NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Assessment Task 1
Term 4, 2011

Name:_____ **Mathematics Class:**_____

Time Allowed: **50 minutes + 2 minutes reading time**
Total Marks: **50**

Instructions:

- Attempt all five questions.
- Questions are of equal value.
- Start each question in a new booklet. Put your name on every booklet.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Work down the page.
- Do not work in columns.
- Each question will be collected separately. Submit a blank booklet if you do not attempt a question.

Question	1ai	1aii	1aiii-b	2abc	2d	3	4	5a	5bc	Total
P4			/8	/7			/10	/4		/29
H6	/1				/3	/10				/14
H7		/1								/1
H9									/6	/6
										/50

Question 1 (10 marks) **Marks**

(a) Multiple choice. Write down the correct alternative on your writing paper.

(i) Part of the curve $y = f(x)$ is increasing and concave down. **1**

Which of the following statements is true for this part of the curve?

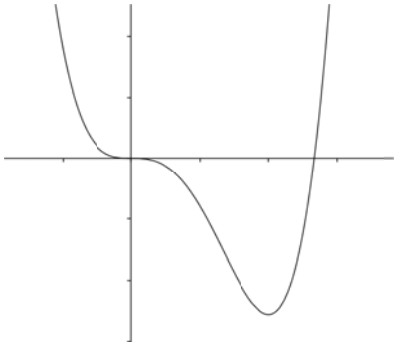
(A) $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

(B) $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$

(C) $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$

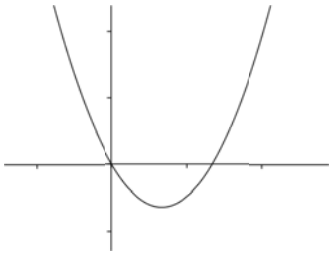
(D) $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$

(ii) The graph $y = f(x)$ is shown below. **1**

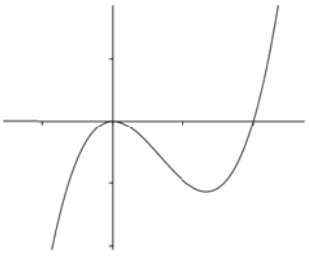


The graph of $y = f''(x)$ could be

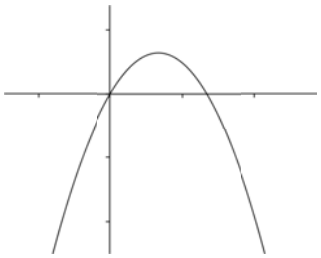
(A)



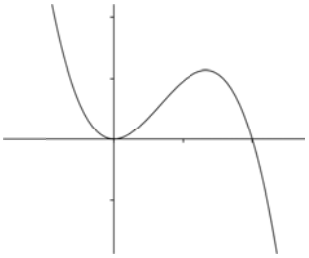
(C)



(B)



(D)



Question 1 continued **Marks**

(iii) The quadratic equation $x^2 - 6x - 9 = 0$ has roots α and β. **1**

α and β are

(A) real and equal.

(B) real, unequal and rational.

(C) real, unequal and irrational.

(D) not real.

(iv) The locus of the set of points whose distance from the *x*-axis is twice the distance from the *y*-axis has its equation given by **1**

(A) $y = 2x$

(B) $x = 2y$

(C) $x^2 = 4y^2$

(D) $y^2 = 4x^2$

(v) The parabola $(x-2)^2 = -8(y+3)$ has

(A) its vertex at $(2, -3)$ and directrix at $y = -1$

(B) its vertex at $(2, -3)$ and directrix at $y = -5$

(C) its focus at $(2, -3)$ and directrix at $y = -1$

(D) its focus at $(2, -3)$ and directrix at $y = -5$

(b) Show all necessary working.

If α and β are the roots of the equation $2x^2 - 5x - 10 = 0$, find the values of

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $(1+\alpha)(1+\beta)$ **1**

(iv) $\alpha^2 + \beta^2$ **2**

Question 2 (10 marks) Start a new booklet **Marks**

(a) Form the quadratic equation with integer coefficients whose roots are $2+\frac{\sqrt{3}}{2}$ and $2-\frac{\sqrt{3}}{2}$. **2**

(b) Find real numbers A, B and C, if **3**

$$2x^2+5x-19 \equiv A(x+3)(x-1)+B(x+3)+C(x-1).$$

(c) Show that the parabola $y=2x^2-2kx+(3k^2+1)$ is positive definite. **2**

(d) The function $f(x)=x^4+ax^3+bx+1$ has a point of inflexion at $(2,3)$. Find the values of a and b. **3**

Question 3 (10 marks) Start a new booklet

Let $f(x)=8x^2-x^4$.

(a) Find the coordinates of the points where the curve crosses the axes. **2**

(b) Find the coordinates of the stationary points and determine their nature. **4**

(c) Find the coordinates of the points of inflexion. **2**

(d) Sketch the graph of $y=f(x)$ indicating clearly the intercepts, stationary points and points of inflexion. **2**

Question 4 (10 marks) Start a new booklet **Marks**

(a) By making a suitable substitution or otherwise solve the equation **3**

$$\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+10=0.$$

Leave your answer(s) in exact form.

(b) Find the values of k for which the equation $x^2-2kx+4k-3=0$ has two real and distinct roots. **3**

(c) Consider the parabola $y^2=2x-4y-2$.

(i) Find the coordinates of the vertex. **2**

(ii) Find the coordinates of the focus. **1**

(iii) Find the equation of the directrix. **1**

Question 5 (10 marks) Start a new booklet

(a) The point $P(x,y)$ moves so that the sum of the squares of the distances from each of the points $A(0,6)$ and $B(8,0)$ is equal to 82. **4**

Show that the locus of $P(x,y)$ is a circle and find its centre and radius.

(b) Let $f(x)=(x-2)^4-32x-7$. **3**

Does $y=f(x)$ have any points of inflexion? Justify your answer.

(c) The roots of the equation $(p+q)x^2-6qx-3p=0$ are reciprocals. **3**

Show that the sum of the roots is 8.

End of paper

MATHEMATICS TERM 4 2011 SOLUTIONS

Question 1.

(a)

(i) D

(ii) A

(iii) C

(iv) D

(v) A

(b)

$2x^2 - 5x - 10 = 0$

(i) $\alpha + \beta = -\frac{b}{a} = \frac{5}{2}$

(ii) $\alpha\beta = \frac{c}{a} = \frac{-10}{2} = -5$

(iii)
$$(1+\alpha)(1+\beta)$$
$$= 1 + (\alpha + \beta) + \alpha\beta$$
$$= 1 + \frac{5}{2} - 5$$
$$= -\frac{3}{2}$$

(iv)
$$\alpha^2 + \beta^2$$
$$= (\alpha + \beta)^2 - 2\alpha\beta$$
$$= \left(\frac{5}{2}\right)^2 - 2 \times (-5)$$
$$= \frac{25}{4} + 10$$
$$= \frac{65}{4}$$

Question 2.

(a)

$$\alpha + \beta = 2 + \frac{\sqrt{3}}{2} + 2 - \frac{\sqrt{3}}{2} = 4$$

$$\alpha\beta = \left(2 + \frac{\sqrt{3}}{2}\right)\left(2 - \frac{\sqrt{3}}{2}\right) = 4 - \frac{3}{4} = \frac{13}{4}$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$x^2 - (4)x + \frac{13}{4} = 0$$

$$4x^2 - 16x + 13 = 0$$

(b)

$$2x^2 + 5x - 19 \equiv A(x+3)(x-1) + B(x+3) + C(x-1)$$

coefficient x^2 : $\boldsymbol{A = 2}$

$x = 1$: $-12 = 4B$

$\boldsymbol{B = -3}$

$x = 0$: $-19 = -3A + 3B - C$

$-19 = -6 - 9 - C$

$\boldsymbol{C = 4}$

(c) $y = 2x^2 - 2kx + (3k^2 + 1)$

$$\Delta = (2k)^2 - 4 \times 2 \times (3k^2 + 1)$$
$$= 4k^2 - 24k^2 - 8$$
$$= -(20k^2 + 8)$$

which is negative for all k.

also $a = 2 > 0$.

$\therefore$ positive definite

(d) $f(x) = x^4 + ax^3 + bx + 1$

$$f'(x) = 4x^3 + 3ax^2 + b$$
$$f''(x) = 12x^2 + 6ax$$
$$f''(2) = 12 \times (2)^2 + 6a \times 2 = 0$$
$$\therefore 12a + 48 = 0$$
$$a = -4$$
$$f(2) = (2)^4 - 4(2)^3 + 2b + 1 = 3$$
$$16 - 32 + 2b + 1 = 3$$
$$2b = 18$$
$$b = 9$$

Question 3.

$f(x) = 8x^2 - x^4$

(a)

$8x^2 - x^4 = 0$

$x^2(8 - x^2) = 0$

$\therefore x = 0, \pm\sqrt{8} \ (= \pm 2\sqrt{2})$

$(0,0) \quad (2\sqrt{2}, 0) \quad (-2\sqrt{2}, 0)$

(b)

$f'(x) = 16x - 4x^3 = 4x(4 - x^2)$

$f''(x) = 16 - 12x^2 = 4(4 - 3x^2)$

stat.pts when $f'(x) = 0$

$4x(4 - x^2) = 0$

$\therefore x = 0, 2, -2$

at $x = 0, y = 0, f''(0) = 16 > 0$

$\therefore$ concave up. Minimum TP at $(0,0)$

at $x = -2, y = 16, f''(-2) = -32 < 0$

$\therefore$ concave down. Maximum TP at $(-2,16)$

at $x = 2, y = 16, f''(2) = -32 < 0$

$\therefore$ concave down. Maximum TP at $(2,16)$

(c)

For inflexions $f''(x) = 0$

$16 - 12x^2 = 0$

$x = \pm\sqrt{\frac{16}{12}} = \pm\frac{2}{\sqrt{3}}$

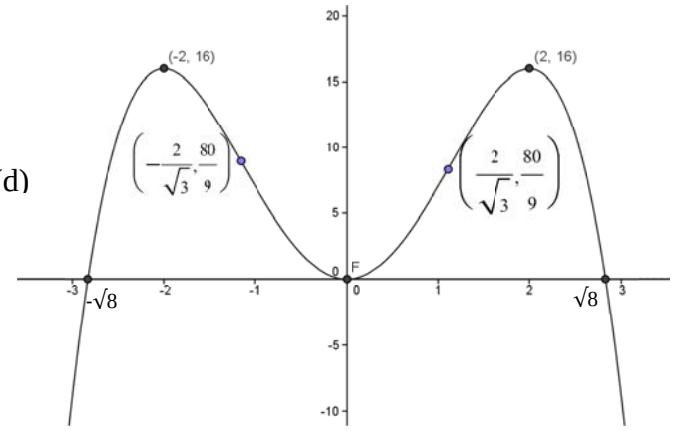
$y = 8 \times \frac{4}{3} - \left(\frac{4}{3}\right)^2 = \frac{80}{9} = 8\frac{8}{9}$

$\therefore \left(\pm\frac{2}{\sqrt{3}}, 8\frac{8}{9}\right)$

x	-2	$-2/\sqrt{3}$	0	$2/\sqrt{3}$	2
y"	-32	0	16	0	-32

$\therefore$concavity changes-inflexions

(d)



Question 4.

(a)

$\left(x + \frac{1}{x}\right)^2 - 7\left(x + \frac{1}{x}\right) + 10 = 0.$

Let $U = x + \frac{1}{x}$

$U^2 - 7U + 10 = 0$

$(U - 2)(U - 5) = 0$

$U = 2, 5$

$x + \frac{1}{x} = 2, \quad x + \frac{1}{x} = 5$

$x^2 + 1 = 2x, \quad x^2 + 1 = 5x$

$x^2 - 2x + 1 = 0, \quad x^2 - 5x + 1 = 0$

$(x - 1)^2 = 0, \quad x = \frac{5 \pm \sqrt{25 - 4 \times 1}}{2}$

$x = 1, \frac{5 \pm 21}{2}$

(b)

$x^2 - 2kx + 4k - 3 = 0$

$\Delta = (-2k)^2 - 4 \times 1 \times (4k - 3)$

$= 4k^2 - 16k + 12$

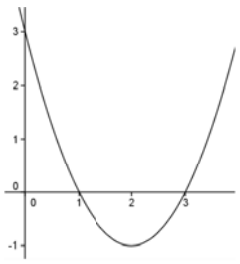
For real,distinct roots $\Delta > 0$

$4k^2 - 16k + 12 > 0$

$k^2 - 4k + 3 > 0$

$(k - 1)(k - 3) > 0$

$\therefore \ k < 1$ or $k > 3$



(c)

$y^2 = 2x - 4y - 2$

(i)

$y^2 + 4y = 2x - 2$

$y^2 + 4y + 4 = 2x + 2$

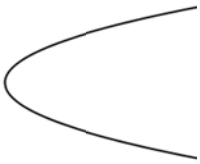
$(y+2)^2 = 2(x+1)$

$\therefore$ vertex $(-1,-2)$

(ii)

Focal length=0.5,

shape



∴Focus (-0.5 , -2)

(iii)

Directrix: x = -1.5

Question 5.

(a)

Let P(x , y) satisfy condition

$PA^2 + PB^2 = 82$

$(x-0)^2 + (y-6)^2 + (x-8)^2 + (y-0)^2 = 82$

$x^2 + y^2 - 12y + 36 + x^2 - 16x + 64 + y^2 = 82$

$2x^2 - 16x + 2y^2 - 12y + 100 = 82$

$x^2 - 8x + y^2 - 6y = -9$

$x^2 - 8x + 16 + y^2 - 6y + 9 = -9 + 16 + 9$

$(x-4)^2 + (y-3)^2 = 4^2$

Circle centre (4 , 3) radius 4

(b)

$f(x) = (x-2)^4 - 32x - 7$

$f'(x) = 4(x-2)^3 - 32$

$f''(x) = 12(x-2)^2$

Possible inflexion at $f''(x) = 0$

i.e. $x = 2$

Check change of concavity

x	1	2	3
y”	12	0	12

No change in concavity. No inflexions.

(c)

$(p+q)x^2 - 6qx - 3p = 0$

let roots α, β

$\alpha\beta = \dfrac{-3p}{p+q}$

but $\beta = \dfrac{1}{\alpha}$

$\therefore \dfrac{-3p}{p+q} = 1$

$-3p = p + q$

$q = -4p$

$\alpha + \beta = -\dfrac{-6q}{p+q}$

$= \dfrac{6 \times (-4p)}{p - 4p}$

$= \dfrac{-24p}{-3p}$

$= 8$ as required