NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Assessment Task 1
Term 4, 2012

Name:_______________________________ Mathematics Class: 11Ma_

Time Allowed: **60 minutes + 2 minutes reading time**
Total Marks: **54**

Instructions:

- Attempt all questions.
  - Section I has five multiple choice questions.
  - Section II has four free response questions of approximately equal value.
- Start each free response question in a new booklet.
- Put your name on every booklet. Submit a blank booklet if you do not attempt a question.
- Show all necessary working. Marks may be deducted for incomplete or poorly arranged work.
- Work down the page. Do not work in columns.

Question	1-3	4-5	6ab	6c	7ab	7c	8a	8b	8c	9a	9bc	9d	Total
P4	/3		/5		/6			/3		/3			/20
H5		/1				/6	/5				/7		/19
H6		/1		/7									/8
H7									/5				/5
H9												/2	/2
													/54

Section I

5 marks
Attempt Questions 1 – 5

Use the multiple-choice answer sheet for Questions $1-5$.

---

1 A parabola has its focus at $(0,2)$. The equation of its directrix is $x=-4$. The equation of the parabola is:

(A) $x^2=16(y-2)$

(B) $(y-2)^2=16x$

(C) $(y-2)^2=8(x+2)$

(D) $(x+2)^2=8y$

2 The minimum value of $x^2-7x+10$ is:

(A) 10

(B) $-\frac{7}{2}$

(C) $-\frac{9}{4}$

(D) 0

3 The locus of point(s) equidistant from the points $(p,0)$ and $(0,p)$ is:

(A) The point $\left(\frac{p}{2},\frac{p}{2}\right)$

(B) The line $x=y$

(C) The line $x=-y$

(D) The line $x=y$ for $p>0$ and $x=-y$ for $p<0$

4 The line $2x+3y=5$ is a tangent to the curve $y=f(x)$ at $(-2,3)$. The same line is also a normal to the curve at $(1,1)$. The value of $f'(1)$ could be:

(A) $-\frac{2}{3}$

(B) $\frac{2}{3}$

(C) $-\frac{3}{2}$

(D) $\frac{3}{2}$

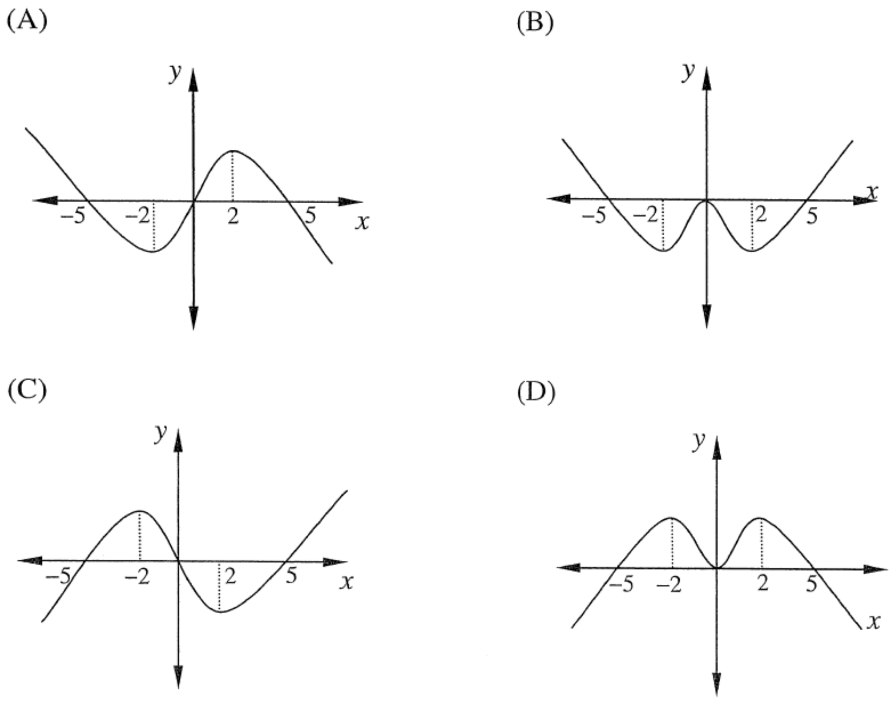
5 Consider a curve with the following properties:

$g(x)$ is odd.

$g(5) = 0$ and $g'(2) = 0$.

$g'(x) > 0$ for $x > 2$.

Which of the following could be the graph of $g(x)$?



Section II

49 marks
Attempt Questions 6 – 9

Answer each question in a SEPARATE writing booklet.

In Questions 6 – 9, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Find the equation of the parabola with vertex at $(-3, 2)$ and focus at $(-3, 6)$. **2**

(b) A point moves such that its distance from $A(-1, 5)$ is twice its distance from $B(2, -4)$. Find the equation of its locus. **3**

(c) Consider the function $f(x) = 3x - x^3 + 5$.

(i) Find the coordinates of the stationary points of $f(x)$ and determine their nature. **3**

(ii) Find any inflexion points of $f(x)$. **2**

(iii) Sketch the graph of $f(x)$ showing all stationary and inflexion points. You are not required to find the x-intercepts. **2**

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Find the value(s) of K, for which the parabola $y = 6x^2$ does not meet the line $16x - y + K = 0$. **2**

Question 7 (continued)

(b) α and β are the roots of the equation $2x^2 - 11x + 3 = 0$. Evaluate :

(i) $\alpha + \beta$. **1**

(ii) $\alpha\beta$. **1**

(iii) $(\alpha - 1)(\beta - 1)$. **2**

(c) Find the primitive function of the following:

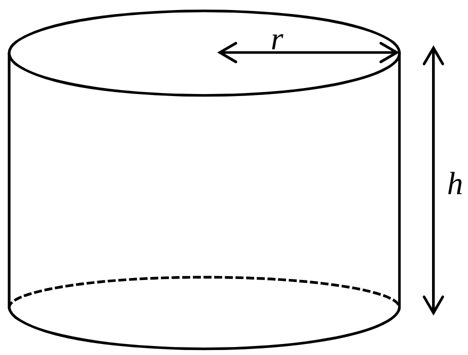
(i) $2x^3 - 3x^2 + 5$. **2**

(ii) $\dfrac{1}{\sqrt{2x+3}}$. **2**

(iii) $\dfrac{3x - 2x^2}{x^4}$. **2**

Question 8 (13 marks) Use a SEPARATE writing booklet.

(a) A cylindrical water tank is open at the top and has a fixed volume of V cm^3.
The radius of the cylinder is r cm and the perpendicular height is h cm.



(i) Write an expression for the height h of the cylinder in terms of V and r. **1**

(ii) Show that the surface area of the tank is given by $S = \dfrac{2V}{r} + \pi r^2$. **1**

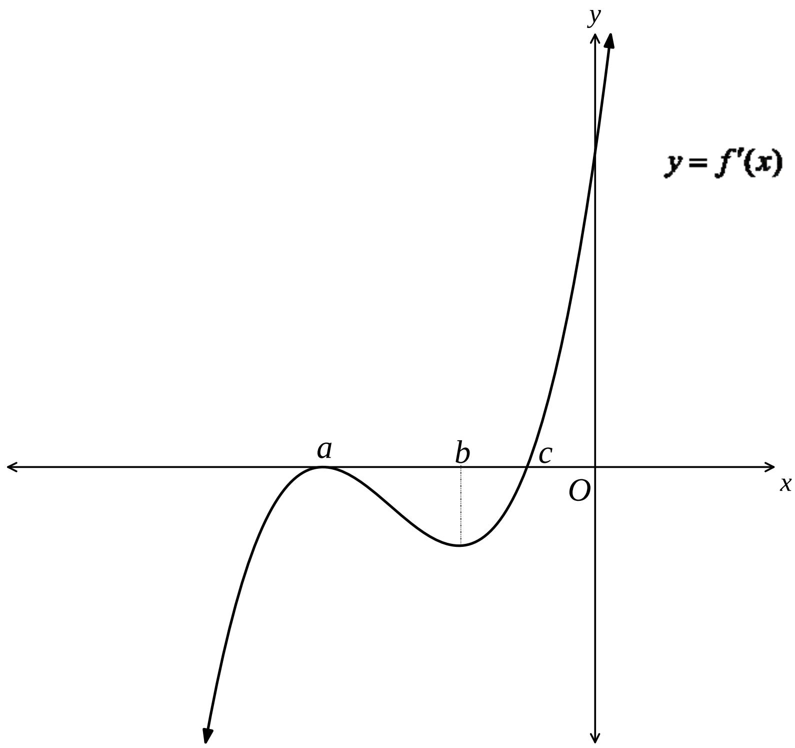
(iii) Show that S is a minimum when $r = \left(\dfrac{V}{\pi}\right)^{\frac{1}{3}}$. **3**

Question 8 (continued)

(b) By using a suitable substitution, or otherwise, solve the following equation for x.

$$2\left(x^2+1\right)^2-11\left(x^2+1\right)+5=0\,. \qquad \textbf{3}$$

(c) The graph of the gradient function $y=f'(x)$ is given below.



(i) For what values of x is the original function $f(x)$ increasing? **1**

(ii) Where is $f(x)$ concave up? **2**

(iii) Where does $f(x)$ have a turning point? **1**

(iv) What feature exists on the graph of $f(x)$ at $x=a$? **1**

Question 9 (12 marks) Use a SEPARATE writing booklet.

(a) Find the values of A, B and C such that :

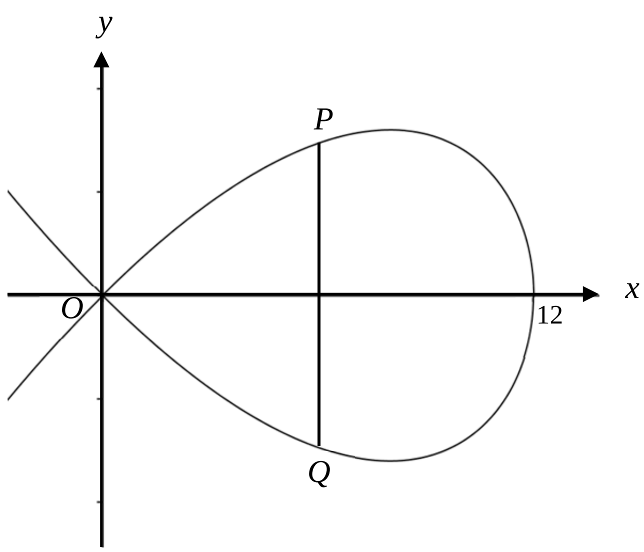
$$x^2 - 3x + 5 \equiv A(x-1) + Bx(x+1) + Cx$$ **3**

(b) The second derivative of a function $f''(x) = 6x - 4$ and $f(x)$ has a stationary point at $\left(\frac{2}{3}, \frac{2}{3}\right)$. Find the equation of $f(x)$. **3**

(c) PQ is a chord, parallel to the y-axis, of the loop of the curve $y^2 = x^2(12 - x)$, as shown below, where $0 < x < 12$.

(i) Show that y^2 has a maximum in this domain at $x = 8$. **3**

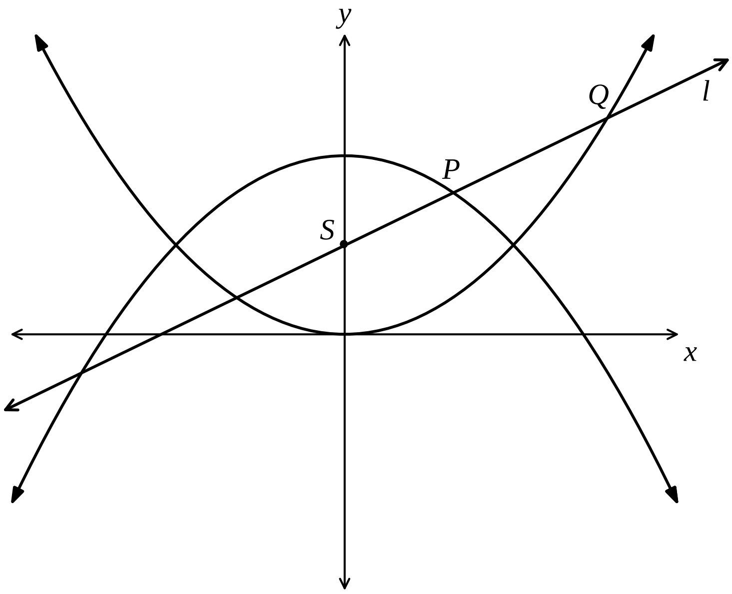
(ii) Hence, or otherwise, find the maximum length of the chord PQ. **1**



Question 9 continues on page 8

Question 9 (continued)

(d) The diagram below, shows two parabolas $x^2 = 4y$ and $x^2 = -4(y-2)$.
They have a common focus at S.



The line l passes through S and cuts the two parabolas at $P(a,b)$ and $Q(c,d)$ as shown.

By considering the locus definition of a parabola, or otherwise, show that the length of $PQ = b + d - 2$. **2**

End of paper

Section I

1 C

Focus to the right of directrix. $\therefore$ Sideways facing parabola with focal length 2 and vertex at $(-2,2)$.

$$(y-2)^2 = 8(x+2)$$

2 C

Minimum occurs on axis of symmetry

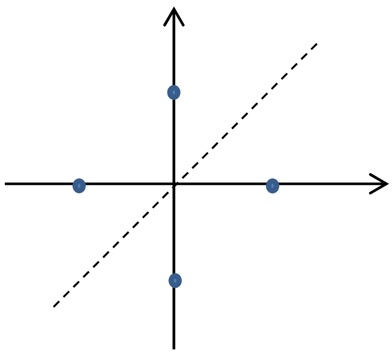
Axis = $-\dfrac{b}{2a} = \dfrac{7}{2}$

Sub $x = \dfrac{7}{2}$ $\quad y = -\dfrac{9}{4}$

3 B

Dots show possible positions of $(p,0)$ and $(0,p)$ for positive and negative p.

Dotted line shows locus



4 D

Gradient of line is $-\dfrac{2}{3}$ which is the gradient of the normal at $x=1$. The gradient of tangent at $x=1$ is $f'(1)$. $\therefore$ gradient of line and $f'(1)$ multiply to -1. $\therefore f'(1) = \dfrac{3}{2}$

5 C

Cannot be B or D as they are even functions (symmetric about y-axis). Only C has

$g'(x) > 0$ for $x > 2$

Section II

Question 6

(a) Focal length = 4 and parabola faces up

$$(x+3)^2 = 16(y-2)$$

(b) $PA = 2.PB \quad \Rightarrow \quad PA^2 = 4.PB^2$

$$(x+1)^2 + (y-5)^2 = 4\left[(x-2)^2 + (y+4)^2\right]$$

$$x^2 + 2x + 1 + y^2 - 10y + 25$$

$$= 4x^2 - 16x + 16 + 4y^2 + 32y + 64$$

$$3x^2 - 18x + 3y^2 + 42y + 54 = 0$$

$$x^2 - 6x + y^2 + 14y + 18 = 0$$

is the equation of the locus of P.

(c) Stationary points occur when $f'(x) = 0$

$$3 - 3x^2 = 0$$

$$3(1 - x^2) = 0$$

$$x = \pm 1$$

$\therefore (-1,3)$ and $(1,7)$ are stationary points.

$$f''(x) = -6x$$

At $(-1,3)$ $f''(x) = -6 \times -1 = 6 > 0$

At $(1,7)$ $f''(x) = -6 \times 1 = -6 < 0$

$\therefore (-1,3)$ is a point of local minimum and $(1,7)$ is a point of local maximum.

(ii) Inflexion points occur when $f''(x) = 0$ and concavity changes.

$f''(x) = 0 \Rightarrow -6x = 0 \quad \Rightarrow \quad x = 0$

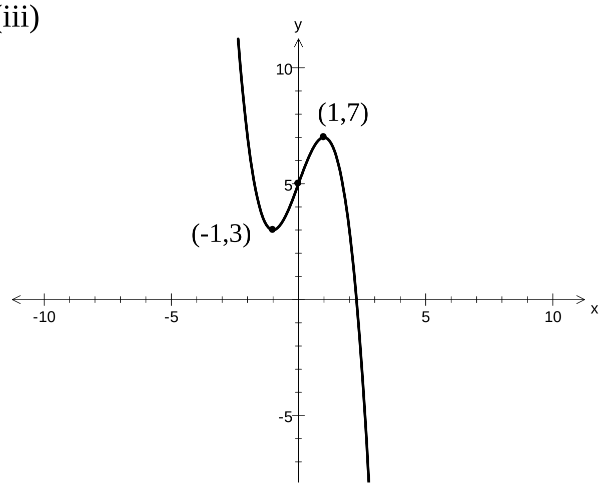
$\therefore$ there is a possible point of inflexion at $x = 0$.

Test change in concavity:

x	-0.5	0	+0.5
$f''(x)$	3	0	-3
	>0		<0
	Concave up		Concave down

There is a change in concavity. $\therefore (0,5)$ is a point of inflexion.

(iii)



Question 7

(a) For points of intersection, we solve the line and the parabola simultaneously.

$$y = 6x^2 \text{ and } y = 16x + K$$
$$6x^2 = 16x + K$$
$$6x^2 - 16x - K = 0$$

If they do not intersect, this equation has no solutions. i.e, $\Delta < 0$

$$256 - 4\times 6\times -K < 0$$
$$24K < -256$$
$$K < -\frac{32}{3}$$

(b) (i) $\alpha + \beta = \frac{11}{2}$

(ii) $\alpha\beta = \frac{3}{2}$

(iii) $(\alpha - 1)(\beta - 1) = \alpha\beta - \alpha - \beta + 1$

$$= \alpha\beta - (\alpha + \beta) + 1$$
$$= \frac{3}{2} - \frac{11}{2} + 1$$
$$= -3$$

(c) (i) Let $f(x) = 2x^3 - 3x^2 + 5$

$$F(x) = \frac{x^4}{2} - x^3 + 5x + C$$

(ii) Let $f(x) = \frac{1}{\sqrt{2x+3}} = (2x+3)^{-\frac{1}{2}}$

$$F(x) = \frac{(2x+3)^{\frac{1}{2}}}{\frac{1}{2}\times 2}$$
$$= \sqrt{2x+3} + C$$

(iii) $f(x) = \frac{3}{x^3} - \frac{2}{x^2} = 3x^{-3} - 2x^{-2}$

$$F(x) = -\frac{3}{2x^2} + \frac{2}{x} + C$$

Question 8

(a) (i) $V = \pi r^2 h \quad \Rightarrow \quad h = \frac{V}{\pi r^2}$

(ii) $S = 2\pi r h + \pi r^2$

$$= 2\pi r \times \frac{V}{\pi r^2} + \pi r^2$$
$$= \frac{2V}{r} + \pi r^2$$

(iii) Stationary points occur when $\frac{dS}{dr} = 0$

i.e. when $-\frac{2V}{r^2} + 2\pi r = 0$

$$2\pi r = \frac{2V}{r^2}$$
$$r^3 = \frac{2V}{2\pi} = \frac{V}{\pi}$$
$$\therefore \quad r = \left(\frac{V}{\pi}\right)^{\frac{1}{3}}$$

$$\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 2\pi > 0 \quad (\because r > 0, V > 0)$$

Or at $r = \left(\frac{V}{\pi}\right)^{\frac{1}{3}}$; $\frac{d^2S}{dr^2} = 4V \div \frac{V}{\pi} + 2\pi = 6\pi > 0$

$\therefore$ S has a minimum at $r = \left(\frac{V}{\pi}\right)^{\frac{1}{3}}$

(b) Let $u = x^2 + 1$

$$2u^2 - 11u + 5 = 0$$
$$(2u-1)(u-5) = 0$$
$$u = \frac{1}{2} \text{ or } u = 5$$
$$x^2 + 1 = \frac{1}{2} \text{ or } x^2 + 1 = 5$$
$$x^2 = -\frac{1}{2} \quad \text{or} \quad x^2 = 4$$

$$x^2 = -\frac{1}{2} \Rightarrow \text{no real solns}$$
$$x^2 = 4 \Rightarrow x = \pm 2$$

(c) (i) $x > c$

(ii) $x < a$ and $x > b$

(iii) A minimum turning point at $x = c$

(iv) A stationary point of inflexion

Question 9

(a) *Compare coefficients of* x^2 *in LHS and RHS*:

$B = 1$

Sub $x = 0$:

$5 = -A \quad \Rightarrow \quad A = -5$

Sub $x = 1$:

$$1 - 3 + 5 = 2B + C$$
$$C = 3 - 2 = 1$$
$$\therefore A = -5;\ B = 1;\ C = 1$$

(b) $f''(x) = 6x - 4$

$$f'(x) = 3x^2 - 4x + C$$

$$f'\left(\frac{2}{3}\right) = 0$$
$$3 \times \frac{4}{9} - 4 \times \frac{2}{3} + C = 0$$
$$C = \frac{4}{3}$$
$$\therefore f'(x) = 3x^2 - 4x + \frac{4}{3}$$
$$f(x) = x^3 - 2x^2 + \frac{4}{3}x + D$$
$$f\left(\frac{2}{3}\right) = \frac{2}{3}$$
$$\frac{8}{27} - 2 \times \frac{4}{9} + \frac{4}{3} \times \frac{2}{3} + D = \frac{2}{3}$$
$$D = \frac{10}{27}$$
$$f(x) = x^3 - 2x^2 + \frac{4}{3}x + \frac{10}{27}$$

(c) (i) Let $s = y^2 = x^2(12 - x)$

$$s = 12x^2 - x^3$$

Stationary points occur when $s' = 0$

$$24x - 3x^2 = 0$$
$$3x(8 - x) = 0$$
$$x = 0 \text{ or } x = 8$$

But $0 < x < 12$

Only stationary point within domain is $x = 8$

$s'' = 24 - 6x$

At $x = 8$ $s'' = 24 - 6 \times 8 = -24 < 0$

$\therefore s$ has a maximum at $x = 8$

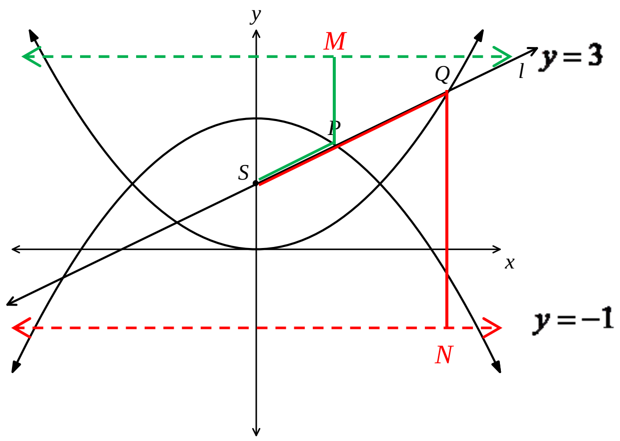
i.e. y^2 has a maximum at $x = 8$.

(ii) At $x = 8$, $y^2 = 8^2 \times (12 - 8) = 256$

$y = \pm 16$

Length of chord PQ = $2 \times 16 = 32$ units

(d) In the diagram below, add the two directrices at $y=-1$ and $y=3$. Draw perpendiculars PM and QN to the directrices from P and Q.



Now, by the locus definition of the parabola
$SQ = QN = d+1$

And $SP = PM = 3-b$

$$\begin{aligned} PQ &= SQ - SP \\ &= (d+1)-(3-b) \\ &= b+d-2 \end{aligned}$$

End of solutions