point $(5, -2)$ and maximum turning point $(15,22)$.

The question asked for 1 cycle
6 points -> 3 marks
4 points -> 2 marks
2 points -> 1 mark
- Centre
- a
- -sine curve
- (5,-2)
- (15,22)
- Period

c.

$x = 10 - 12\sin\frac{\pi}{10}t$

$20 = 10 - 12\sin\frac{\pi}{10}t$

$\sin\frac{\pi}{10}t = -\frac{10}{12} = -\frac{5}{6}$

$\frac{\pi}{10}t = \pi + \sin^{-1}\frac{5}{6} = 4.1267\ldots$

$t = 4.1267\ldots \times \frac{10}{\pi} = 13.1357$

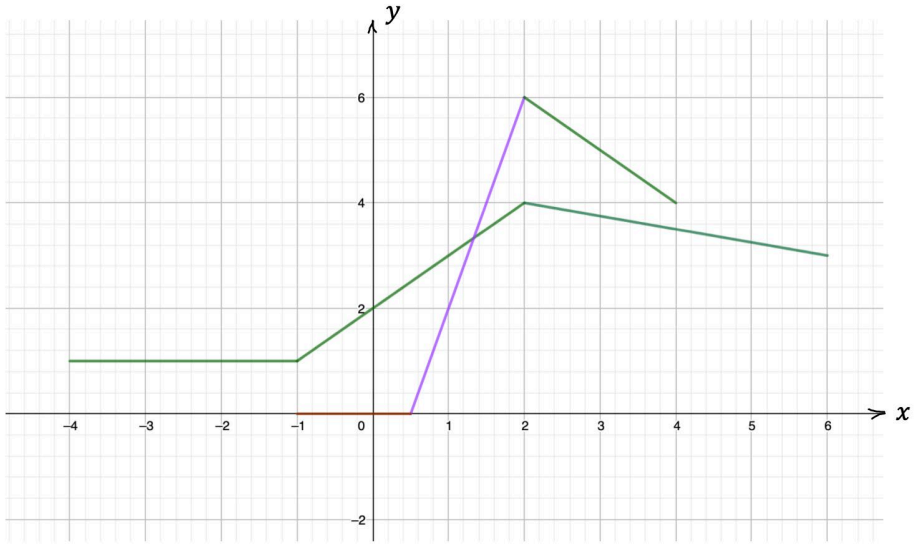
Approximately 13 minutes

2 marks: correctly solves the equation

1 mark: must give $\sin\frac{\pi}{10}t = -\frac{10}{12} = -\frac{5}{6}$
May have incorrect solution, but calculates the time correctly

You need to use radian mode.

Question 16



Halving the x- value was of concern to students

Many students had no working lines/point calculated. This created so much confusion.

$g(x) = 2f\big(2(x-1)\big) - 2$

4 marks: correctly follows the order and plots the correct points and hence the correct graph
Halves x, increases x by 1, double y, reduce y by 2.
3 marks: Follows the correct order of transformation and makes minor mistakes

2 mark: performs the dilations before translations, but gets the directions mixed up.

1 mark: At least one part of the graph is correct

Note: The easiest way of doing this question is to make a table of mapped points.

Points to check:

x	F(x)	F(2x)	F(2(x-1))	2f(2(x-1))	2f(2(x-1))-2
-4	(-4,1)	(-2,1)	(-1,1)	(-1,2)	(-1,0)
-1	(-1,1)	(-1/2,1)	(1/2,1)	(1/2,2)	(1/2,0)
2	(2,4)	(1,4)	(2,4)	(2,8)	(2,6)
6	(6,5)	(3,5)	(4,4)		