NORTH SYDNEY BOYS HIGH SCHOOL

2016 HSC ASSESSMENT TASK 2

Mathematics

Reading Time – 5 minutes
Working Time – 60 minutes
Total Time – 65 minutes

General Instructions

Questions are to be answered in the booklet provided, showing ALL necessary working.

- Write using blue or black pen
- Board approved calculators may be used
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher: Please tick

- O Mr Berry
- O Mr Hwang
- O Mr Ireland
- O Dr Jomaa
- O Ms Lee
- O Ms Ziaziaris
- O Mr Zuber

Student Number:____________________________

(To be used by the exam markers only.)

Question No	1	2	3	4	5	Total	Total %
Mark	/10	/8	/7	/9	/8	/42	/100

Question 1 (10 Marks) **Marks**

(a) Find the primitive of each of the following:

(i) $\int(2x^3 - 3x^2 + 1)\,dx$ **2**

(ii) $\int(2 - 3x)^5\,dx$ **2**

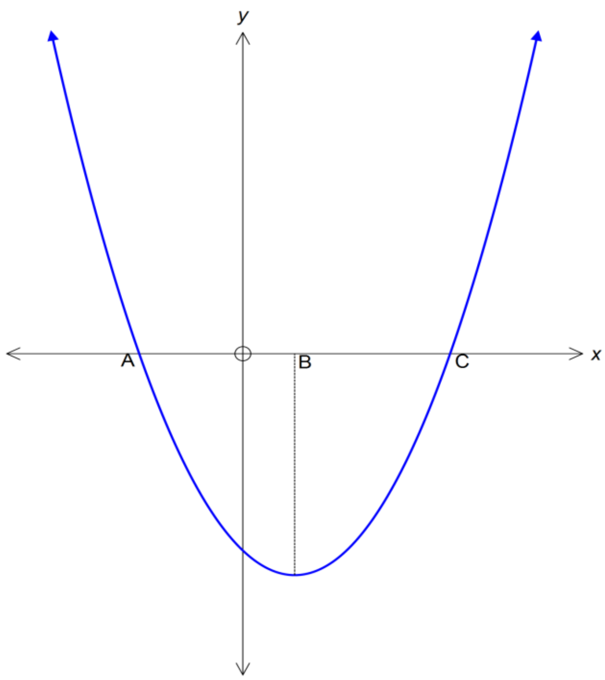
(b) Evaluate:

(i) $\displaystyle\int_1^2 \frac{t^4+1}{t^2}dt$ **2**

(ii) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}}$ **2**

(c)

Sketch a possible primitive of the function shown below: **2**



Question 2 (8 Marks)

(a) Evaluate

$$\int_0^2 (e^{3x} + 1)dx$$ **2**

(b) Differentiate

$$y = \frac{x^2 + e^x}{x - 1}$$ **3**

(c) Find the equation of the tangent to the curve $y = e^{3x}$ at the point whose *x* coordinate is 2. **3**

Question 3 (7 Marks)

(a) Consider the function $y = x^2 e^x$:

(i) For what values of *x* is this function defined? **1**

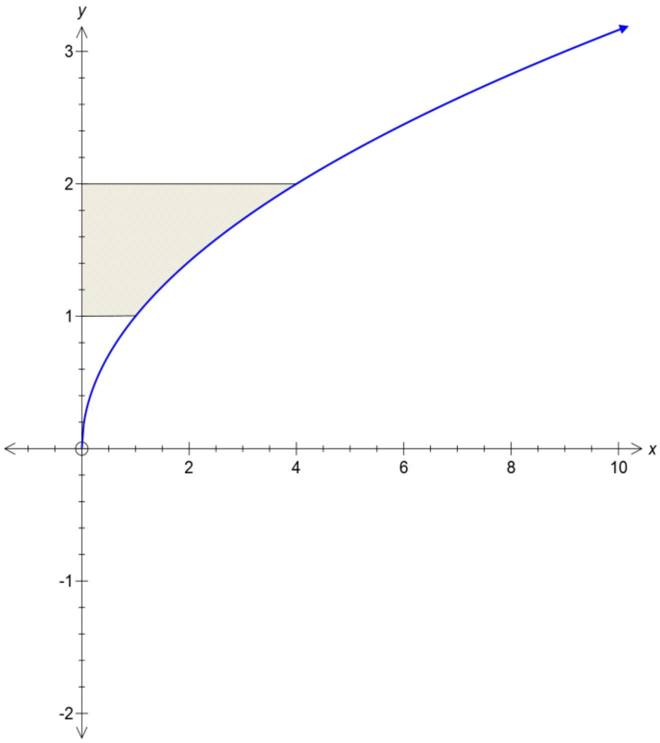
(ii) Describe the behavior of the function as *x*:
α) approaches $+\infty$ **1**
β) approaches $-\infty$ **1**

(iii) Find any stationary points and determine their nature. **3**

(iv) Sketch the curve of this function on the number plane. **1**

Question 4 (9 Marks)

(a)



The region above is the area between $y = \sqrt{x}$, $y = 1$, $y = 2$ and the $y-$ axis.

Find the exact area of the shaded region. **3**

(b) (i) Copy and complete the following table: **1**

x	-2	-1	0	1	2
$4 - x^2$					

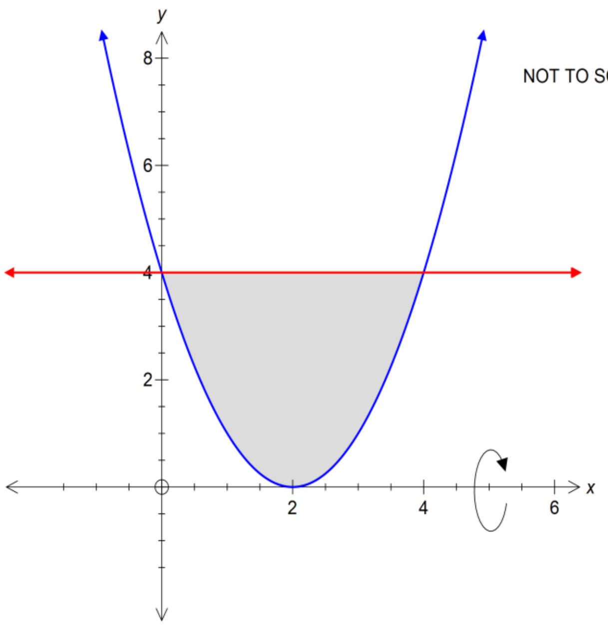
(ii) Hence estimate $\int_{-2}^{2}(4 - x^2)dx$ using four applications of the trapezoidal rule. **2**

(iii) Find the exact value of $\int_{-2}^{2}(4 - x^2)dx$ **2**

(iv) Did you over estimate or under estimate the exact integral in (ii)? Give a reason for your answer. **1**

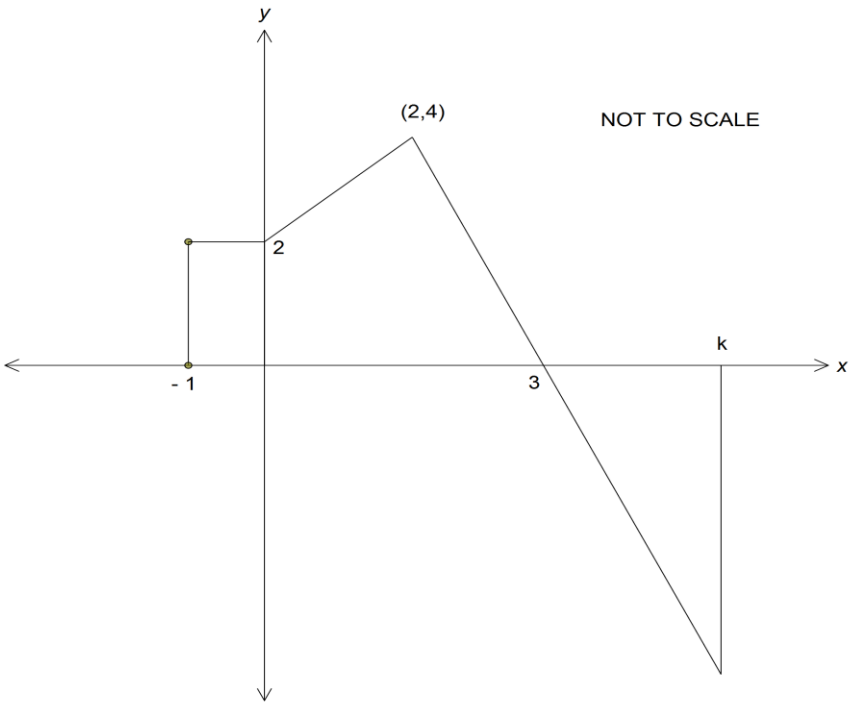
Question 5 (8 Marks)

(a) The shaded region bounded by the graph $y = (x-2)^2$ and the line $y = 4$ is rotated about the x-axis to form a solid of revolution as show in the diagram.



Find the volume of the solid formed. Leave your answer in exact form. **4**

(b) Consider the graph $y = f(x)$ below.



If $\int_{-1}^{k} f(x)dx = 0$ find the value of k. **4**

Question 1

(a) (i)

$$\int (2x^3 - 3x^2 + 1)\ dx = \frac{x^4}{2} - \frac{x^3}{3} + x + C$$

✓ -1 for each error
✓

(a) (ii)

$$\int (2 - 3x)^5\ dx = \frac{(2-3x)^6}{-3 \times 6} + C$$

✓ Correct integral

$$= -\frac{(2-3x)^6}{18} + C$$

✓ Simplified solution

(b) (i)

$$\int_1^2 \frac{t^4 + 1}{t^2}\,dt = \int_1^2 \left(t^2 + \frac{1}{t^2}\right) dt$$

✓ Correct integral

$$= \left[\frac{t^3}{3} - \frac{1}{t}\right]_1^2$$

$$= \left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right)$$

$$= \frac{17}{6}$$

✓ Solution

(b) (ii)

$$\int_1^4 \frac{dx}{\sqrt{x}} = \int_1^4 x^{-\frac{1}{2}}\ dx$$

✓ Correct integral

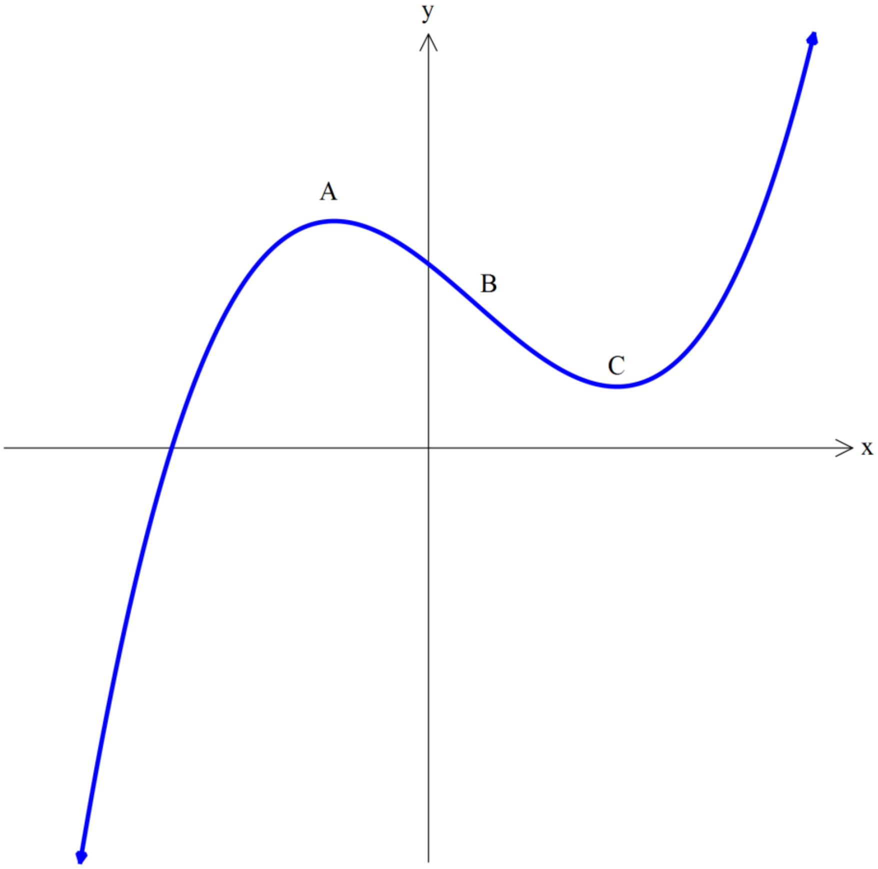
$$= \left[\frac{1}{2}x^{\frac{1}{2}}\right]_1^4$$

$$= \frac{1}{2}(\sqrt{4} - \sqrt{1})$$

$$= \frac{1}{2}$$

✓ Solution

(c)



- ✓ Correct shape
- ✓ Points in correct positions

note the vertical translation does not matter

Question 2

(a)

$$\int_0^2 (e^{3x} + 1)\,dx = \left[\frac{e^{3x}}{3} + x \right]_0^2$$

✓ Integral

$$= \left(\frac{e^6}{3} + 2 \right) - \left(\frac{1}{3} + 0 \right)$$

$$= \frac{e^6 + 5}{3}$$

✓ Solution

(b)

$$\text{let } y = \frac{u}{v}$$

$$\therefore u = x^2 + e^x$$

$$u' = 2x + e^x$$

$$v = x - 1$$

$$v' = 1$$

$$y' = \frac{u'v - v'u}{v^2}$$

✓ Quotient rule

$$= \frac{(2x + e^x)(x - 1) - (1)(x^2 + e^x)}{(x - 1)^2}$$

✓ Substitution into formula

$$= \frac{2x^2 - 2x + xe^x - e^x - x^2 - e^x}{(x - 1)^2}$$

$$= \frac{x^2 + xe^x - 2e^x - 2x}{(x - 1)^2}$$

✓ Final answer

(c)

$$y = e^{3x}$$

$$\frac{dy}{dx} = 3e^{3x}$$

when $x = 2$

$$m = 3e^{6}$$

✓ Gradient

and $y = e^{6}$

$$y - y_1 = m(x - x_1)$$

$$y - e^{6} = 3e^{6}(x - 2)$$

✓ Substitution into formula

$$y - e^{6} = 3e^{6}x - 6e^{6}$$

$$\therefore \quad y = 3e^{6}x - 5e^{6}$$

or

$$3e^{6}x - y - 5e^{6} = 0$$

✓ Final answer

Question 3

(a) (i) All real x ✓ Answer

(ii) (α) $\lim_{x \to \infty} x^2 e^x = \infty$ ✓ Answer

(β) $\lim_{x \to -\infty} x^2 e^x = 0$ ✓ Answer

(a) (iii)

$$\text{let } y = uv$$

$$\begin{aligned} u &= x^2 \\ u' &= 2x \\ v &= e^x \\ v' &= e^x \end{aligned}$$

$$\begin{aligned} y' &= u'v' + v'u \\ &= (2x)(e^x) + (x^2)(e^x) \\ &= xe^x(x + 2) \end{aligned}$$

✓ Derivative

$$\begin{aligned} \text{when } y' &= 0 \\ xe^x(x+2) &= 0 \\ \therefore \quad x &= -2 \text{ or } x = 0 \end{aligned}$$

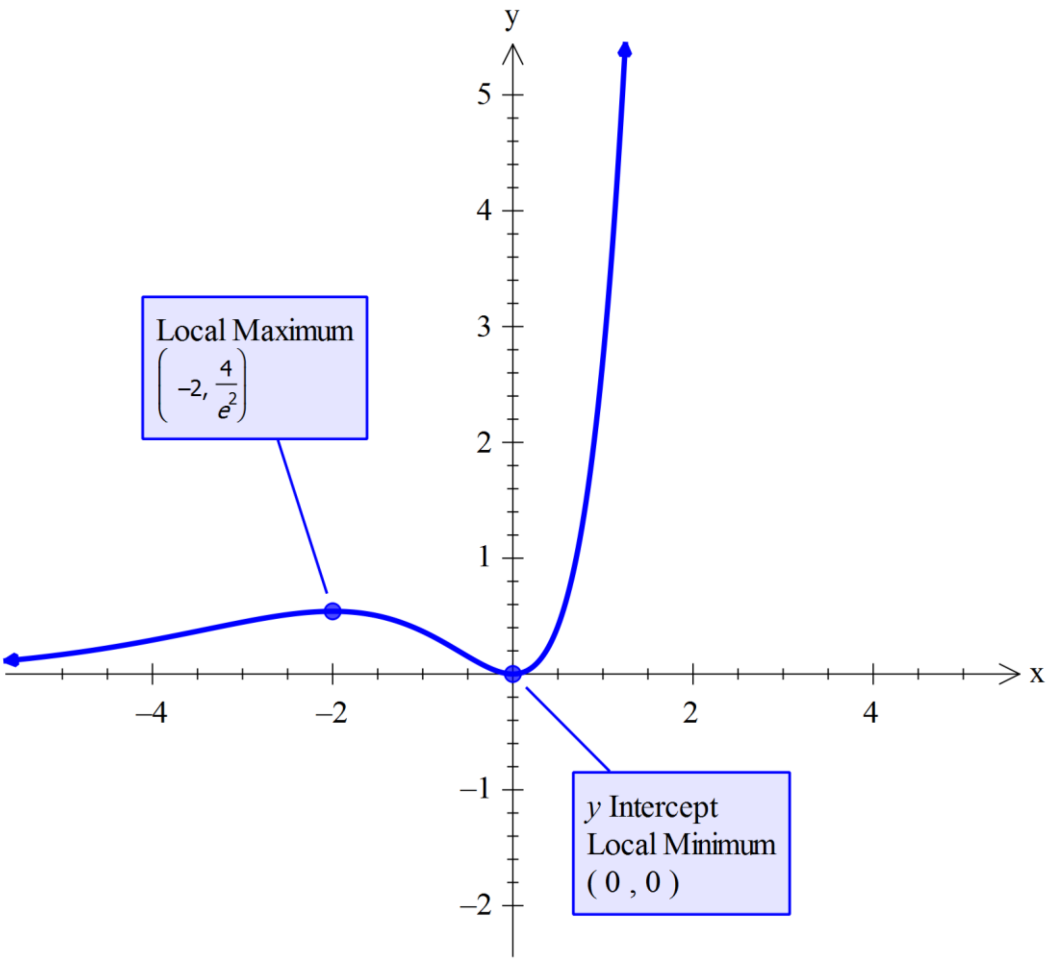
$\therefore$ stationary points occur at $\left(-2, \dfrac{4}{e^2}\right)$ and $(0,0)$ ✓ Stationary points

test the nature:

x	-3	-2	-1	0	1
y'	$+\dfrac{3}{e^3}$	0	$-\dfrac{1}{e}$	0	$+3e$

$\therefore$ $(-2, \frac{4}{e^2})$ is a local maxima and $(0,0)$ is a local minima. ✓ Testing nature

(a) (iv)



✓ Correct graph with details labelled

Question 4

(a) (i)

$$y = \sqrt{x}$$
$$\therefore x = y^2$$

✓ Making x the subject

$$A = \int_1^2 x\,dy$$
$$= \int_1^2 y^2\,dy$$
$$= \left[\frac{y^3}{3}\right]_1^2$$

✓ Correct integral

$$= \frac{8}{3} - \frac{1}{3}$$
$$= \frac{7}{3}\text{ units}^2$$

✓ Correct answer (must say units^2).

(b) (i)

x	-2	-1	0	1	2
$4 - x^2$	0	3	4	3	0

✓ Correct table

(b) (ii)

$$A \approx \frac{h}{2}\left(y_0 + y_4 + 2(y_1 + y_2 + y_3)\right)$$

✓ Trapezoidal rule

$$= \frac{1}{2}(0 + 0 + 2(3 + 4 + 3))$$
$$= 10\text{ units}^2$$

✓ Answer

(b) (iii)

$$A = \int_{-2}^{2} (4 - x^2)dx$$
$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

✓ Correct integral

$$= \left((4)(2) - \frac{8}{3}\right) - \left((4)(-2) - \frac{(-8)}{3}\right)$$
$$= \frac{24}{3} - \frac{8}{3} + \frac{24}{3} - \frac{8}{3}$$
$$= \frac{32}{3}\text{ units}^2$$

✓ Correct answer

(b) (iv)

Underestimated because the function is concave down.

✓ Must give valid reason

Question 5

(a)

First find points of intersection

$$y = (x-2)^2 \ldots (1)$$
$$y = 4 \ldots (2)$$
$$(1) = (2)$$

$$\therefore \quad (x-2)^2 = 4$$
$$x^2 - 4x + 4 = 4$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 4$$

✓ Intercepts

Subtract the volumes under the parabola from the volume of the cylinder

$$V = \pi r^2 h - \left(\pi \int_2^4 ((x-2)^2)^2 \, dx + \pi \int_0^2 ((x-2)^2)^2 \, dx \right)$$

✓ Forming equation

$$= \pi(4^2)(4) - \pi \int_0^4 (x-2)^4 \, dx$$

$$= 64\pi - \pi \left[\frac{(x-2)^5}{5} \right]_0^4$$

✓ Substituting values into integrals/formula

$$= 64\pi - \pi \left(\frac{2^5}{5} - \frac{(-2)^5}{5} \right)$$

$$= 64\pi - \frac{64\pi}{5}$$

$$= \frac{241\pi}{5} \text{ units}^3$$

✓ Answer

(b)

$$\int_{-1}^{3} f(x)\,dx = \text{Rectangle} + \text{Trapezium} + \text{Triangle}$$

$$= (1 \times 2) + \frac{2}{2}(2+4) + \left(\frac{1}{2}\right)(1)(4)$$

$$= 10$$

$$\therefore \int_{3}^{k} f(x)\,dx = 10$$

✓ Area above x axis

To find the area under the *x* –axis in terms of *k*.

$$y = -4x + 12$$

$$\text{when } x = k$$

$$y = -4k + 12$$

$$\int_{3}^{k} f(x)\ dx = -10$$

✓ Area below axis the same magnitude

$$\therefore \left(\frac{1}{2}\right)(k-3)(12-4k) = -10$$

✓ Setting up equation

$$(k-3)(6-2k) = -10$$

$$6k - 2k^2 - 18 + 6k = -10$$

$$2k^2 - 12k + 8 = 0$$

$$k^2 - 6k + 4 = 0$$

$$k = \frac{6 \pm \sqrt{(-6)^2 - (4)(4)(1)}}{(2)(1)}$$

$$\therefore \quad k = \frac{6+\sqrt{20}}{2} \quad (\text{since k} > 3)$$

$$= \frac{6+2\sqrt{5}}{2}$$

$$= 3 + \sqrt{5}$$

✓ Answer