MATHEMATICS DEPARTMENT

Student Number

2023 **Year 12 Examination**

Mathematics Advanced

Task 3

Date Monday, 29/05/2023

Q	Marks
MC	/5
6	/8
7	/2
8	/3
9	/6
10	/3
11	/3
12	/5
13	/6
Total	/41

General Instructions

- Reading time – 55 minutes
- Working time – 5 minutes
- Write using blue or black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks: 41

Section I – 5 **marks**

- Allow about 7 minutes for this section

Section II - 36 **marks**

- Allow about 48 for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 25% of the course.

Section I

5 marks
Attempt Questions 1–5
Allow about 7 minutes for this section.

Use the multiple-choice sheet for Question 1–5.

---

1 $\int 3^x \, dx =$

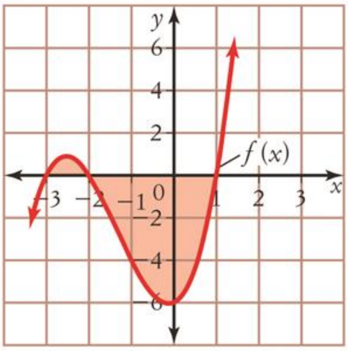
A. $\dfrac{3^x}{\ln 3} + c$

B. $\dfrac{3x}{\ln 3} + c$

C. $3^x \ln 3 + c$

D. $3x \ln 3 + c$

2 The correct definite integral that represents the shaded area of the function below is:



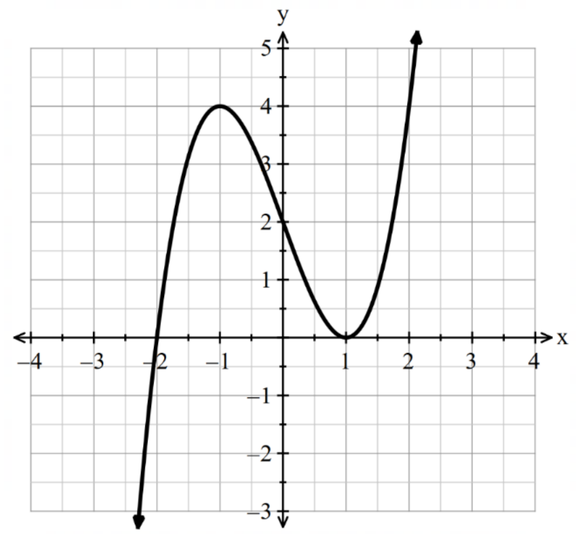
A. $\int_{-3}^{1} f(x) dx$

B. $-\int_{-3}^{1} f(x) dx$

C. $\int_{-3}^{-2} f(x) dx + \int_{-2}^{1} f(x) \, dx$

D. $\int_{-3}^{-2} f(x) dx - \int_{-2}^{1} f(x) dx$

3 Describe the concavity of the curve between $-1 \leq x \leq 1$.



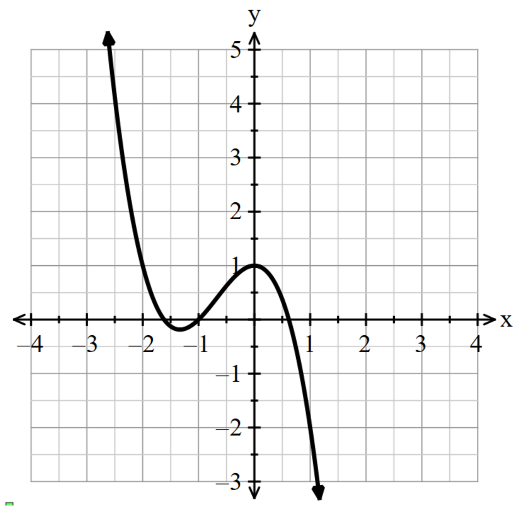
A. Concave up in [-1, 0) and concave up in (0,1]

B. Concave up in [-1, 0) and concave down in (0,1]

C. Concave down in [-1, 0) and concave up in (0,1]

D. Concave down in [-1, 0) and concave down in (0,1]

4 Determine the global maximum and minimum of the curve over the domain $-2\frac{1}{2} \leq x \leq 1$.



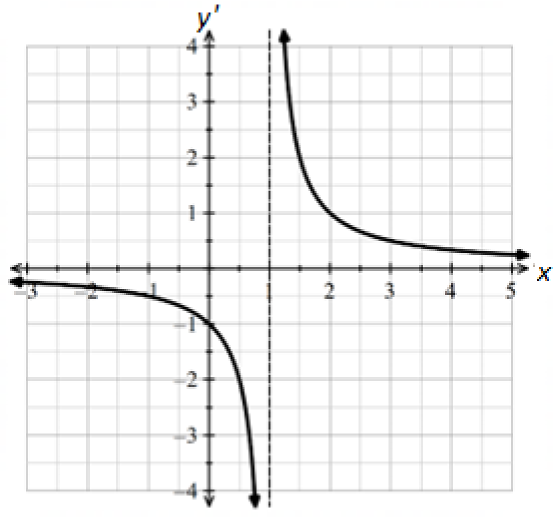
A. Global maximum = 1 and global minimum = -2

B. Global maximum = 4 and global minimum = -2

C. Global maximum = 1 and global minimum = -0.2

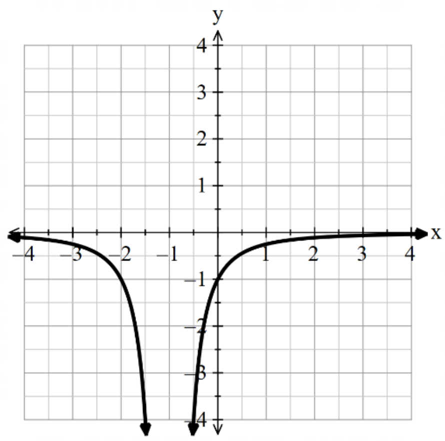
D. Global maximum = 4 and global minimum = -0.2

5 The graph given below is derivative function $f'(x)$.



The correct graph of $f(x)$ is

A.



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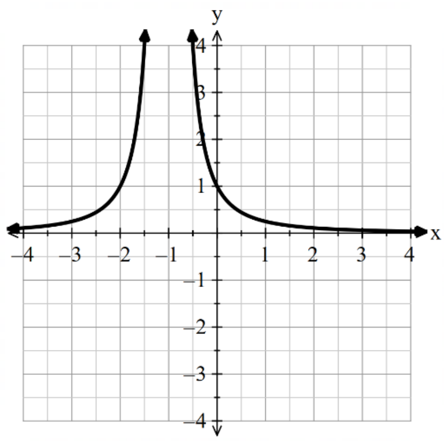

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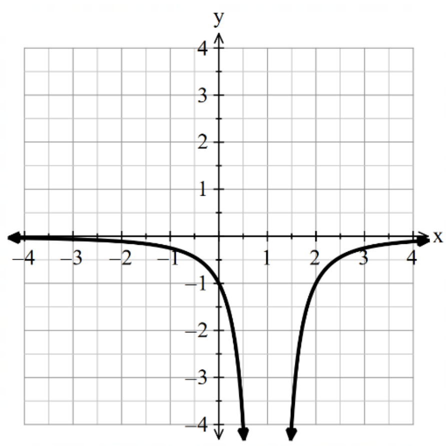

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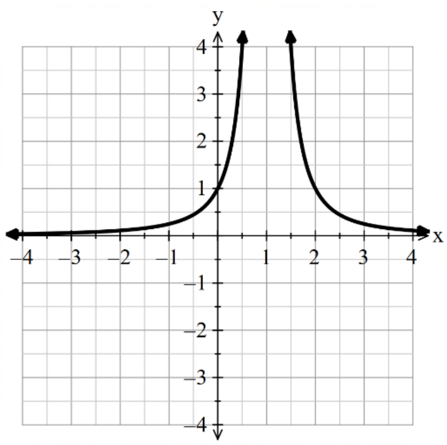


End of Section I

Section II

34 marks
Attempt Questions 6–14
Allow about 48 minutes for this section.

In Questions 6 – 14, your response should include relevant mathematical reasoning and/or calculations.

Question 6 (8 marks)

Find the following:

(a) $\int \frac{2}{x^2}\,dx$ **2**

(b) $\int \frac{3x^4 + 2x^3}{x^2}\,dx$ **2**

(c) $\int_0^3 \frac{4}{2x+3}\,dx$ **2**

(d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(x - \frac{\pi}{3}\right) dx$ **2**

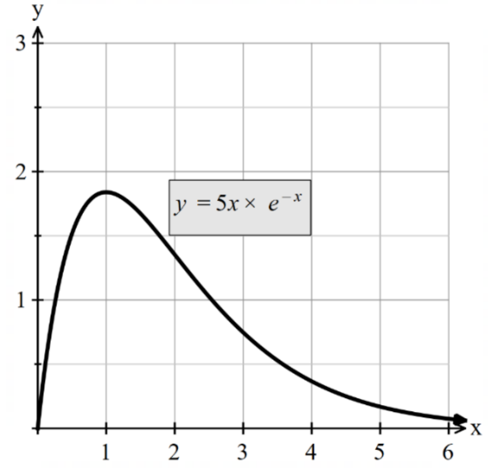
Question 7 (2 marks)

Evaluate **2**

$$\int_0^4 3x\sqrt{5x^2+1}\,dx$$

Question 8 (3 marks)

By using the graph below, approximate $\int_0^6 5xe^{-x}dx$ by using trapezoidal rule with 3 subintervals. Answer correct to one decimal place. **3**



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Question 9 (6 marks)

Consider the curve $y = \frac{1}{10}(x^3 - 9x^2 + 15x)$,

(a) Find the coordinates of any stationary points, and determine their nature. **2**

(b) Find the coordinates of any inflection points. **2**

Do NOT write in this area.

(c) Hence, sketch the curve, labelling stationary points, inflection points and the y-intercept. You do NOT need to find the other x-intercepts. **2**

Question 10 (3 marks)

For a curve $y = f(x)$, it is known that $f''(x) = 6x - 6$ and $f(x)$ has a local minimum at $(3, -16)$. Find the equation of $f(x)$. **3**

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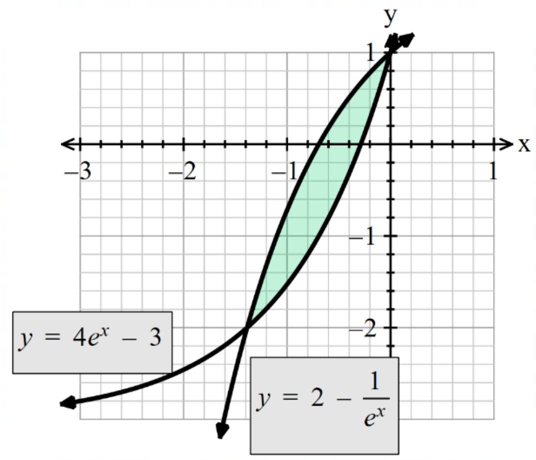
Question 11 (3 marks)

The functions $f(x) = 2 - \frac{1}{e^x}$ and $g(x) = 4e^x - 3$ intersect at two points as shown.



(a) Show that the points of intersection occur at $x = 0$ and $x = -2\ln 2$. **1**

(b) Hence find the area enclosed by $f(x)$ and $g(x)$. Answer correct to three decimal places. **2**

Question 12 (5 marks)

The velocity of a particle travelling along the $x-$ axis is given by $v = 2 - \frac{10}{t+2} ms^{-1}$, where t is time in seconds.

(a) When is the particle stationary? **1**

(b) What happens to the velocity as $t \to \infty$? **1**

(c) Find the distance travelled in the first 5 seconds, correct to 2 decimal places. **3**

Do NOT write in this area.

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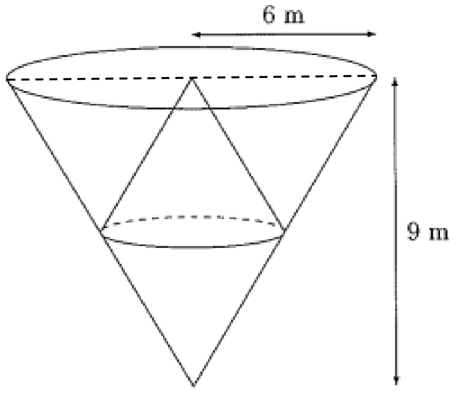
Question 13 (6 marks)

An inverted right circular cone holds a smaller right circular cone inside it.
The small cone touches the large cone at the circle around its base and the apex point as shown below.



The radius of the bigger cone is 6 m and the height is 9 m. The radius of the smaller cone is r m and the height is h m.

(a) Show that $h = 9 - \frac{3}{2}r$. **1**

(b) Show that the volume of the smaller cone is $V = \frac{\pi}{2}(6r^2 - r^3)$. **1**

(c) Find the maximum volume of the small cone, showing why this is the maximum volume. **4**

Do NOT write in this area.

End of paper

Section II extra writing space

If you use this space, clearly indicate which question you are answering.

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Solutions and Marking Scheme

Killara HIGH SCHOOL
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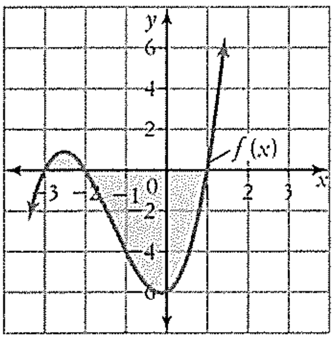

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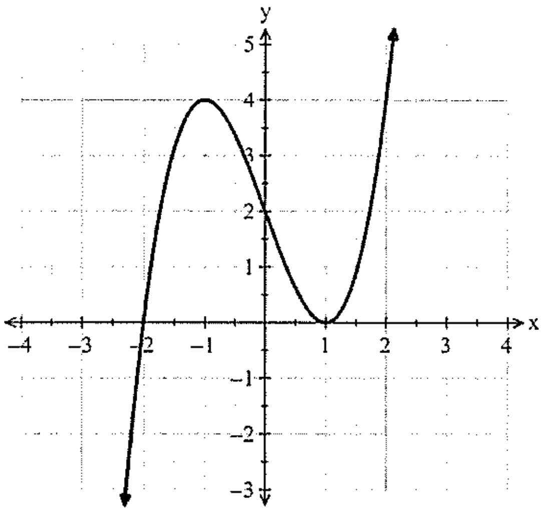

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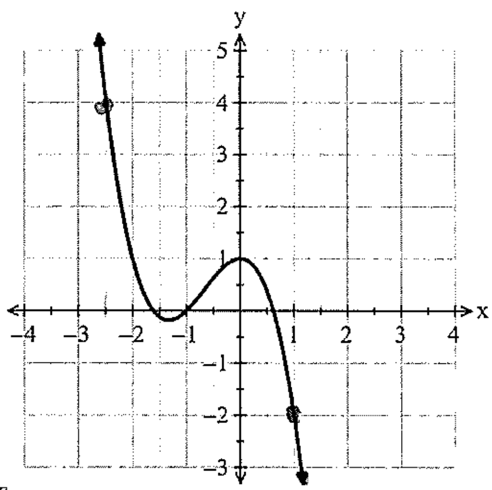

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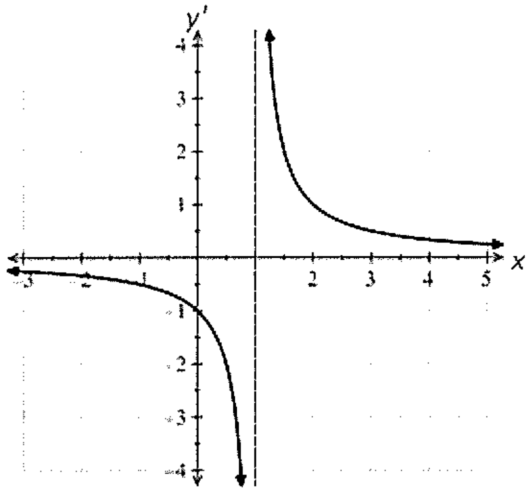

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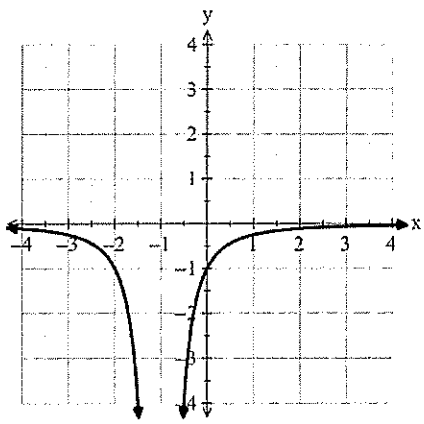

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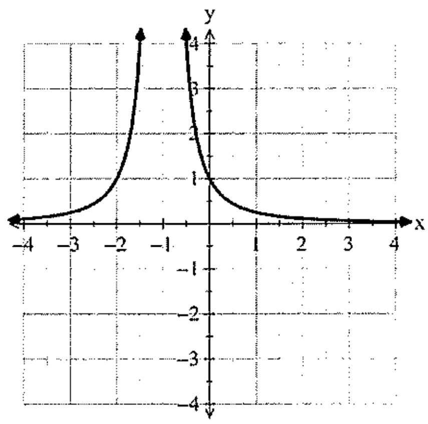

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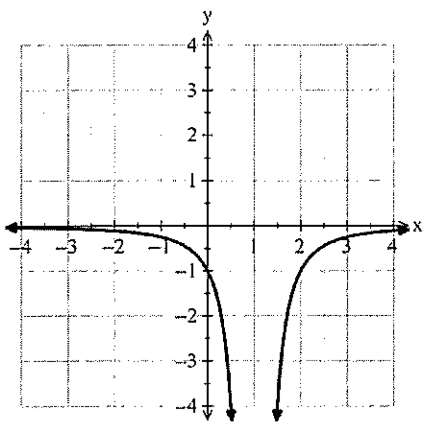

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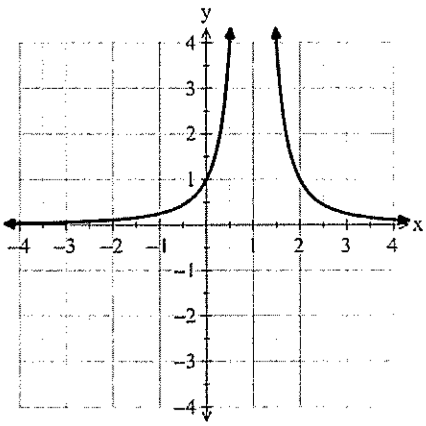

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Question 6 (8 marks)

Find the following:

(a) $\int \frac{2}{x^2}\,dx$ $= \int 2x^{-2}\,dx$ ← 1 mark for index form **2**

$= \frac{-2}{x} + C$ ← 1 mark

Penalise for not having + C once throughout the paper

(b) $\int \frac{3x^4 + 2x^3}{x^2}\,dx$ **2**

$= \int 3x^2 + 2x\,dx$ ← 1 mark for $3x^2 + 2x$

$= x^3 + x^2 + C$ ← 1 mark

(c) $\int_0^3 \frac{4}{2x+3}\,dx$ **2**

1 mark for antiderivative from correct working

$= \left[2\ln(2x+3)\right]_{x=0}^{x=3}$

$= 2\ln 3$ or $\ln 9$

1 mark

(d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(x - \frac{\pi}{3}\right) dx$ **2**

1 mark for antiderivative

$= \left[\cos\left(x - \frac{\pi}{3}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{6}}$ OR $\left[-\cos\left(x - \frac{\pi}{3}\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{\sqrt{3}}{2} - 1$ $\qquad = \frac{\sqrt{3}}{2} - 1$

1 mark for final correct Answer

Question 7 (2 marks)

Evaluate **2**

$$\int_0^4 3x\sqrt{5x^2+1}\,dx$$

$= \frac{3}{10} \int_0^4 \sqrt{5x^2+1} \cdot 10x \, dx$

$= \frac{3}{10} \int_0^4 (5x^2+1)^{\frac{1}{2}} \, d\,5x^2+1$

$= \frac{3}{10} \cdot \frac{2}{3} (5x^2+1)^{\frac{3}{2}} \Big|_{x=0}^{x=4}$

$= \frac{1}{5} \times 728$

$= \frac{728}{5}$

1 mark for ~~introducing~~ $10x \cdot \frac{3}{10}\,dx$.

or sth equivalent. say $f(x) = 5x^2+1$

This can be the correct use

of $\int f'(x)\,[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C$ formula.

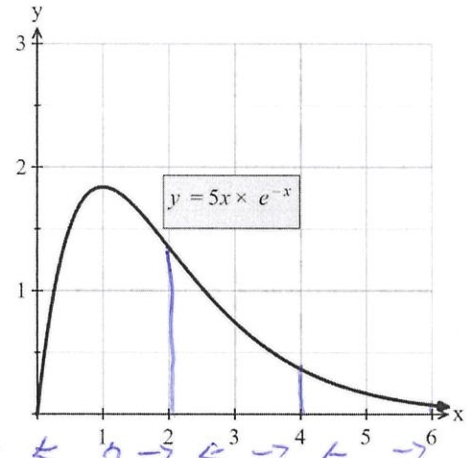
1 mark for correct answer.

Do NOT write in this area.

Do NOT write in this area.

Question 8 (3 marks)

By using the graph below, approximate $\int_0^6 5xe^{-x}dx$ by using trapezoidal rule with 3 subintervals. Answer correct to one decimal place. **3**



$\frac{6-0}{3} = 2$ subinterval ~~size~~

Size ← 2 → ← 2 → ← 2 →

Height of each trapezium ← 1 mark

$\frac{1}{2}[f(0) + 2f(2) + 2f(4) + f(6)] \times 2$ ← 1 mark

$= f(0) + 2f(2) + 2f(4) + f(6)$

$= 0 + 20e^{-2} + 40e^{-4} + 30e^{-6}$

≈ 3.5 units^2 ← 1 mark

1 mark for finding the correct interval width $\frac{6-0}{3} = 2$

1 mark for correct use of the trapezoidal rule

1 mark for correct answer

Exact value $\int_0^6 5xe^{-x}dx = [-e^{-x} \cdot 5x]_0^6 + 5\int_0^6 e^{-x}dx$

$= -30e^{-6} + 5[e^{-x}]_6^0 = 5 - 35e^{-6}$ units^2

For your interest, not a part of the course.

Question 9 (6 marks)

Consider the curve $y = \frac{1}{10}(x^3 - 9x^2 + 15x)$,

(a) Find the coordinates of any stationary points, and determine their nature. **2**

$\frac{dy}{dx} = \frac{1}{10}(3x^2 - 18x + 15)$

$\frac{dy}{dx} = 0$

$x^2 - 6x + 5 = 0$

$x = 5$ or $x = 1$

$\therefore (1, \frac{7}{10})$ and $(5, -\frac{5}{2})$ are stationary points

$x = 1 \quad y = \frac{7}{10}$

$x = 5 \quad y = -\frac{5}{2}$

x	<1	1	$1<x<5$	5	$x>5$
$\frac{dy}{dx}$	>0	0	<0	0	>0

increasing decreasing increasing

$(1, \frac{7}{10})$ $(5, -\frac{5}{2})$

From the table $(1, \frac{7}{10})$ is a local maximum and $(5, -\frac{5}{2})$ is a local minimum

1 mark for finding $(1, \frac{7}{10})$ and $(5, -\frac{5}{2})$ from correct working

1 mark for determining the nature. The second derivative test is accepted as well.

(b) Find the coordinates of any inflection points. **2**

$\frac{d^2y}{dx^2} = \frac{1}{10}(6x - 18)$

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 3$

When $x = 3 \quad y = -\frac{9}{10}$

There is a sign change on either side of $x = 3$ from the 2nd derivative test.

$\therefore (3, -\frac{9}{10})$ is a point of inflection

x	$x<3$	$x=3$	$x>3$
$\frac{d^2y}{dx^2}$	<0	0	>0

concave down concave up

1 mark for finding $(3, -\frac{9}{10})$

1 mark for using the table to confirm it is a point of inflection

Do NOT write in this area.

(c) Hence, sketch the curve, labelling stationary points, inflection points and the y-intercept. You do NOT need to find the other x-intercepts. **2**

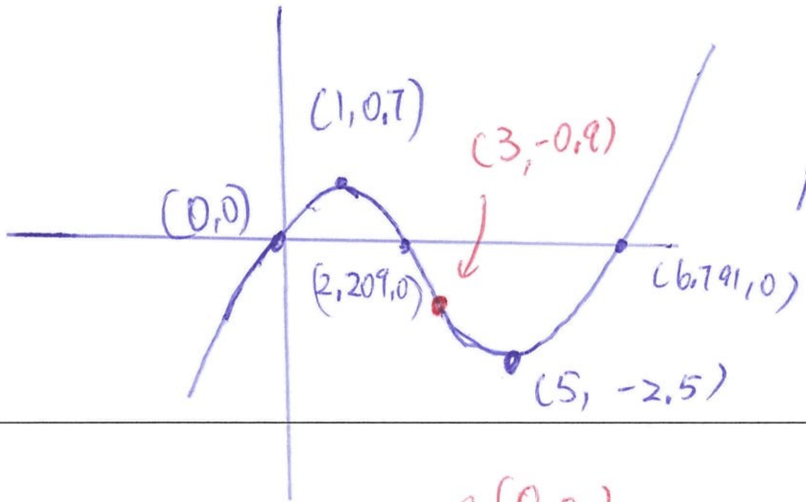
$y = \frac{1}{10}(x^3 - 9x^2 + 15x)$

y intercept $(0,0)$

$x(x^2 - 9x + 15) = 0$

$x = 0$ OR $x = \frac{9 \pm \sqrt{21}}{2}$

$(0,0)$ $\left(\frac{9 \pm \sqrt{21}}{2}, 0\right) \Rightarrow$ x intercepts (not required for this Question)



NOT TO SCALE

4 important points $\begin{cases} (0,0) \\ (1, 0.7) \\ (3, -0.9) \\ (5, -2.5) \end{cases}$

1 mark for correct shape and any two important points labelled.

2 marks for correct shape and all 4 important points labelled.

Please give student marks for errors carried forward

Question 10 (3 marks)

For a curve $y = f(x)$, it is known that $f''(x) = 6x - 6$ and $f(x)$ has a local minimum at $(3, -16)$. Find the equation of $f(x)$. **3**

$f'(x) = \int 6x - 6 \, dx$

$= 3x^2 - 6x + c$

$(3, -16)$ is a stationary point

$\therefore f'(3) = 0$

$27 - 18 + c = 0 \quad c = -9$

$f'(x) = 3x^2 - 6x - 9$

$f(x) = x^3 - 3x^2 - 9x + c$

$f(3) = -16$

$27 - 27 - 27 + c = -16$

$c = 11$

$\therefore f(x) = x^3 - 3x^2 - 9x + 11$

1 ~~Point~~ mark for finding $\int 6x - 6 \, dx = 3x^2 - 6x + c$ and recognising $(3, -16)$ is a stationary point by establishing $f'(3) = 0$

2 ~~poi~~ marks for finding $f'(x) = 3x^2 - 6x - 9$ and $f(x) = x^3 - 3x^2 - 9x + c$

3 marks for finding $f(x) = x^3 - 3x^2 - 9x + 11$

Do NOT write in this area.

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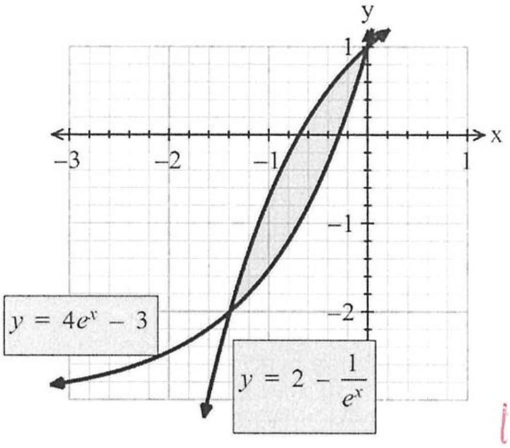
Question 11 (3 marks)

The functions $f(x) = 2 - \frac{1}{e^x}$ and $g(x) = 4e^x - 3$ intersect at two points as shown.



1 mark for any correct working

(a) Show that the points of intersection occur at $x = 0$ and $x = -2\ln 2$. **1**

$4e^x - 3 = 2 - e^{-x}$

$4e^{2x} - 5e^x + 1 = 0$

$(4e^x - 1)(e^x - 1) = 0$

$\rightarrow e^x = \frac{1}{4}$ or $e^x = 1$

$x = \ln\frac{1}{4}$ or $x = 0$

$x = -2\ln 2$

(b) Hence find the area enclosed by $f(x)$ and $g(x)$. Answer correct to three decimal places. **2**

$\int_{-2\ln 2}^{0} 2 - e^{-x} - 4e^x + 3 \, dx$

$= \left[5x + e^{-x} - 4e^x\right]_{-2\ln 2}^{0}$ OR $\left[4e^x - e^{-x} - 5x\right]_{0}^{-2\ln 2}$

$= 10\ln 2 - 6 \approx 0.931$

1 mark for correct antiderivative

2 marks for correct answer

Question 12 (5 marks)

The velocity of a particle travelling along the x – axis is given by $v = 2 - \frac{10}{t+2} ms^{-1}$, where t is time in seconds.

(a) When is the particle stationary? **1**

$V = 0$

$t = 3s$ $\quad 2 = \frac{10}{t+2} \Rightarrow t = 3s$

1 mark for correct answer from correct working

(b) What happens to the velocity as $t \rightarrow \infty$? **1**

$\lim_{t \to \infty} V = 2$ m/s

1 mark ~~from~~ for correct answer

(c) Find the distance travelled in the first 5 seconds, correct to 2 decimal places. **3**

$\int_0^3 \frac{10}{t+2} - 2 \, dt + \int_3^5 2 - \frac{10}{t+2} \, dt$

$= \left[10\ln(t+2) - 2t\right]_{t=0}^{t=3} + \left[10\ln(t+2) - 2t\right]_{t=5}^{t=3}$

$= (10\ln 5 - 6) - 10\ln 2 + (10\ln 5 - 6) - 10\ln 7 + 10$

$= 20\ln 5 - 10\ln 2 - 10\ln 7 - 2$

≈ 3.80 m

1 mark for correct estabishment of integration with the correct upper and lower bonds.

2 marks for correct antiderivative

3 marks for correct answer

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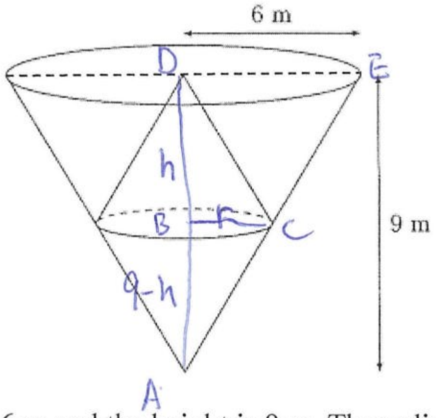
An inverted right circular cone holds a smaller right circular cone inside it.
The small cone touches the large cone at the circle around its base and the apex point as shown below.



The radius of the bigger cone is 6 m and the height is 9 m. The radius of the smaller cone is r m and the height is h m.

(a) Show that $h = 9 - \frac{3}{2}r$. **1**

$\Delta ABC \ ||| \ \Delta ADE$ $\quad \frac{9-h}{9} = \frac{r}{6}$ 1 mark for correct proof

$9 - h = \frac{9r}{6}$

Domain

$h = 9 - \frac{3r}{2}$ $\quad 0 < r < 6$

(b) Show that the volume of the smaller cone is $V = \frac{\pi}{2}(6r^2 - r^3)$. **1**

$V = \frac{1}{3}\pi r^2 (9 - \frac{3}{2}r)$ 1 mark for correct proof

$V = 3\pi r^2 - \frac{1}{2}\pi r^3$

$V = \frac{\pi}{2}(6r^2 - r^3)$

$0 < r < 6$

(c) Find the maximum volume of the small cone, showing why this is the maximum volume. **4**

$V = \frac{\pi}{2}(6r^2 - r^3) \quad 0 < r < 6$

$\frac{dV}{dr} = \frac{\pi}{2}(12r - 3r^2)$ 1

$\frac{dV}{dr} = 0 \quad 12r - 3r^2 = 0$

$3r(4 - r) = 0$

$r = 0 \quad r = 4m$

Reject 2

$\therefore r > 0$

$\frac{d^2V}{dt^2} = \frac{\pi}{2}(12 - 6r)$

when $r = 4m \quad \frac{d^2V}{dt} = \frac{\pi}{2}(12 - 24) = -6\pi < 0$

Concave down $\Rightarrow$ local maximum 3

Volume max $= \frac{\pi}{2}(6 \times 4^2 - 4^3) = 16\pi$ m^3

4

1 $\boxed{1 \text{ mark}}$ for 1st derivative

2 $\boxed{1 \text{ mark}}$ for finding $r=0$ (Rejected) & $r=4m$

3 $\boxed{1 \text{ mark}}$ for either the 2nd derivative test or a table to justify why it is a max at $r=4$ m

4 $\boxed{1 \text{ mark}}$ for final max volume

Do NOT write in this area.