NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Assessment Task 1
Term 4, 2013

Name:____________________________ **Mathematics Class:**_________

Student Number:____________________________

Time Allowed: 50 minutes + 2 minutes reading time
Total Marks: 41 marks

Instructions:

- Attempt all questions.

SECTION 1

- Answer on the Multiple Choice answer sheet provided.

SECTION 2

- Start each question in a new booklet. Put your number on every booklet.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Work down the page.
- Do not work in columns.
- Each question will be collected separately.
- Submit a **blank booklet** if you do not attempt a question.

	Q1&2	**Q3**	**Q4&5**	**Q6 a)b)c)**	**Q6 d)**	**Q7a)**	**Q7b)**	**Q8a)**	**Q8b)**	**Q8c)**	
P4	**/2**			/10				**/2**			**/14**
H5							**/3**		**/3**	**/7**	**/13**
H6						**/9**					**/9**
H7		**/1**	**/2**		**/2**						**/5**
											/41

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Section I

5 marks
Attempt Questions 1–5
Allow about 7 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5

1. The parabola $y = kx^2 - 2x - 1$ is negative definite. What are the possible values of k?

(A) $k < -1$

(B) $k \leq -1$

(C) $k < 1$

(D) $k \leq 1$

2. What is the equation of a parabola that has a vertex of (0,0) and a focus (2,0)?

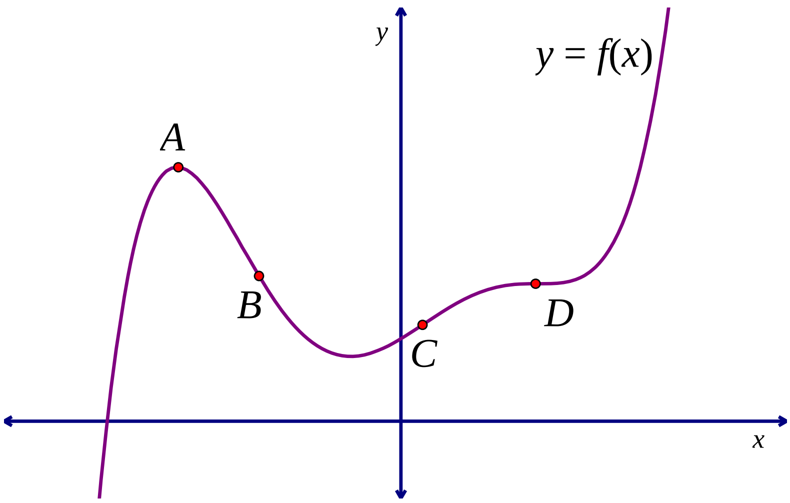
(A) $(x-2)^2 = -8y$

(B) $x^2 = -8y$

(C) $(y-2)^2 = 8x$

(D) $y^2 = 8x$

3. Below is the graph of $y = f(x)$



Which point has $f'(x) < 0$ and $f''(x) = 0$

(A) A

(B) B

(C) C

(D) D

4. What is the primitive function of $f(x) = 2x^2 + \dfrac{5}{x^2}$?

(A) $4x - \dfrac{10}{x^3} + C$

(B) $6x^3 - \dfrac{5}{x} + C$

(C) $\dfrac{2x^3}{3} + \dfrac{5}{3x^3} + C$

(D) $\dfrac{2x^3}{3} - \dfrac{5}{x} + C$

5. If $x^2 + 2x + 2 \equiv A\left(x^2 - 1\right) + B(x+1) + 3C$, what is the value of C?

(A) $C = -\dfrac{1}{3}$

(B) $C = -\dfrac{5}{3}$

(C) $C = \dfrac{5}{3}$

(D) $C = \dfrac{1}{3}$

End of Section I

Section II

36 marks
Attempt Questions 6–8
Allow about 42 minutes for this section

Answer each question in the appropriate writing booklet. Extra pages are available.

In Questions 6 - 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (12 Marks) Use the Question 6 Writing Booklet.

(a) The equation $3x^2 - 4x - 2 = 0$ has roots α and β.
Find the value of:

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

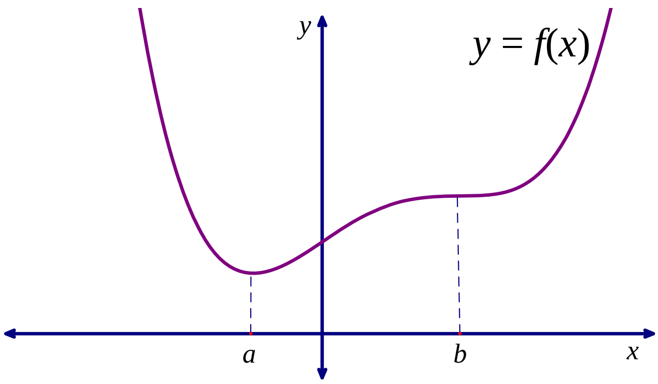
(iii) $\alpha^2 + \beta^2$ **2**

(b) Use the substitution $u = x^2 - 2$ or otherwise to solve **3**

$$4\left(x^2 - 2\right)^2 - 29\left(x^2 - 2\right) + 7 = 0$$

(c) The point $P(x, y)$ moves so that it is equidistant from $(2,1)$ and $(3,3)$. **3**
Find the equation of the locus of P.

(d) Below is the graph of $y = f(x)$ **2**



Sketch the graph of $y = f'(x)$, clearly indicating the points where $x = a$ and $x = b$.

End of Question 6

Question 7 (12 Marks) Use the Question 7 Writing Booklet.

(a) Consider the function $f(x) = x^3 - 3x^2$.

(i) Find the coordinates of any stationary points of $f(x)$ and determine their nature. **3**

(ii) Find the coordinates of any inflexion points. **2**

(iii) Hence sketch the graph of $y = f(x)$. Clearly indicate the stationary points, any points of inflexion and any intercepts with the axes. **3**

(iv) Hence determine the values of k so that the equation $x^3 - 3x^2 = k$ has three solutions. **1**

(b)

(i) Express the equation of the parabola $x^2 + 2x + 8y + 25 = 0$ in the form: **2**

$$(x-h)^2 = -4a(y-k)$$

(ii) Hence find the equation of the directrix of the parabola. **1**

End of Question 7

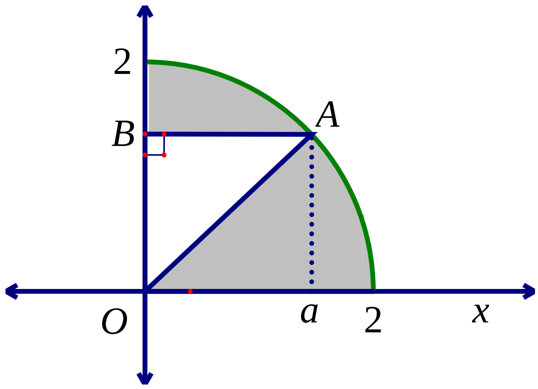
Question 8 (12 Marks) Use the Question 8 Writing Booklet.

(a) Find the values of k such that $y = x^2 + 8x - 5$ and $y = -2x^2 + x + k$ Do not intersect. 2

(b) The derivative of a function $f(x)$ is $f'(x) = 2x + 1$. The line $y = -3x + 2$ is a tangent to the graph of $f(x)$. 3

Find the function $f(x)$.

(c) The diagram represents the curve $x^2 + y^2 = 4$ where $x \geq 0$ and $y \geq 0$.
The point A lies on the curve and B is the foot of the perpendicular from A to the $y-$axis.
The x-coordinate of A is a.



(i) Show that the shaded area S in terms of a is **2**

$$S = \pi - \frac{a\sqrt{4 - a^2}}{2}$$

(ii) Show that $\dfrac{dS}{da} = \left(\dfrac{a^2 - 2}{\sqrt{4 - a^2}} \right)$ **2**

(iii) Find the minimum area of the shaded region. **3**

End of Paper

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SECTION *I*

Q1 A

$y = kx^2 - 2x - 1$

$\Delta = (-2)^2 - 4(k)(-1)$

Negative definite when:

$\Delta < 0$ leading coefficient < 0

$4 + 4k < 0$

$4k < -4$

$k < -1$

Q2 D

Form is $y^2 = 4ax$

Focal length is 2

Solution is $y^2 = 8x$

Q3 B

Gradient is negative at B and it is a change in concavity.

Q4 D

$f(x) = 2x^2 + \frac{5}{x^2}$

$f(x) = 2x^2 + 5x^{-2}$

$F(x) = 2x^3 - 5x^{-1} + C$

$F(x) = 2x^3 - \frac{5}{x} + C$

Q5 D

$x^2 + 2x + 2 \equiv A(x^2 - 1) + B(x+1) + 3C$

Sub $x = -1$ into both LHS and RHS

$(-1)^2 + 2(-1) + 2 = A((-1)^2 - 1) + B((-1)+1) + 3C$

$1 - 2 + 2 = 3C$

$C = \frac{1}{3}$

SECTION *II*

Question 6

$3x^2 - 4x - 2 = 0$

a)i)

$\alpha + \beta = -\frac{b}{a}$

$= -\frac{-4}{3} = \frac{4}{3}$

a)ii)

$\alpha\beta = \frac{c}{a}$

$= \frac{-2}{3}$

a)iii)

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(\frac{4}{3}\right)^2 - 2\left(-\frac{2}{3}\right)$

$= \left(\frac{16}{9}\right) + \left(\frac{4}{3}\right)$

$= \frac{28}{9}$

b)

$4(x^2 - 2)^2 - 29(x^2 - 2) + 7 = 0$

$u = x^2 - 2$

$4(u)^2 - 29(u) + 7 = 0$

$(4u - 1)(u - 7) = 0$

$u = \frac{1}{4}$	$u = 7$
$x^2 - 2 = \frac{1}{4}$	$x^2 - 2 = 7$
$x^2 = \frac{9}{4}$	$x^2 = 9$
$x = \pm\frac{3}{2}$	$x = \pm 3$

c)

let the distance from $P(x,y)$ to (2,1) be d_1

let the distance from $P(x,y)$ to (3,3) be d_2

$d_1 = \sqrt{(x-2)^2 + (y-1)^2}$

$d_2 = \sqrt{(x-3)^2 + (y-3)^2}$

If $d_1 = d_2$

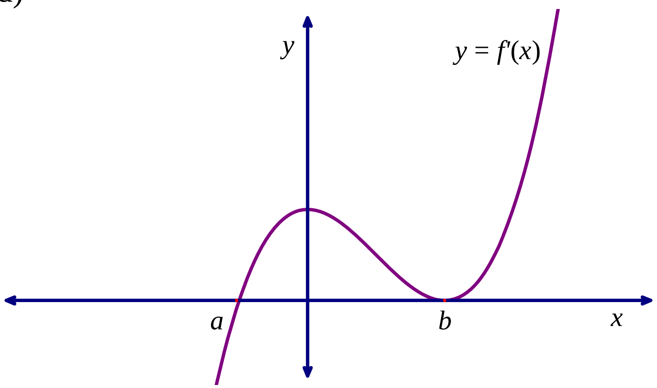
$(x-2)^2 + (y-1)^2 = (x-3)^2 + (y-3)^2$

$x^2 - 4x + 4 + y^2 - 2y + 1 = x^2 - 6x + 9 + y^2 - 6y + 9$

Locus of *P*

$2x + 4y - 13 = 0$

d)



Question 7

a) i)

$f(x)=x^3-3x^2$

$f'(x)=3x^2-6x$

$f''(x)=6x-6$

Stationary points where

$f'(x)=0$

$f'(x)=3x^2-6x$

$0=3x^2-6x$

$0=3x(x-2)$

Stationary Points at

$x=0 \quad and \quad x=2$

$y-Values$

$f(0)=(0)^3-3(0)^2=0$

$f(2)=(2)^3-3(2)^2=-4$

Nature of stationary points

Test in 2nd derivative

$f''(0)=6(0)-6=-6$

$\therefore$ MAX turning point at (0,0)

$f''(2)=6(2)-6=6$

$\therefore$ MIN turning point at $(2,-4)$

a) ii)

Inflection point at change in concavity

Possible point of inflexion where

$f''(x)=0$

$0=6x-6$

$6=6x$

$x=1$

Test either side of $x=1$ in second derivative

x	0	1	2
$f''(x)$	-6	0	6

Change of concavity at $x=1$

When $x=1$ $f(1)=(1)^3-3(1)^2=-2$

Point of inflexion at (1,-2)

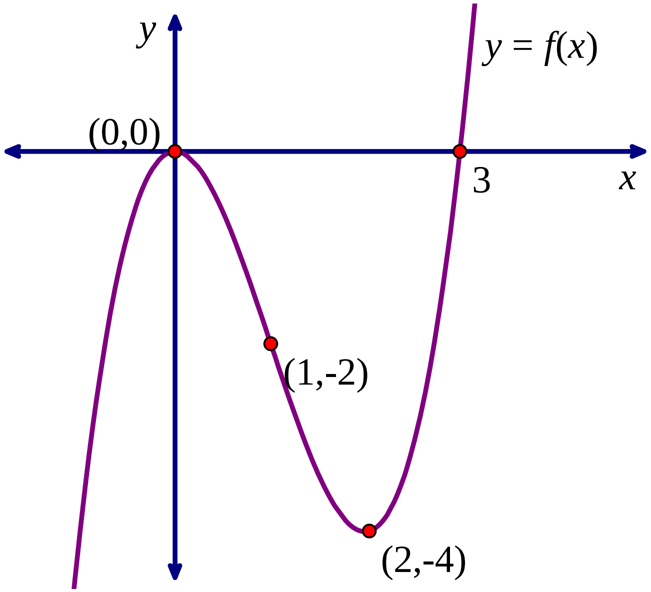
a) iii)

x – intercepts

$0=x^3-3x^2$

$0=x^2(x-3)$

$x=0 \quad amd \quad x=3$



a) iv)

$x^3-3x^2=k$

From the graph the equation will have three solutions if $-4<k<0$

b) i)

$x^2+2x+8y+25=0$

Complete the square.

$x^2+2x+1=-8y-24$

$(x+1)^2=-8(y+3)$

b) ii)

Vertex is at (-1,-3)

Focal Length is 2

Concave Down Parabola

Directrix is at

$y=-1$

Question 8

a)

Solve the two $y=x^2+8x-5$ and $y=-2x^2+x+k$ equations simultaneously

$-2x^2+x+k=x^2+8x-5$

$0=3x^2+7x-k-5$

No intersections if $\Delta<0$

$\Delta=7^2-4\times3\times(-k-5)$

$\Delta=49+12k+60$

$\Delta=109+12k$

$109+12k<0$

$12k<-109$

$k<\frac{-109}{12}$

$k<-9\frac{1}{12}$

b)

$f'(x) = 2x + 1$

$F(x) = x^2 + x + C$

METHOD 1

As $y = -3x + 2$ is a tangent then where $f'(x) = -3$ will be the x-value of the point of intersection.

Let

$-3 = 2x + 1$

$-4 = 2x$

$x = -2$

y- Value of the point of intersection is

$y = -3(-2) + 2$

$y = 8$

$\therefore (-2{,}8)$ lies on the curve

$y = x^2 + x + C$

$\therefore 8 = (-2)^2 + (-2) + C$

$8 = 4 - 2 + C$

$C = 6$

$\therefore\ f(x) = x^2 + x + 6$

METHOD 2

The tangent $y = -3x + 2$ and function $F(x) = x^2 + x + C$ will have only one intersection point. Solving simultaneously will only provide one solution.

$x^2 + x + C = -3x + 2$

$x^2 + 4x + C - 2 = 0$

$\Delta = 4^2 - 4 \times 1 \times (C - 2)$

$\Delta = 16 - 4C + 8$

$\Delta = 24 - 4C$

One solution when . $\Delta = 0$

$24 - 4C = 0$

$C = 6$

$\therefore\ f(x) = x^2 + x + 6$

c) i)

$\text{Area}_{Shaded} = \text{Area}_{QuarterCircle} - \text{Area}_{Triangle}$

$\text{Area}_{Shaded} = \dfrac{\pi \times r^2}{4} - \dfrac{1}{2} \times OB \times BA$

$OB = \sqrt{4 - BA^2}$ by Pythagoras' Theorem

$\text{Area}_{Shaded} = \dfrac{\pi \times 2^2}{4} - \dfrac{1}{2} \times \sqrt{4 - a^2} \times a$

$\text{Area}_{Shaded} = \pi - \dfrac{a\sqrt{4 - a^2}}{2}$

c) ii)

$S = \pi - \dfrac{a\sqrt{4 - a^2}}{2}$

$\dfrac{dS}{da} = u'v + v'u$

$= -\dfrac{1}{2}\left[1 \times \sqrt{4 - a^2} + \dfrac{1}{2}\left(4 - a^2\right)^{-\frac{1}{2}}(-2a)(a)\right]$

$= -\dfrac{1}{2}\left[\sqrt{4 - a^2} - \dfrac{a^2}{\sqrt{4 - a^2}}\right]$

$= -\dfrac{1}{2}\left[\dfrac{4 - a^2 - a^2}{\sqrt{4 - a^2}}\right]$

$= \dfrac{a^2 - 2}{\sqrt{4 - a^2}}$

C iii)

Possible min when $\dfrac{dS}{dt} = 0$

$0 = \dfrac{a^2 - 2}{\sqrt{4 - a^2}}$

$0 = a^2 - 2$

$a^2 = 2$

$a = \pm\sqrt{2}$

But $a > 0$

When $a = \sqrt{2}$ the shaded area is a possible minimum.

Test either side of $a = \sqrt{2}$ in $\dfrac{dS}{dt}$

a	1.4	$\sqrt{2}$	1.5
$\dfrac{dS}{dt}$	$\dfrac{(1.4)^2 - 2}{\sqrt{4 - (1.4)^2}} < 0$	0	$\dfrac{(1.5)^2 - 2}{\sqrt{4 - (1.5)^2}} > 0$

$\therefore$ When $a = \sqrt{2}$ the shaded area is a minimum

When $a = \sqrt{2}$

$S = \pi - \dfrac{\sqrt{2}\sqrt{4 - \left(\sqrt{2}\right)^2}}{2}$

$= \pi - \dfrac{\sqrt{2}\sqrt{2}}{2}$

$= \pi - 1$ units^2