NORTH SYDNEY BOYS HIGH SCHOOL

2012 HSC ASSESSMENT TASK 1

Mathematics

General Instructions

- Working time – 50 minutes
- Write on the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Lin
- O Mr Berry
- O Mr Lam
- O Mr Lowe
- O Mr Ireland
- O Mr Fletcher
- O Mr Weiss

Student Number:________________________

(To be used by the exam markers only.)

Question No	1	2	3	4	Total	Total %
Mark	/12	/15	/15	/11	/53	/100

Question 1 (12 marks) Start a new page. **Marks**

(a) Consider the arithmetic progression 5, 12, 19, ..., 292.
  (i) Calculate the number of terms in the series. 2
  (ii) Find the sum of the series. 2

(b) In a geometric series the first term is 12 and the fourth term is 768.
  (i) Find the common ratio. 2
  (ii) Hence find the sum of the first 10 terms of the series. 2

(c) Evaluate $\sum_{k=1}^{5}(-1)^k k^2$ 2

(d) If the sum of the first n terms of a series is $S_n = 16n - n^2$ find an expression for the n^{th} term T_n. 2

Question 2 (15 marks) Start a new page.

(a) Solve the equation $(2x+1)^2 = 5$ 2

(b) Solve the inequality $x^2 - x - 12 > 0$ 2

(c) Solve for x: $4^x - 7 \cdot 2^x - 8 = 0$ 3

(d) If α and β are the roots of $2x^2 - 6x - 7 = 0$, evaluate the following:
  (i) $\alpha + \beta$ 1
  (ii) $\alpha\beta$ 1
  (iii) $\frac{1}{\alpha} + \frac{1}{\beta}$ 2
  (iv) $\alpha^2 - 3\alpha$ 1

(e) Given that $3x^2 + 7x + 14 \equiv A(x+2)^2 + B(x+2) + C$, find the values of A, B and C. 3

Question 3 (15 marks) Start a new page.

(a) A layer of plastic cuts out 15% of the light that strikes it.

(i) Show that two layers let through 72·25% of light. **1**

(ii) How many layers are needed to cut out at least 90% of the light? **2**

(b) Consider the series: $\sin^2 x + \sin^4 x + \ldots.$ $\quad (0 < x < \frac{\pi}{2})$

(i) Show that a limiting sum exists. **1**

(ii) Find the limiting sum, expressing the answer in simplest form. **2**

(c) For what values of m is the quadratic $y = mx^2 - 8x + m$ positive definite? **3**

(d) Consider the quadratic equation: $x^2 - 3x + k = 0.$

(i) Find the value of k if 2 is a root of the equation. **2**

(ii) Find the value of k for which the equation has two equal roots. **2**

(iii) Find the value of k if one of the roots exceeds the other root by two. **2**

Question 4 (11 marks) Start a new page.

(a) Find the number which when added to each of 2, 6 and 13 will give a set of three numbers in geometric progression. **2**

(b) $A(-1, 3)$ and $B(3, 1)$ are two points on the plane. Show that the locus of $P(x, y)$ such that PA is perpendicular to PB is a circle with centre $C(1, 2)$ and radius $\sqrt{5}$. **3**

(c) The first three terms in an arithmetic series are $a, \frac{1}{a}, 1$ (where $a < 0$.)

(i) Find the values of a and the common difference, d. **3**

(ii) Find the smallest value of n for which the sum of the first n terms is > 1000. **3**

END OF EXAMINATION

Suggested Solutions

Question 1 (Lowe)

(a) i. (2 marks)

✓ [1] for formula of AP terms.

✓ [1] for final answer.

$$\begin{aligned} a = 5 \qquad d &= 7 \\ T_n &= a + d(n-1) \\ \underset{-5}{292} &= \underset{-5}{5} + 7(n-1) \\ \underset{\div 7}{287} &= \underset{\div 7}{7}(n-1) \\ n - 1 &= 41 \\ \therefore n &= 42 \end{aligned}$$

ii. (2 marks)

✓ [1] for formula of AP sum.

✓ [1] for final answer.

$$\begin{aligned} S_n &= \frac{n}{2}(a + \ell) \\ &= \frac{42}{2}(5 + 292) \\ &= 6\,237 \end{aligned}$$

Alternatively, use

$$S_n = \frac{n}{2}\Big(2a + d(n-1)\Big)$$

(b) i. (2 marks)

✓ [1] for formula of GP terms.

✓ [1] for final answer.

$$\begin{aligned} a = 12 \qquad T_4 &= 768 \\ T_n &= ar^{n-1} \\ T_4 = \underset{\div 12}{768} &= \underset{\div 12}{12}\, r^3 \\ r^3 &= 64 \\ \therefore r &= 4 \end{aligned}$$

ii. (2 marks)

✓ [1] for formula of GP sum.

✓ [1] for final answer.

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ S_{10} &= \frac{12\left(4^{10} - 1\right)}{4 - 1} \\ &= 4\,194\,300 \end{aligned}$$

(c) (2 marks)

✓ [1] for writing out terms of summation.

✓ [1] for final answer.

$$\begin{aligned} \sum_{k=1}^{5} (-1)^k k^2 &= -1^2 + 2^2 - 3^2 + 4^2 - 5^2 \\ &= -15 \end{aligned}$$

(d) (2 marks)

✓ [1] for correct substitution into $T_n = S_n - S_{n-1}$.

✓ [1] for final answer.

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= \left(16n - n^2\right) - \left(16(n-1) - (n-1)^2\right) \\ &= \cancel{16n} - \cancel{n^2} - \Big(\cancel{16n} - 16 - \cancel{n^2} + 2n - 1\Big) \\ &= 17 - 2n \end{aligned}$$

Question 2 (Lam)

(a) (2 marks)

✓ [1] for each correct root. No other marks awarded.

$$\begin{aligned} (2x+1)^2 &= 5 \\ 2x + 1 &= \pm\sqrt{5} \\ x &= -\frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

(b) (2 marks)

✓ [1] for each correct inequality. No other marks awarded.

$$\begin{aligned} x^2 - x - 12 &> 0 \\ (x-4)(x+3) &> 0 \\ \therefore x < -3 \quad &\text{or} \quad x > 4 \end{aligned}$$

(c) (3 marks)

✓ [1] for transforming into quadratic equation.

✓ [1] for justifying $2^x = -1$ is invalid.

✓ [1] for $x = 3$ only.

$$(2^x)^2 - 7 \times 2^x - 8 = 0$$

Let $m = 2^x$,

$$\begin{aligned} m^2 - 7m - 8 &= 0 \\ (m-8)(m+1) &= 0 \\ m &= -1, 8 \\ \therefore 2^x &= -1, 8 \end{aligned}$$

As there are no real solutions to $2^x = -1$,

$$\begin{aligned} \therefore 2^x &= 8 \\ x &= 3 \end{aligned}$$

(d) $2x^2 - 6x - 7 = 0$

i. (1 mark)

$$\alpha + \beta = -\frac{(-6)}{2} = 3$$

ii. (1 mark)

$$\alpha\beta = -\frac{7}{2}$$

iii. (2 marks)

✓ [1] for transforming into $\frac{\alpha+\beta}{\alpha\beta}$.

✓ [1] for final answer.

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{3}{-\frac{7}{2}} = -\frac{6}{7}$$

iv. (1 mark)

$$\begin{aligned} \alpha + \beta &= 3 \\ \therefore \beta &= 3 - \alpha \\ \alpha(3-\alpha) &= -\frac{7}{2} \\ \alpha(\alpha - 3) &= \frac{7}{2} \\ \therefore \alpha^2 - 3\alpha &= \frac{7}{2} \end{aligned}$$

(e) (3 marks)

✓ [1] for each of A, B and C. No variations permitted.

$$3x^2 + 7x + 14 \equiv A(x+2)^2 + B(x+2) + C$$

By inspection,

$$A = 3$$

Substitute $x = -2$,

$$\begin{aligned} 3(-2)^2 + \cancel{7(-2)} + \cancel{14} &= C \\ \therefore C &= 12 \end{aligned}$$

Substitute $x = 0$,

$$\begin{aligned} 14 &= 4A + 2B + C \\ 14 &= 12 + 2B + 12 \\ \therefore 2B &= -10 \\ \therefore B &= -5 \end{aligned}$$

Alternatively, expand and equate coefficients to obtain simultaneous equations.

Question 3 (Berry)

(a) i. (1 mark)

$$0.85 \times 0.85 = 0.722\,5$$

ii. (2 marks)

✓ [1] for setting up inequation.

✓ [1] for final answer. Trial and error is acceptable.

Let n be the number of layers required.

$$\begin{aligned} 0.85^n &\le 0.1 \\ n \log 0.85 &\le \log 0.1 \\ \therefore n &\ge \frac{\log 0.1}{\log 0.85} = 14.68\cdots \end{aligned}$$

Hence 15 layers is required.

(b) i. (1 mark)

$$r = \sin^2 x$$

Since $0 < \sin x < 1$ for $0 < x < \frac{\pi}{2}$,

$$\begin{aligned} \therefore 0 < \sin^2 x < 1 \\ \therefore -1 < r < 1 \end{aligned}$$

and limiting sum exists.

ii. (2 marks)
✓ [1] for correct limiting sum formula.
✓ [1] for final answer.

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{\sin^2 x}{1-\sin^2 x} \\ &= \frac{\sin^2 x}{\cos^2 x} = \tan^2 x \end{aligned}$$

(c) (3 marks)
✓ [1] for correct condition for m and Δ to make a positive definite quadratic.
✓ [1] for inequalities.
✓ [1] for final justification.

Positive definite implies $m > 0$ and $\Delta < 0$.

$$\begin{aligned} \therefore m > 0 \qquad \text{and } 64 - 4m^2 &< 0 \\ 16 - m^2 &< 0 \\ (4-m)(4+m) &< 0 \\ m < -4 \quad \text{or} \quad m &> 4 \end{aligned}$$

As $m > 0$ **and** $m > 4$, therefore $m > 4$ fulfills all conditions.

(d) i. (2 marks)
✓ [1] for correct substitution.
✓ [1] for value of k.

$$x^2 - 3x + k = 0$$

As $x = 2$ is a solution,

$$\begin{aligned} 2^2 - 3(2) + k &= 0 \\ 4 - 6 + k &= 0 \\ \therefore k &= 2 \end{aligned}$$

ii. (2 marks)
✓ [1] for constructing correct inequality.
✓ [1] for value of k.

2 equal roots implies $\Delta = 0$.

$$\begin{aligned} \Delta = b^2 - 4ac &= 0 \\ \therefore (-3)^2 - 4(1)(k) &= 0 \\ 9 - 4k &= 0 \\ \therefore k &= \frac{9}{4} \end{aligned}$$

Alternatively, k completes the square, i.e. $k = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

iii. (2 marks)
✓ [1] for $\alpha = \frac{1}{2}$.
✓ [1] for final answer.

Let the roots be α and $\alpha + 2$.

$$\begin{aligned} \alpha + (2+\alpha) &= -\frac{b}{a} = 3 \\ 2\alpha &= 1 \\ \therefore \alpha &= \frac{1}{2} \end{aligned}$$

So the roots are $\frac{1}{2}$, $\frac{5}{2}$. By the product of roots,

$$\begin{aligned} \frac{1}{2} \times \frac{5}{2} &= k \\ \therefore k &= \frac{5}{4} \end{aligned}$$

Question 4 (Lin)

(a) (2 marks)
✓ [1] for correct substitution into definition of GP.
✓ [1] for final answer.

Let the number be x.

$$\begin{aligned} \frac{T_2}{T_1} &= \frac{T_3}{T_2} \\ \frac{6+x}{2+x} &= \frac{13+x}{6+x} \\ (6+x)^2 &= (13+x)(6+x) \\ \cancel{x^2} + 12x + 36 &= \cancel{x^2} + 15x + 26 \\ 3x &= 10 \\ \therefore x &= \frac{10}{3} \end{aligned}$$

(b) (3 marks)

- ✓ [1] for correct substitution into gradients.
- ✓ [1] for correct manipulation.
- ✓ [1] for final answer.

$PA \perp PB$ implies $m_{PA} \times m_{PB} = -1$.

$$\begin{aligned} \frac{y-3}{x+1} \times \frac{y-1}{x-3} &= -1 \\ (y-1)(y-3) &= -(x-3)(x+1) \\ y^2 - 4y + 3 &= -(x^2 - 2x - 3) \\ x^2 - 2x + y^2 - 4y &= 0 \\ x^2 - 2x + 1 + y^2 - 4y + 4 &= 5 \\ (x-1)^2 + (y-2)^2 &= 5 \end{aligned}$$

which produces a circle of centre $(1, 2)$ and radius $\sqrt{5}$.

(c) i. (3 marks)

- ✓ [1] for correct substitution into AP definition.
- ✓ [1] for $a = -2$.
- ✓ [1] for $d = \frac{3}{2}$.

By definition of an AP,

$$\begin{aligned} T_3 - T_2 &= T_2 - T_1 \\ 1 - \frac{1}{a} &= \frac{1}{a} - a \\ \frac{2}{a} &= 1 + a \\ 2 &= a + a^2 \\ a^2 + a - 2 &= 0 \\ (a-1)(a+2) &= 0 \\ \therefore a &= -2, 1 \end{aligned}$$

As $a < 0$, $\therefore a = -2$.

$$\begin{aligned} \therefore d &= 1 - \frac{1}{a} \\ &= 1 - \frac{1}{-2} \\ &= \frac{3}{2} \end{aligned}$$

ii. (3 marks)

- ✓ [1] for correct substitution into sum formula exceeding 1 000.
- ✓ [1] for simplification.
- ✓ [1] for final answer.

$$\begin{aligned} S_n &= \frac{n}{2}\left(2a + d(n-1)\right) \\ \frac{n}{2}\left(2(-2) + \frac{3}{2}(n-1)\right) &> 1\,000 \\ \frac{n}{4}\left(-8 + 3n - 3\right) &> 1\,000 \\ n(3n - 11) &> 4\,000 \\ 3n^2 - 11n - 4\,000 &> 0 \\ n &= \frac{11 \pm \sqrt{121 + 4(3)(4\,000)}}{2(3)} \\ &= \frac{11 \pm \sqrt{121 + 48\,000}}{6} \\ \therefore n < \frac{11 - \sqrt{48\,121}}{6} &\text{ or } n > \frac{11 + \sqrt{48\,121}}{6} \end{aligned}$$

As $n > 0$,

$$\therefore n > \frac{11 + \sqrt{48\,121}}{6} \approx 38.39\cdots$$

Hence 39 terms required to exceed 1 000.