NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Assessment Task 1
Term 4, 2014

Name:_______________________________ Mathematics Class: 11Ma__

Time Allowed: **55 minutes + 2 minutes reading time**
Total Marks: **48**

Instructions:

- Attempt all questions.
  - Section I has five multiple choice questions.
  - Section II has three free response questions of approximately equal value.
- Start each free response question in a new booklet.
- Put your name on every booklet. Submit a blank booklet if you do not attempt a question.
- Show all necessary working. Marks may be deducted for incomplete or poorly arranged work.
- Work down the page. Do not work in columns.

Question	1-4	5	6a-c	6d	7 a	7b	7c	8 a	8bc	Total
P4	/4		/8		/3			/3		/18
H5				/6		/3				/9
H6							/9			/9
H7		/1								/1
H9									/11	/11
										48

Section I

5 marks
Attempt Questions 1 – 5

Use the multiple-choice answer sheet for Questions $1-5$.

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1 Which of the following expressions is positive definite?

(A) $3+2x-x^2$

(B) $5-3x+x^2$

(C) $4+8x+3x^2$

(D) $(x-1)(x+2)$

2 What is the equation of the locus of points equidistant from the point $(3,1)$ and the line $x=-1$?

(A) $(y-1)^2=8(x-1)$

(B) $(y-1)^2=16(x-3)$

(C) $(x-3)^2=4y$

(D) $(x-3)^2=8(y-1)$

3 The roots of the quadratic equation $kx^2-x+m=0$ are 1 and -3. The value of m is

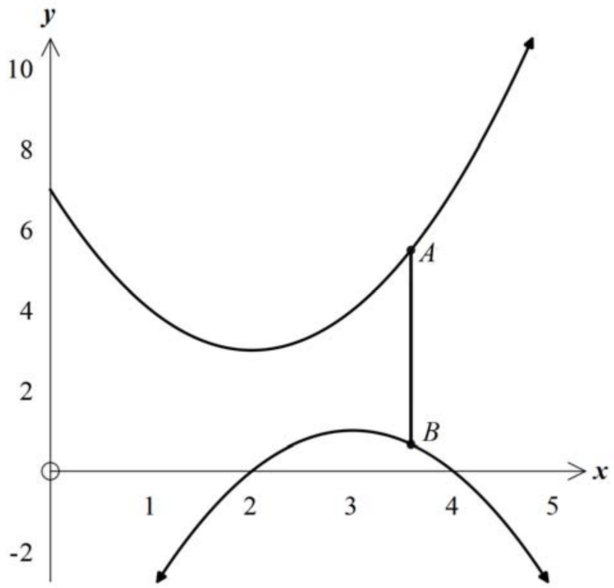
(A) 6

(B) -3

(C) $\frac{3}{2}$

(D) -2

4 The diagram shows the graphs of $y=(x-2)^2+3$ and $y=1-(x-3)^2$.



Points A and B have the same x coordinate. Which expression gives the length of the vertical interval AB?

(A) $2x-1$

(B) $-10x-1$

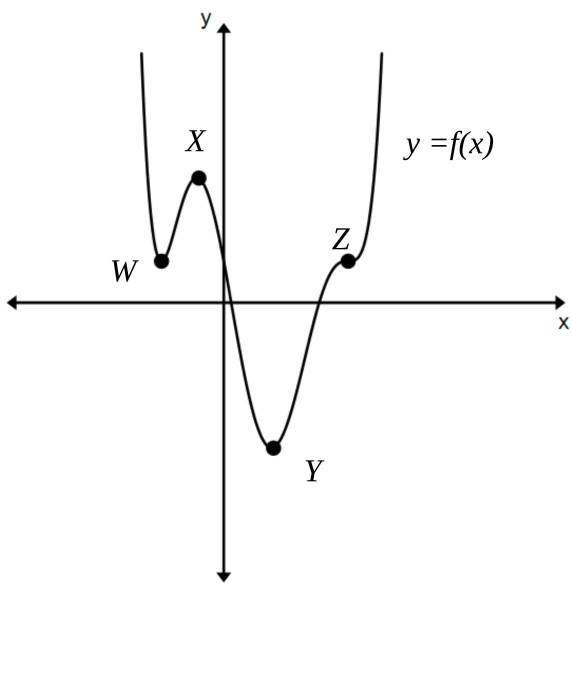
(C) $2x^2-10x+15$

(D) $2x^2+2x+15$

5 The diagram below shows the graph of $y=f(x)$. Points W, X, Y and Z are stationary points and Z is a point of inflexion.

Which of the points on $y=f(x)$ corresponds to the description:

$$y>0,\quad \frac{dy}{dx}=0,\quad \frac{d^2y}{dx^2}>0$$



(A) W

(B) X

(C) Y

(D) Z

Section II

43 marks
Attempt Questions 6 – 8

Answer each question in a SEPARATE writing booklet.

In Questions 6 – 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (14 marks) Use a separate writing booklet

(a) Solve $x^4 + 48 = 19x^2$ **2**

(b) Find the constants *A, B* and *C* such that **3**

$$2x^2 - 9x + 14 = A(x-1)(x-2) + B(x-1) + C$$

(c) Find the range of values of *p* for which the roots of the equation $x^2 + px + 4 = 0$ are

(i) real **2**

(ii) real and positive **1**

(d) During the Ice Age, the amount of sunshine in summer at latitude 50° N (in Frankfurt in Germany and Vancouver in Canada) is given by the equation

$$C = -0.09t^3 - 4.45t^2 - 62t + 745 \qquad \text{for } -25 \le t \le -5$$

The amount of sunshine *C* is measured in calories per cm^2, and time *t* is measured in thousands of years with $t = 0$ representing the present time.

(i) Show that when $t = -10$ the amount of summer sunshine was at its maximum. **3**

(ii) What was the value of *C* at that time? **1**

(iii) How many thousands of years ago was the value of *C* at its minimum? **2**

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) By writing the equation of the parabola $y = x^2 - 2x + 2$ in the form $(x-h)^2 = 4a(y-k)$, find the coordinates of the focus and the equation of the directrix. **3**

(b) Find the primitive function of the following

(i) $2 + 3x - x^2$ **2**

(ii) $\dfrac{3}{x^2}$ **1**

(c) Consider the curve given by $y = 3x^4 - 4x^3 - 5$

(i) Find the coordinates of the two stationary points. **3**

(ii) Find all the values of x for which $\dfrac{d^2y}{dx^2} = 0$. **2**

(iii) Determine the nature of the stationary points. **2**

(iv) Sketch the curve for the domain $-1 \le x \le 2$. **2**

Question 8 (14 marks) Use a SEPARATE writing booklet.

(a) The quadratic equation $ax^2 + bx + c = 0$ has roots α and 3α.

(i) Write down expressions for the sum and the product of roots in terms of the coefficients *a, b* and c. **1**

(ii) Show that $3b^2 = 16ac$ **2**

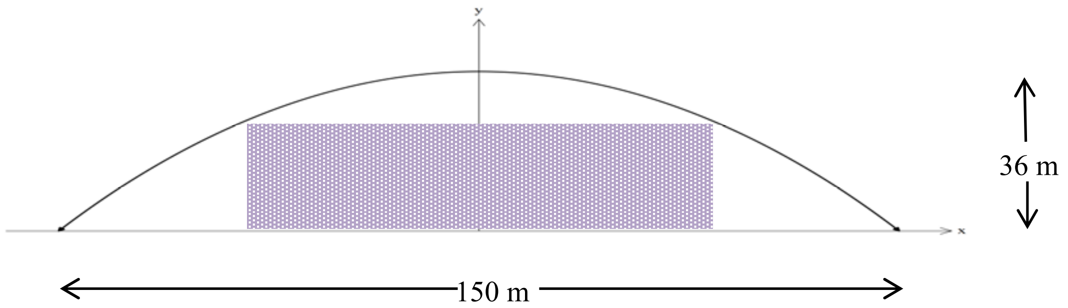
(b) The graphs of $y = x^2 - 3x - 5$ and $y = x + k$ have only one point of intersection, at *P*.

(i) Show that the *x* coordinate of *P* satisfies the equation $x^2 - 4x - 5 - k = 0$ **1**

(ii) Find the value of *k*. **2**

(iii) Find the coordinates of *P*. **2**

(c) The cross section of a large building with a parabolic roof is shown. The height of the tallest point of the roof is 36 metres while the arc spans a horizontal distance of 150 metres. A rectangular structure is to be built below the roof, as shown.



(i) If the diagram is placed on the Cartesian plane, with the *y* axis being the axis of the parabola, show that the area of the rectangle is given by **2**

$$A = \frac{8x}{625}(75 - x)(75 + x)$$

(ii) Find the dimensions of the rectangle so that its area is maximised. **4**

End of Paper

1. Positive definite : $a > 0$ and $\Delta < 0$ (no roots)

Try B : $5 - 3x + x^2$

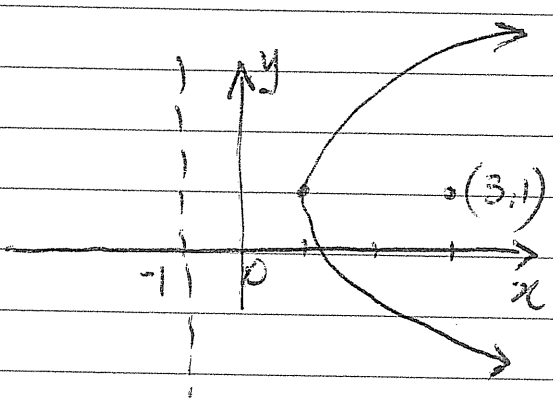
$\Delta = (-3)^2 - 4 \times 5 \times 1$

$= -11$

B

Yes, its positive definite

2. Sketch :



(3,1) is the focus

$x = -1$ is the directrix

Vertex is (1, 1)

Focal length = 2

Form $(y-k)^2 = 4a(y-k)$

A

3. Sum of roots : $1 + -3 = -\frac{(-1)}{2k}$

$-2 = \frac{1}{2k}$

$k = -\frac{1}{2}$

Product of roots = $1 \times -3 = \frac{m}{k}$

$-3 = \frac{m}{k}$

(c)

$m = -3 \times -\frac{1}{2}$

$= \frac{3}{2}$

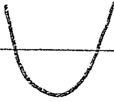
4. $AB = y_2 - y_1$

$= (x-2)^2 + 3 - \left[1 - (x-3)^2\right]$

$= x^2 - 4x + 4 + 3 - \left[1 - (x^2 - 6x + 9)\right]$

$= x^2 - 4x + 7 - 1 + x^2 - 6x + 9$

$= 2x^2 - 10x + 15$ (C)

5. $y > 0$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} > 0$  concave up

$\therefore$ W (A)

Question 6

a) $x^4 - 19x^2 + 48 = 0$

$(x^2 - 16)(x^2 - 3) = 0$

$x^2 = 16$ or $x^2 = 3$

$x = \pm 4, \pm\sqrt{3}$

2

Done very well. Only silly mistakes made.

b) $2x^2 - 9x + 14 = A(x-1)(x-2) + B(x-1) + C$

Sub $x=1$ into both sides: $2 - 9 + 14 = 0 + 0 + C$

$7 = C$

Sub $x=2$: $8 - 18 + 14 = 0 + B + C$

$4 = B + 7$

$B = -3$

3

Equate coefficients of x^2 on both sides : $2 = A$

$\therefore A = 2, \quad B = -3, \quad C = 7$

Check: equate constants : $14 = 2A - B + C$

$= 2(2) - (-3) + 7$

$= 14$ true.

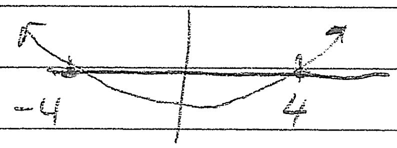
Please note that the RHS does not need to be expanded for us to be able to find A, B, C. Keeping the factored form on RHS allows for suitable substitutions $x=1$, $x=2$ to be used first.
Ext 1 and Ext 2 students particularly need to stop expanding RHS, unless necessary.

c) $x^2 + px + 4 = 0$

Real roots if $\Delta \geq 0$

$p^2 - 16 \geq 0$

$(p-4)(p+4) \geq 0$



$p \leq -4$ or $p \geq 4$. 2

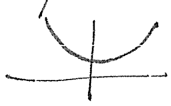
(ii) If roots are both positive then sum must be positive

$\therefore \dfrac{-p}{1} > 0$

$\therefore p \leq -4$ only solution 1

Most students did not think about the fact that p is the coefficient of x, and therefore must be related to the sum of roots.

Many students confused roots being positive with a quadratic that is "positive definite". The latter of course has no roots.



d) $C = -0.09t^3 - 4.45t^2 - 62t + 745 \qquad -25 \leq t \leq -5$

(i) $\frac{dC}{dt} = -0.27t^2 - 8.9t - 62$

When $\frac{dC}{dt} = 0$ Max and min C occur

$\therefore -0.27t^2 - 8.9t - 62 = 0$

$$t = \frac{8.9 \pm \sqrt{(-8.9)^2 - 4\times(-0.27)(-62)}}{-0.54}$$

$$= \frac{8.9 \pm \sqrt{12.25}}{-0.54}$$

$= -22.963$ (3 d.p) or -10

$\frac{d^2C}{dt^2} = -0.54t - 8.9$

When $t = -10$, $\frac{d^2C}{dt^2} = -3.5 < 0$

$\therefore$ C is maximised when $t = -10$

Many students correctly worked from $\frac{dC}{dt}$ only, without solving. To get full marks, you needed to sub $t=-10$ into $\frac{dC}{dt}$ to show it was a zero AND find values of $\frac{dC}{dt}$ either side:

eg

t	-11	-10	-9
$\frac{dC}{dt}$	3.23	0	-3.77

(ii) $C = 1010$ calories per cm^2 by substitution of $t = -10$ into C

Students were awarded half a mark for
$C = -0{\cdot}09(-10)^3 - 4{\cdot}45(-10)^2 - 62(-10) + 745$
Quite a few had calculator errors. Use brackets for negatives.

iii) $\frac{dC}{dt} = 0$ when $t = -22{\cdot}963$

$\frac{d^2C}{dt^2} = -0{\cdot}54(-22{\cdot}963) - 8{\cdot}9$

$= 3{\cdot}5$ (1 d.p) 1

> 0

$\therefore C$ is a minimum when $t = -22{\cdot}963$

Since t is measured in thousands of years, then C was a minimum 22963 years ago (nearest year) 1

Those students who used the quadratic formula in (i) already had $t = -22{\cdot}96$ (2 d.p) 1 mark.
The second mark was awarded for recalling that t is measured in thousands of years!

Question 7

a) $y = x^2 - 2x + 2$

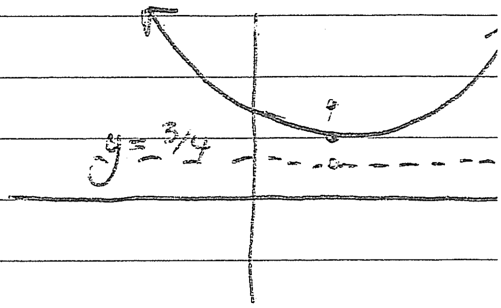
$= x^2 - 2x + 1 + 1$

$= (x-1)^2 + 1$

$\therefore (y-1) = (x-1)^2$ which is in the form $(x-h)^2 = 4a(y-k)$

where (h,k) is $(1,1)$ and $4a = 1$

$a = 1/4$



Focus is $(1, 1\tfrac{1}{4})$

Directrix $y = \frac{3}{4}$

- A minority of students could not successfully complete the square and find vertex.

b) (i) $f(x) = 2 + 3x - x^2$

$F(x) = 2x + \frac{3x^2}{2} - \frac{x^3}{3} + C$

(ii) $f(x) = 3x^{-2}$

$F(x) = \frac{3x^{-1}}{-1} = -\frac{3}{x}$

- Don't forget to $+ C$

c) $y = 3x^4 - 4x^3 - 5$

(i) $\frac{dy}{dx} = 12x^3 - 12x^2$
$= 12x^2(x-1)$

Stat points occur when $\frac{dy}{dx} = 0$

Stat points occur when $x = 0, 1$

Stat points $(0,-5)$ and $(1,-6)$

• well done.

(ii) $\frac{d^2y}{dx^2} = 36x^2 - 24x$
$= 12x(3x-2)$

$\frac{d^2y}{dx^2} = 0$ when $x = 0, \frac{2}{3}$

• well done

(iii)

x	-1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$
$\frac{dy}{dx}$	-24	0	$-\frac{3}{2}$	0	$\frac{27}{2}$

Curve is a polynomial, ie continuous

There is a horizontal point of inflexion at $(0, -5)$ and a minimum turning point at $(1, -6)$

Alternatively using the second derivative

x	-1	0	$\frac{1}{2}$	1
$\frac{d^2y}{dx^2}$	60	0	-3	12

There is a change in concavity when $x = 0$
$\therefore (0, -5)$ is a horizontal point of inflexion

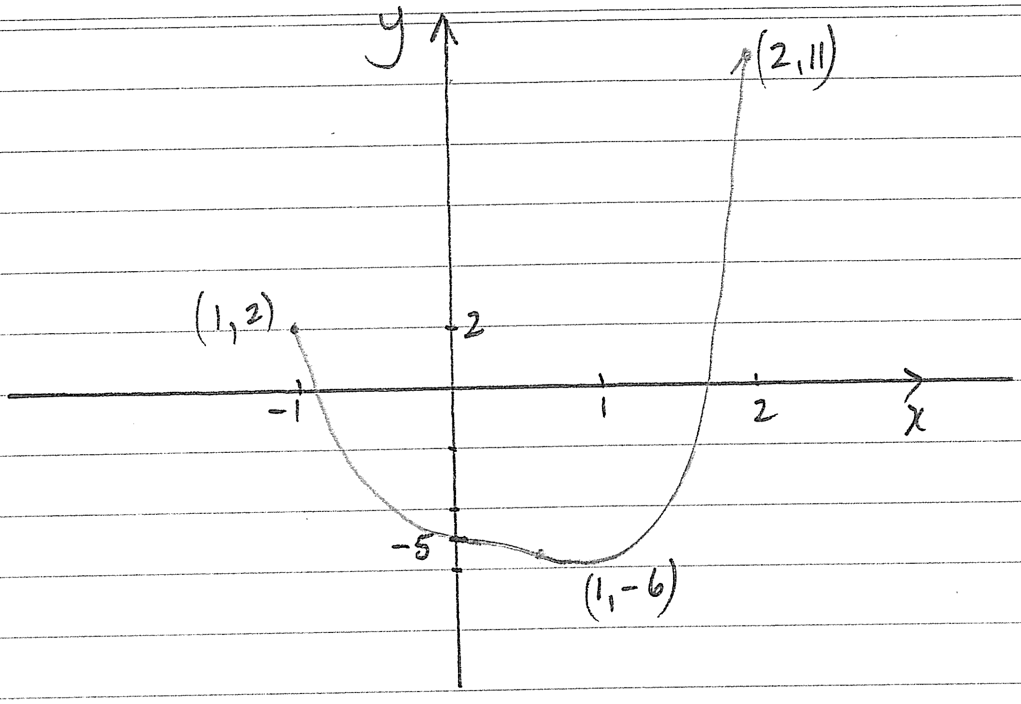
At $(1, -6)$ $\frac{d^2y}{dx^2} > 0$ $\Longrightarrow$ $(1, -6)$ is a minimum turning point

- Need to test using first or second derivative either side of $x = 0$ to confirm horizontal pt. of inflexion.

- When choosing test points, make sure that you don't choose points that have other inflexion points (2nd derivative) or stationary points (1st derivative) in between.

(iv) Boundary points: $x = -1,\ y = 2$

$x = 2,\ y = 11$



- be careful of scale.
- consider the domain.
- draw a <u>horizontal</u> pt of inflexion at $x = 0$.

Question 8

a) $ax^2+bx+c=0$ has roots α and 3α

(i) Sum of roots: $\alpha+3\alpha = -\frac{b}{a}$

$$4\alpha = \frac{-b}{a} \quad \text{.... (1)}$$

Product of roots: $3\alpha^2 = \frac{c}{a}$ .... (2)

(ii) Eliminate α: Sub $\alpha = \frac{-b}{4a}$ into (2)

$$3\left(-\frac{b}{4a}\right)^2 = \frac{c}{a}$$

$$\frac{3b^2}{16a^2} = \frac{c}{a}$$

$$3b^2 = \frac{16a^2c}{a}$$

$$3b^2 = 16ac$$

b) Points of intersection occur when $x^2-3x-5 = x+k$

ie $x^2-4x-5-k=0$

8(b)

If there's only one point of intersection then

$\Delta = 0$

ie $(-4)^2 - 4(-5-k) = 0$

$16 + 20 + 4k = 0$

$4k = -36$

$k = -9$

Alternatively
$x^2 - 4x - 5 - k$ must be a perfect square

$x^2 - 4x + 4 - 4 - 5 - k = 0$

$(x-2)^2 - 9 - k = 0$

$\therefore k = -9$

(iii) If $k = -9$ then

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$

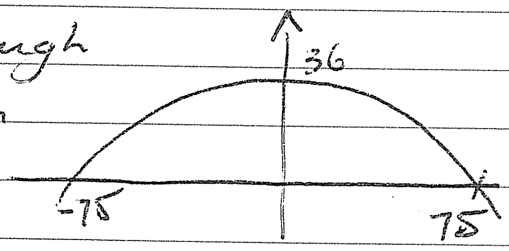
Sub into $y = x^2 - 3x - 5$

$= 4 - 6 - 5$

$= -7$

$\therefore$ P has coordinates $(2, -7)$

c) Equation of parabola passing through $(-75, 0)$, $(0, 36)$, $(75, 0)$ has form



$y = a(75+x)(75-x)$

Sub $x = 0$, $y = 36$

$36 = a(75)(75)$

$\therefore a = \frac{36}{5625}$

$\therefore y = \frac{36}{5625}(75+x)(75-x)$

$= \frac{4}{625}(75+x)(75-x)$

Now rectangle has width $2x$ and height y

$\therefore$ Area $= 2x \cdot \frac{4}{625}(75+x)(75-x)$

$= \frac{8x}{625}(75+x)(75-x)$

(ii) Expanding we get

$$A = \frac{8x}{625}\left(5625 - x^2\right)$$

$$= 72x - \frac{8x^3}{625}$$

$$\frac{dA}{dx} = 72 - \frac{24x^2}{625}$$

Maximum will occur when $\frac{dA}{dx} = 0$

$$72 - \frac{24x^2}{625} = 0$$

$$x^2 = \frac{72 \times 625}{24}$$

$$x = 25\sqrt{3} \quad , \quad x > 0$$

$$\frac{d^2A}{dx^2} = -\frac{48x}{625}$$

When $x = 25\sqrt{3}$, $\frac{d^2A}{dx^2} = -\frac{48\sqrt{3}}{25}$

$$< 0$$

$\therefore$ Area of rectangle is maximised when $x = 25\sqrt{3}$

$$y = \frac{4}{625}\left(75^2 - x^2\right)$$

$$= 24$$

Rectangle has dimensions $50\sqrt{3}$ and 24