NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED

2020 Year 12 Course Assessment Task 1

Monday, 2 December 2019

General instructions

- Working time – 1 hour (plus 5 minutes reading time).
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt all questions.
- All necessary working should be shown in every question.
- Marks may be deducted for illegible or incomplete working.
- A NESA Reference Sheet is provided.

NESA STUDENT #: ....

Class: (please ✓)

- ○ 12MAT.1 – Ms Ham
- ○ 12MAT.2 – Mr Lam
- ○ 12MAT.3 – Mr Sekaran
- ○ 12MAT.4 – Ms Nam
- ○ 12MAT.5 – Mrs Gan
- ○ 12MAT.6 – Ms Park

Marker's use only

QUESTION	**1**	**2**	**3**	**Total**	**%**
MARKS	$\overline{}$ 11	$\overline{}$ 18	$\overline{}$ 24	$\overline{}$ 53	

Question 1 (11 marks) **Marks**

(a) Differentiate with respect to x:

i. $y = (3 + e^x)^5$ **2**

....

....

....

....

ii. $y = \frac{e^{2x}}{2x+1}$ **2**

....

....

....

....

(b) Find the equation of the tangent to $y = e^{-x}$ at the point $(0,1)$. **2**

....

....

....

....

....

....

....

....

....

(c) Sketch the graph $y = 2e^{-2x} - 2$, showing all important features. **2**

....

....

....

....

(d) Let $f(x) = e^x + k$, where k is a real number. The tangent to the graph of $y = f(x)$ at the point where $x = a$ passes through the point $(0, 0)$.

i. Show that the equation of the tangent is $y = e^a x$. **1**

....

....

....

....

....

....

ii. Find the value of k in terms of a. **2**

....

....

....

....

....

....

....

....

Question 2 (18 marks) **Marks**

(a) Evaluate the following: **2**

$$\sum_{k=2}^{10} 4 \times \left(\frac{1}{2}\right)^{k-1}$$

....

....

....

....

....

....

....

....

....

(b) A geometric series has common ratio $\frac{1}{p}$ and limiting sum $\frac{1}{1-p}$. **2**

Find the first term of this series, a, in simplest form.

....

....

....

....

....

....

....

....

(c) $2m-8$, $2m+4$ and $5m-2$ are successive terms of a geometric sequence. Find the value of m. **3**

(d) i. Show that $\log_{10} 27 + \log_{10} 9 + \log_{10} 3 + \cdots$ is an arithmetic series. **1**

ii. Show that the sum of the first seven terms of this series is 0. **2**

(e) A pattern is made up of a series of concentric circles with centre O. The radius of the first (innermost) circle is 10 cm. The distance between two adjacent concentric circles is 2 cm.

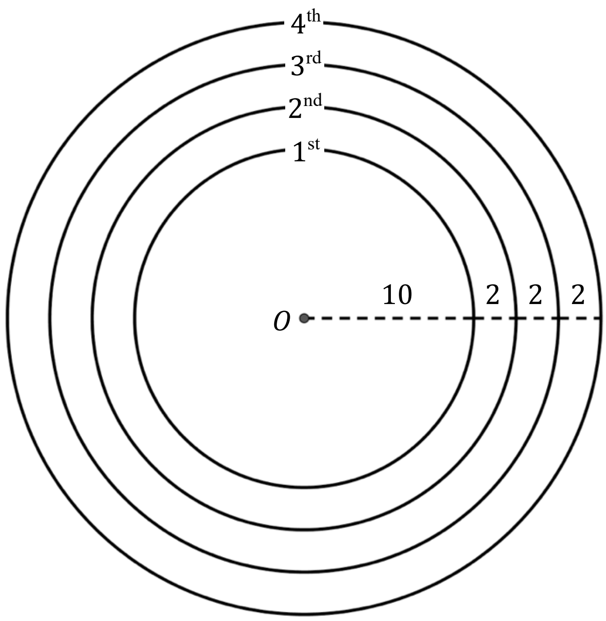


Diagram not drawn to scale

i. What is circumference of the first circle? **1**

....

....

ii. Show that the circumference, C, of the n^{th} circle is given by $C = 4\pi(4 + n)$. **2**

....

....

....

....

....

....

....

iii. Show that the total circumference of n circles is given by $L = 2\pi n(9 + n)$. **2**

iv. If $L = 1584\pi$, find the value of n and hence find the radius of the n^{th} circle. **3**

Question 3 (24 marks) **Marks**

(a) 8 members of a rowing team were on a training row.

During the training row 6 members of the team were sunburnt and 2 were sunburnt and bitten by sand-flies. 1 member had nothing happen to him.

i. Draw a Venn diagram showing this information. **1**

ii. Hence find the probability that he was sunburnt but not bitten by sand-flies if a student was selected at random from the rowing team. **1**

....

....

(b) The chance Janine can solve a probability problem is $\frac{9}{10}$ and the chance Naomi can solve it is $\frac{4}{5}$.
If they work on the problem independently, what is the probability that:

i. They both solve the problem? **2**

....

....

ii. At least 1 of them is able to solve the problem? **2**

....

....

....

....

(c) Four identical balls are numbered $1, 2, 3$ and 4 and put into a box.

A ball is randomly drawn from the box, and not returned to the box. A second ball is then randomly drawn from the box.

By drawing a diagram or otherwise, find the probability that:

i. The first ball drawn is numbered 4 and the second ball drawn is numbered 1. **1**

....

....

....

....

ii. The sum of the numbers on the two balls is 5. **1**

....

....

....

....

....

....

iii. The second ball drawn is numbered 1, given that the sum of the numbers on the two balls is 5. **2**

....

....

....

....

....

....

(d) A and B are events of a sample space. Given that $P(A|B) = p$, $P(B) = p^2$ and $P(A) = p^{\frac{1}{3}}$:

i. Show that $P(A \cap B) = p^3$. **1**

....

....

....

....

....

....

ii. Find $P(B|A)$ in simplest form. **1**

....

....

....

....

....

(e) From a packet of mixed seed, it was estimated that the probability of any seed planted yielding a white carnation was 0.02.

i. Calculate the probability that from any two seeds planted there will be at least 1 white carnation. **2**

....

....

....

....

ii. How many seeds must be planted to be at least 95% certain of obtaining at least 1 white carnation? **3**

....

....

....

....

....

....

....

....

(f) Reung has invented a game for one person. He throws 2 ordinary dice repeatedly until the sum of the two numbers shown is either a 7 or 9.

If the sum is 9, Reung wins. If the sum is 7, Reung loses. If the sum is any other number, he continues to throw until it is a 7 or 9.

i. Show that the probability that Reung wins on his first throw is $\frac{1}{9}$. **1**

....

....

....

....

....

....

....

ii. Calculate the probability that a second throw is needed for Reung to win. **2**

....

....

....

....

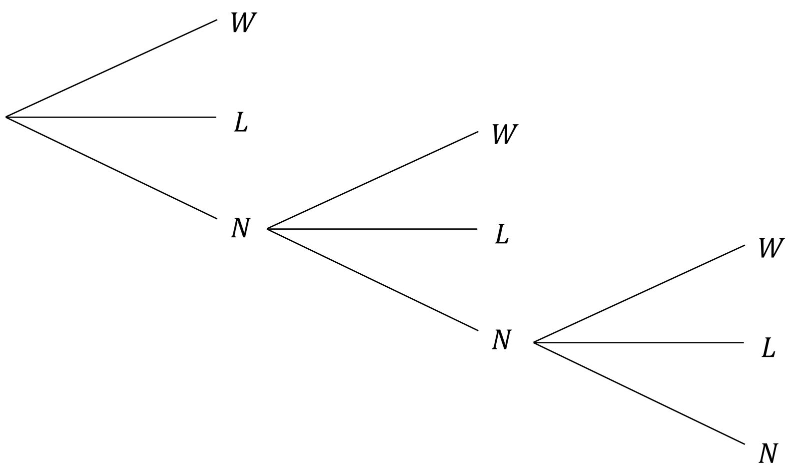
....

....

....

Examination continues overleaf…

iii. By completing the tree diagram or otherwise, find the probability that Reung wins on his first, second or third throw. **2**



....

....

....

....

....

....

....

iv. Calculate the probability that Reung wins the game. **2**

....

....

....

....

....

....

....

....

End of Examination

Additional writing space

SOLUTIONS

NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED

2020 Year 12 Course Assessment Task 1

Monday, 2 December 2019

General instructions

- Working time – 1 hour (plus 5 minutes reading time).
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt all questions.
- All necessary working should be shown in every question.
- Marks may be deducted for illegible or incomplete working.
- A NESA Reference Sheet is provided.

NESA STUDENT #: ....

Class: (please ✓)

O 12MAT.1 – Ms Ham
O 12MAT.2 – Mr Lam
O 12MAT.3 – Mr Sekaran
O 12MAT.4 – Ms Nam
O 12MAT.5 – Mrs Gan
O 12MAT.6 – Ms Park

Marker's use only

QUESTION	1	2	3	Total	%
MARKS	$\overline{11}$	$\overline{18}$	$\overline{24}$	~~55~~ 53	

Question 1 (11 marks) **Marks**

(a) Differentiate with respect to x:

i. $y = (3 + e^x)^5$ **2**

$\frac{dy}{dx} = 5(3+e^x)^4 e^x$

$= 5e^x (3+e^x)^4$

ii. $y = \frac{e^{2x}}{2x+1}$ **2**

$\frac{dy}{dx} = \frac{(2x+1)2e^{2x} - e^{2x}(2)}{(2x+1)^2}$

$= \frac{4xe^{2x} + 2e^{2x} - 2e^{2x}}{(2x+1)^2}$

$= \frac{4xe^{2x}}{(2x+1)^2}$

(b) Find the equation of the tangent to $y = e^{-x}$ at the point $(0,1)$. **2**

$y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$

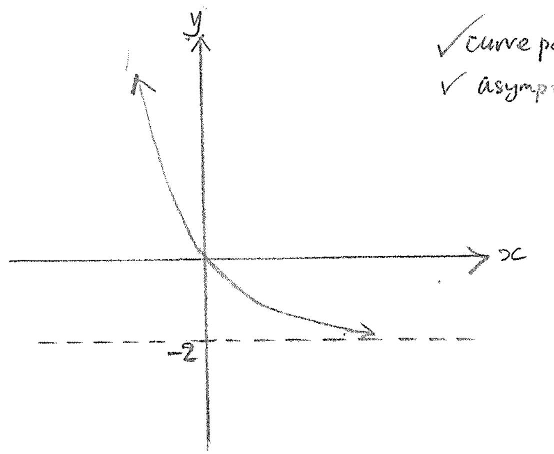
$= -1$ at $(0,1)$

∴ equation of tangent is:

$y - 1 = -1(x - 0)$

$y = 1 - x$

(c) Sketch the graph $y = 2e^{-2x} - 2$, showing all important features. **2**



✓ curve passes through (0,0)
✓ asymptote labelled

(d) Let $f(x) = e^x + k$, where k is a real number. The tangent to the graph of $y=$ $f(x)$ at the point where $x = a$ passes through the point $(0, 0)$.

i. Show that the equation of the tangent is $y = e^a x$. **1**

$f(x) = e^x + k$

$f'(x) = e^x$

$= e^a$ at $x = a$ ✓

The tangent passes through $(0,0)$

$\therefore$ equation of tangent is $y = e^a x$

ii. Find the value of k in terms of a. **2**

Substitute $x = a$ into $y = e^a x$,

$y = e^a a$

$f(a) = e^a + k$ ✓

$\therefore e^a + k = e^a a$

$k = e^a a - e^a$

$= e^a(a-1)$ ✓

Question 2 (18 marks) **Marks**

(a) Evaluate the following: 2

$$\sum_{k=2}^{10} 4 \times \left(\frac{1}{2}\right)^{k-1}$$

$\sum_{k=2}^{10} 4\times(\frac{1}{2})^{k-1} = 4\times\frac{1}{2} + 4\times(\frac{1}{2})^2 + 4\times(\frac{1}{2})^3 + \cdots$

$= \dfrac{4\times\frac{1}{2}[1-(\frac{1}{2})^9]}{1-\frac{1}{2}}$

$= 4(1-\frac{1}{2^9})$

$= 3\frac{127}{128}$

(b) A geometric series has common ratio $\frac{1}{p}$ and limiting sum $\frac{1}{1-p}$. 2

Find the first term of this series, a, in simplest form.

$\dfrac{1}{1-p} = \dfrac{a}{1-\frac{1}{p}}$

$a(1-p) = 1-\frac{1}{p}$

$a = \dfrac{p-1}{p} \times \dfrac{1}{1-p}$

$= -\dfrac{1}{p}$

(c) $2m-8$, $2m+4$ and $5m-2$ are successive terms of a geometric sequence. Find the value of m. 3

$\dfrac{2m+4}{2m-8} = \dfrac{5m-2}{2m+4}$

$(2m+4)^2 = (5m-2)(2m-8)$

$4m^2+16m+16 = 10m^2-44m+16$

$6m^2-60m=0$

$6m(m-10)=0$

$m=0$ or 10

(d) i. Show that $\log_{10}27+\log_{10}9+\log_{10}3+\cdots$ is an arithmetic series. 1

$T_2-T_1 = \log_{10}9-\log_{10}27 = \log_{10}\frac{9}{27}$

$= \log_{10}3^{-1} = -\log_{10}3$

$T_3-T_2 = \log_{10}3-\log_{10}9 = \log_{10}\frac{9}{3}$

$= \log_{10}3^{-1} = -\log_{10}3$

ii. Show that the sum of the first seven terms of this series is 0. 2

$S_7 = \frac{7}{2}[2\log_{10}27+6(-\log_{10}3)]$

$= \frac{7}{2}[2\log_{10}3^3 - 6\log_{10}3]$

$= \frac{7}{2}(6\log_{10}3 - 6\log_{10}3)$

$= 0$

(e) A pattern is made up of a series of concentric circles with centre O. The radius of the first (innermost) circle is 10 cm. The distance between two adjacent concentric circles is 2 cm.

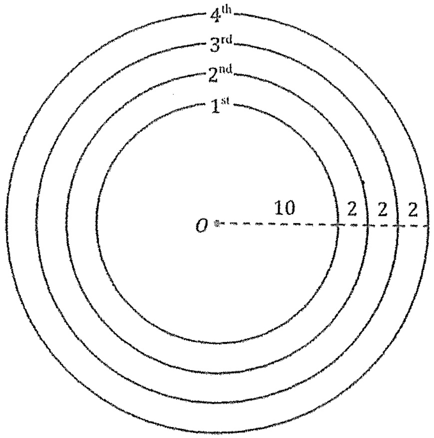


Diagram not drawn to scale

i. What is circumference of the first circle? **1**

20π cm ✓

ii. Show that the circumference, C, of the n^{th} circle is given by $C = 4\pi(4 + n)$. **2**

Radius : $10, 12, 14, 16, \ldots$ AP, $a=10$, $d=2$

Radius of the n^{th} circle

$= 10 + (n-1)2$

$= 8 + 2n$ ✓

$\therefore C = 2\pi(8+2n)$

$= 4\pi(4+n)$ ✓

Alternative Solution

$C : 20\pi, 24\pi, 28\pi, \ldots$ AP, $a = 20\pi, d = 4\pi$

$C = 20\pi + (n-1)4\pi$ ✓

$= 20\pi + 4n\pi - 4\pi$

$= 16\pi + 4n\pi$

$= 4\pi(4+n)$ ✓

iii. Show that the total circumference of n circles is given by $L = 2\pi n(9 + n)$. **2**

$C : 20\pi, 24\pi, 28\pi, \ldots$ AP, $a = 20\pi$, $d = 4\pi$

$\therefore L = \frac{n}{2}\left[20\pi + 4\pi(4+n)\right]$ ← using (e)(ii) ✓

$= n\left[10\pi + 2\pi(4+n)\right]$

$= n(18\pi + 2\pi n)$

$= 2\pi n(9+n)$ ✓

Alternative solution

$L = \frac{n}{2}\left[2\times 20\pi + (n-1)4\pi\right]$ ✓

$= 20n\pi + 2n^2\pi - 2n\pi$

$= 18n\pi + 2n^2\pi$

$= 2\pi n(9+n)$ ✓

iv. If $L = 1584\pi$, find the value of n and hence find the radius of the n^{th} circle. **3**

$2\pi n(9+n) = 1584\pi$

$9n + n^2 = 792$

$n^2 + 9n - 792 = 0$

$(n-24)(n+33) = 0$ ✓

$n = 24$ as $n > 0$ ✓

$\therefore$ radius of the 24^{th} circle

$= 8 + 2n$ using (ii)

$= 8 + 2\times 24$

$= 56$ cm ✓

Alternative solution

$C = 4\pi(4+24)$

$= 112\pi$

$2\pi r = 112\pi$

$\therefore r = 56$

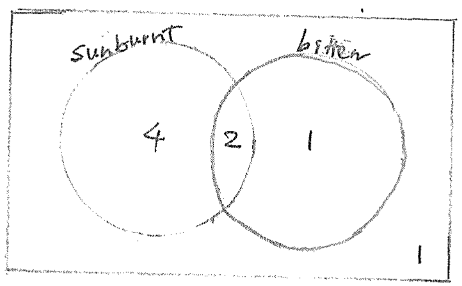
Question 3 (24 marks)

Marks

(a) 8 members of a rowing team were on a training row.

During the training row 6 members of the team were sunburnt and 2 were sunburnt and bitten by sand-flies. 1 member had nothing happen to him.

i. Draw a Venn diagram showing this information. **1**



ii. Hence find the probability that he was sunburnt but not bitten by sand-flies if a student was selected at random from the rowing team. **1**

$\frac{4}{8} = \frac{1}{2}$

(b) The chance Janine can solve a probability problem is $\frac{9}{10}$ and the chance Naomi can solve it is $\frac{4}{5}$. If they work on the problem independently, what is the probability that:

i. They both solve the problem? **2**

$\frac{9}{10} \times \frac{4}{5}$

$= \frac{18}{25}$

ii. At least 1 of them is able to solve the problem? **2**

$1 - \frac{1}{10} \times \frac{1}{5}$

$= \frac{49}{50}$

(c) Four identical balls are numbered 1, 2, 3 and 4 and put into a box.

A ball is randomly drawn from the box, and not returned to the box. A second ball is then randomly drawn from the box.

By drawing a diagram or otherwise, find the probability that:

i. The first ball drawn is numbered 4 and the second ball drawn is numbered 1. **1**

$P(4,1) = \frac{1}{4} \times \frac{1}{3}$

$= \frac{1}{12}$

ii. The sum of the numbers on the two balls is 5. **1**

Sum $= 5$: $(1,4), (2,3), (3,2), (4,1)$

$P(\text{Sum} = 5) = \frac{1}{12} \times 4$

$= \frac{1}{3}$

iii. The second ball drawn is numbered 1, given that the sum of the numbers on the two balls is 5. **2**

$P(2^{nd}\text{ball is } 1 \mid \text{sum} = 5)$

$= \frac{P[2^{nd}\text{ball is } 1 \text{ and sum} = 5]}{P(\text{Sum} = 5)}$

$= \frac{\frac{1}{12}}{\frac{1}{3}}$

$= \frac{1}{4}$

(d) A and B are events of a sample space. Given that $P(A|B) = p$, $P(B) = p^2$ and $P(A) = p^{\frac{1}{3}}$:

i. Show that $P(A \cap B) = p^3$. **1**

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$p = \frac{P(A \cap B)}{p^2}$

$\therefore P(A \cap B) = p^3$

ii. Find $P(B|A)$ in simplest form. **1**

$P(B|A) = \frac{P(B \cap A)}{P(A)}$

$= \frac{p^3}{p^{\frac{1}{3}}}$

$= p^{\frac{8}{3}}$

(e) From a packet of mixed seed, it was estimated that the probability of any seed planted yielding a white carnation was 0.02.

i. Calculate the probability that from any two seeds planted there will be at least 1 white carnation. **2**

$1 - P(\overline{W}\overline{W})$

$= 1 - (0.98)^2$

$= 0.0396$

ii. How many seeds must be planted to be at least 95% certain of obtaining at least 1 white carnation? **3**

$1 - (0.98)^n = 0.95$

$(0.98)^n = 0.05$

$n = \frac{\log_{10} 0.05}{\log_{10} 0.98}$

$= 148.28\ldots$

$\therefore$ 149 seeds must be planted.

(f) Reung has invented a game for one person. He throws 2 ordinary dice repeatedly until the sum of the two numbers shown is either a 7 or 9.

If the sum is 9, Reung wins. If the sum is 7, Reung loses. If the sum is any other number, he continues to throw until it is a 7 or 9.

i. Show that the probability that Reung wins on his first throw is $\frac{1}{9}$. **1**

Sum = 9 : (3,6), (4,5), (5,4), (6,3)

$\therefore P(\text{sum}=9) = \frac{4}{36}$

$= \frac{1}{9}$

ii. Calculate the probability that a second throw is needed for Reung to win. **2**

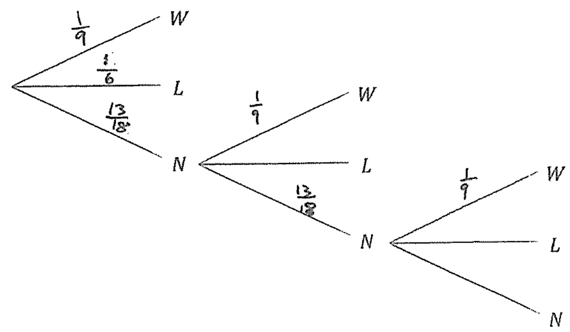
Sum = 7 : (1,6), (2,5), (3,4), (4,3) (5,2), (6,1)

$P(\text{sum}=7) = \frac{6}{36} = \frac{1}{6}$

$\therefore P(\text{Sum is neither } 7 \text{ nor } 9) = 1 - \frac{1}{9} - \frac{1}{6}$

$= \frac{13}{18}$

iii. By completing the tree diagram or otherwise, find the probability that Reung wins on his first, second or third throw. **2**



P (wins on 1st, 2nd or 3rd throw)

= P (wins on 1st throw) + P (wins on 2nd throw) + P (wins on 3rd throw)

$= \frac{1}{9} + \frac{13}{18} \times \frac{1}{9} + \frac{13}{18} \times \frac{13}{18} \times \frac{1}{9}$

$= \frac{1}{9}\left[1 + \frac{13}{18} + \left(\frac{13}{18}\right)^2\right]$

$\doteqdot 0.2493$

iv. Calculate the probability that Reung wins the game.
~~That is, that Reung wins on his first or second or third or nth throw.~~ **2**

$\frac{1}{9}\left[1 + \frac{13}{18} + \left(\frac{13}{18}\right)^2 + \left(\frac{13}{18}\right)^3 + \cdots\right]$

$= \frac{1}{9} \times \frac{1}{1 - \frac{13}{18}}$

$= 0.4$