North Sydney Girls High School

2022 **PRELIMINARY ASSESSMENT TASK 3**

Mathematics Advanced

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 84

Section I – 10 marks (pages 3 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – **74 marks** (pages 9 – 36)

- Attempt Questions 11 – 35
- Allow about 1 hour and 45 minutes for this section

NAME:_____ **TEACHER:**_____

STUDENT NUMBER:

Question	1–10	11–27	28–35	Total
Mark	/10	/45	/29	/84

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Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10

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1 Which of the following is equivalent to $\left(2\sqrt{5}-3\right)^2$?

A. 11

B. $29-12\sqrt{5}$

C. $11-12\sqrt{5}$

D. 29

2 Which expression is equivalent to $\log_a(m)-\log_a(2m)$?

A. $-\log_a(m)$

B. $\log_a(-m)$

C. $\log_a\left(\frac{1}{m}\right)$

D. $\log_a\left(\frac{1}{2}\right)$

3 Which of the following functions is a many-to-one function?

A. $f(x)=\sqrt{8-x^2}$

B. $f(x)=\sqrt{8-x}$

C. $f(x)=x^3+8$

D. $f(x)=\dfrac{8}{x}$

4 What is the correct set of solutions to $3x^2+3x\le 18$?

A. $x\in\{(-\infty,-2]\cup[3,\infty)\}$

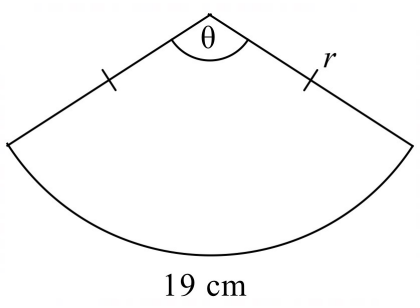
B. $x\in[-2,3]$

C. $x\in\{(-\infty,-6]\cup[1,\infty)\}$

D. $x\in[-3,2]$

5 The following sector has an arc length of 19 cm and an area of 76 cm^2.

What is the exact length of the radius r?



A. $8\sqrt{2}$ cm

B. 8 cm

C. 4 cm

D. $2\sqrt{2}$ cm

6 If $\cot\theta = \frac{24}{7}$ and $180^\circ < \theta \leq 270^\circ$, what is the exact value of $\sin\theta$?

A. $\frac{24}{25}$

B. $\frac{7}{25}$

C. $-\frac{7}{25}$

D. $-\frac{25}{24}$

7 The number of hours (h) taken to clean an apartment block is inversely proportional to the square of the number of cleaners (C). If it takes 6 cleaners 12.5 hours on average to clean an apartment block, which of the following equations represents this situation?

A. $h = \frac{C^2}{75}$

B. $C = \frac{\sqrt{75}}{h^2}$

C. $h = \frac{450}{C^2}$

D. $C = \frac{450}{h^2}$

8 What is the value of $\lim_{x \to -3} \frac{2x+6}{x^2-9}$?

A. $-\frac{1}{3}$

B. 0

C. $\frac{2}{3}$

D. 1

9 Which statement is true of the function $f(x)=\begin{cases} x+2, & x<2 \\ x^2, & x\geq 2 \end{cases}$ at the point $x=2$?

A. Continuous and differentiable

B. Continuous but not differentiable

C. Not continuous but differentiable

D. Neither continuous nor differentiable

10 Let $g(x)=x^3+3$ and $f(x)=\sqrt{x}$. What is the simplified expression for $f'(g(x))$?

A. $x\sqrt{3}$

B. $\dfrac{1}{2\sqrt{x^3+3}}$

C. $\sqrt{x^3+3}$

D. $x\sqrt{x}+3$

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Student Number

Mathematics Advanced
Section II Answer Booklet 1

1

Section II

74 marks
Attempt Questions 11–35
Allow about 1 hour and 45 minutes for this section

Booklet 1 – Attempt Questions 11–27 (45 marks)
Booklet 2 – Attempt Questions 28–35 (29 marks)

Instructions

- Write your student number at the top of this page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your responses should include relevant mathematical reasoning and/or calculations.
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Please turn over

Question 11 (2 marks)

Solve $|5-3x|=6$. **2**

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Question 12 (2 marks)

State the domain and range of $f(x)=3-\dfrac{2}{x+4}$. **2**

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Question 13 (1 mark)

Convert $\dfrac{23\pi}{12}$ radians into degrees. **1**

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Question 14 (2 marks)

Solve the equation $\sqrt{3}\operatorname{cosec} x = 2$ for $-2\pi \le x \le \pi$. **2**

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Question 15 (3 marks)

Differentiate the following:

(a) $y = 5 - 7x^2 + 3x^{-4}$ **1**

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(b) $y = \dfrac{5x^2}{x+2}$ **2**

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Question 16 (5 marks)

A small manufacturer makes and sells power generator units with the following finances:

- Fixed costs of $600 per week;
- Each unit costs $15 to produce;
- The selling price of each unit is $30.

(a) Write a linear equation to represent the Cost function (C) for the number of power generator units (u) produced. **1**

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(b) Find the number of units to be sold for the manufacturer to break even. **1**

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(c) The manufacturer sells 24 units in a particular week. **1**
Determine the amount of profit or loss incurred during this week.

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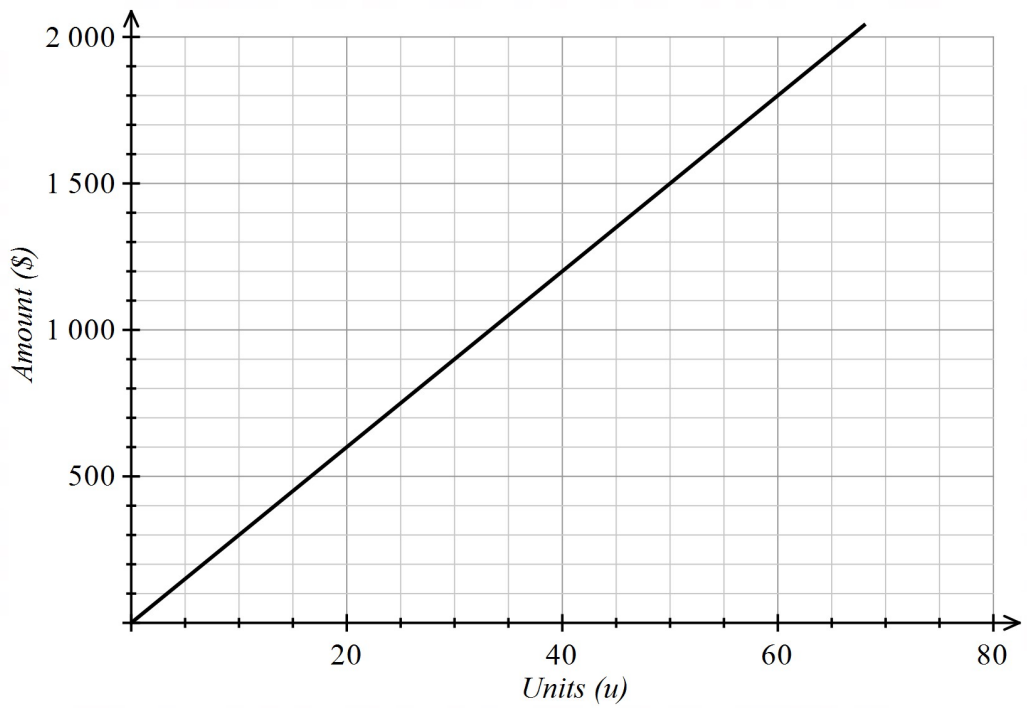
Question 16 continues on page 13

(d) The graph of the Revenue function (R) is provided in the diagram below. **1**

Draw the graph of the Cost function on the same axes, clearly labelling the break-even point.



(e) The manufacturer needs to increase their fixed costs to \$900. **1**

Write a new equation for the Revenue function so that the break-even point remains at the same number of units sold.

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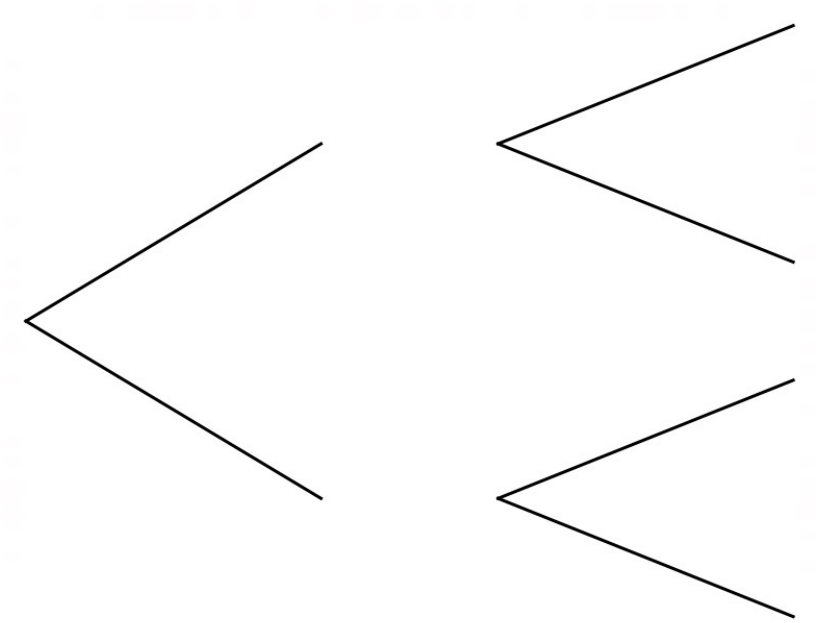
End of Question 16

Question 17 (3 marks)

A bag contains 9 blue marbles and 22 red marbles. Two marbles are drawn at random without replacement and their colours are noted.

(a) Complete the tree diagram to represent the possible outcomes and their respective probabilities. **1**



(b) Find the probability that at least one red marble is drawn. **2**

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Question 18 (2 marks)

Find the minimum value of the quadratic function $k(x) = 2x^2 - 5x + 18$. **2**

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Question 19 (2 marks)

Differentiate $f(x) = 3x - 4x^2$ using first principles, given $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. **2**

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Question 20 (4 marks)

Consider the function $f(x) = 3\cos\left(\frac{x}{2}\right)$.

(a) State the period and amplitude of $f(x)$. **1**

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(b) Sketch the graph $y = f(x)$ for the domain $[-2\pi, 2\pi]$. **2**

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(c) Hence, or otherwise, state how many values of x satisfy the equation $f(x) = 1$ in this domain. **1**

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Question 21 (2 marks)

Let $f(x) = \sqrt{1-x^2}$. Determine whether $f(f(x))$ is odd, even, or neither. Show all working. **2**

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Question 22 (2 marks)

Show that $\cos\theta + \tan\theta \sin\theta = \sec\theta$. **2**

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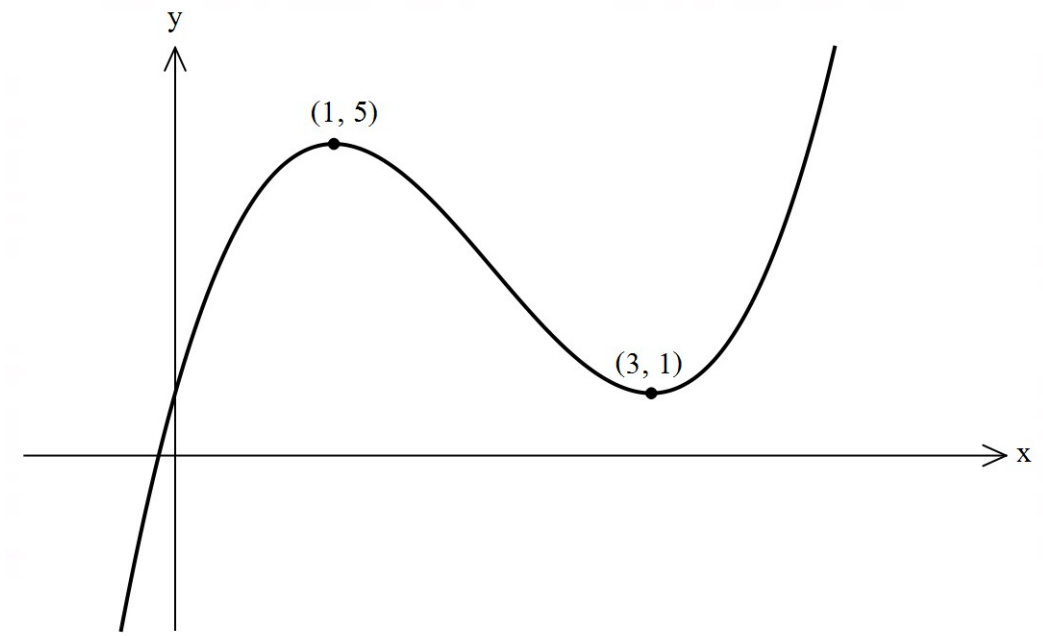
Question 23 (3 marks)

The graph of a function $y = f(x)$ passes through the points $(1,5)$ and $(3,1)$ as shown below.



(a) For what values of x is $f(x)$ decreasing? **1**

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(b) On the number plane above, sketch the gradient function $y = f'(x)$, indicating all necessary features. **2**

Question 24 (4 marks)

At a recent classical music concert, 650 people heard a Mozart symphony and a Bach concerto. Of these, 327 liked the Mozart piece, 438 liked the Bach piece, and 48 did not like either piece.

(a) Draw a Venn diagram to represent this situation. **1**

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(b) A person is selected at random. What is the probability that the person liked the Bach piece? **1**

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(c) Two people are now selected at random. What is the probability that both people liked only the Mozart piece? **2**

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Question 25 (3 marks)

(a) Differentiate $f(x) = x^6(1-x)^3$. **2**

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(b) Hence find the values of x such that $f'(x) = 0$. **1**

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Question 26 (2 marks)

If $a = \log_{\frac{1}{x}}\left(\frac{1}{N}\right)$ such that $x > 0$ and $N > 0$, show that $a = \log_x(N)$. **2**

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Question 27 (3 marks)

The Richter scale is used to rate the magnitude (M) of an earthquake based on the amount of energy (E) released by the earthquake, measured in Joules. They are related by the equation

$$\log_{10}(E) = a + 1.5M$$

where a is a real constant.

The San Francisco earthquake in 1906 registered 8.2 on the Richter scale and released 5.011×10^{16} Joules of energy.

(a) Use this information to determine the value of a, correct to 1 decimal place. **1**

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(b) Hence, determine the amount of energy released by an earthquake that registered 4.7 on the Richter scale. **2**

Give your answer in scientific notation to 3 significant figures.

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End of Booklet 1

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Student Number

Mathematics Advanced
Section II Answer Booklet 2

2

Booklet 2 – Attempt Questions 28–35 (29 marks)

Instructions

- Write your student number at the top of this page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided on pages at the end of each booklet. If you use this space, clearly indicate which question you are answering.

Please turn over

Question 28 (3 marks)

Consider the circle with equation $x^2 + y^2 - 4y + 6x = 12$.

(a) Find the centre and radius of the circle. **2**

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(b) Show that the point $(2, 4)$ lies outside of the circle. **1**

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Question 29 (3 marks)

Solve $\sin^2 x - 3\cos x - 3 = 0$ in the domain $[0, 2\pi]$. **3**

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Question 30 (4 marks)

A parabola with equation $y = f(x)$ has a vertex at $(-1, 3)$ and passes through the origin.

(a) Write an equation for $f(x)$. **2**

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(b) Prove that the gradients of the tangents to the parabola at the x-intercepts are equal in magnitude yet opposite in sign. **2**

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Question 31 (4 marks)

Consider the function $f(x) = |x| - 3$.

(a) On the same axes, sketch the graphs of $y = f(x)$ and $y = -f(x+4)$, showing all points of intersection. **3**

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(b) Hence, or otherwise, solve $f(x) + f(x+4) \geq 0$. **1**

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Question 32 (4 marks)

Consider the function $f(x) = -\sqrt{x-4}$.

(a) Determine the equation of the normal to the curve at the point where $x = 8$. **3**
Write your answer in gradient-intercept form.

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(b) Hence, find the angle of inclination of the normal with the positive x - axis. **1**
Give your answer to the nearest degree.

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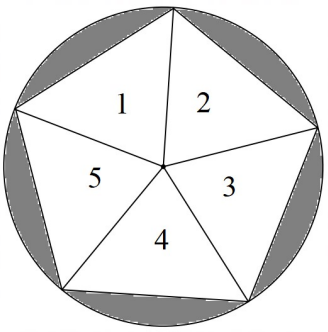
Question 33 (4 marks)

A circular dartboard has a regular pentagon inscribed in it such that the vertices of the pentagon lie on the circumference of the circular game board. **4**

The pentagon is divided into five identical triangles labelled 1 to 5 as shown in the diagram. The segments between the circumference of the dartboard and the edges of the pentagon are shaded.



A player throws a single dart at the board.

The player wins if the dart lands in the shaded section next to an odd-numbered triangle, or inside an even numbered triangle.

Assuming the dart is equally likely to land at any point on the board, find the probability that the player wins the game.

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Question 34 (3 marks)

Prove that the quadratic function $f(x) = x^2 + (m-3)x - m$ has real roots for all real values of m. **3**

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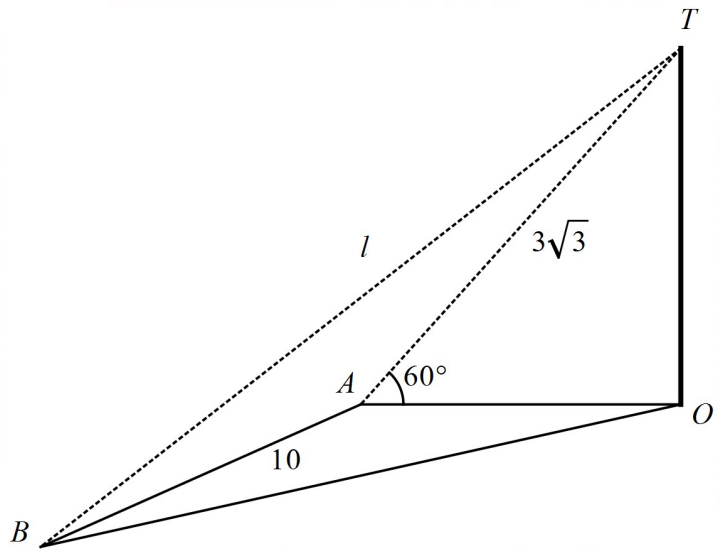
Question 35 (4 marks)

A tall pole OT is held in place vertically by ropes anchored to two pegs in the ground 10 metres apart at A and B as shown. A is due west of O and B is at a bearing of $240°$T from O. **4**

The rope connecting the top of the pole T to the peg at A is $3\sqrt{3}$ metres long and makes an angle of $60°$ with the ground.



Find the length of the rope connecting T and B. Give your answer to the nearest metre.

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Question 35 continues on page 33

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North Sydney Girls High School

2022 **PRELIMINARY ASSESSMENT TASK 3**

Mathematics Advanced

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 84

Section I – 10 marks (pages 3 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 74 marks (pages 9 – 36)

- Attempt Questions 11 – 35
- Allow about 1 hour and 45 minutes for this section

NAME: Solutions **TEACHER:** _____

STUDENT NUMBER: | | | | | | | | |

Question	1–10	11–27	28–35	Total
Mark	/10	/45	/29	/84

North Sydney Girls High School

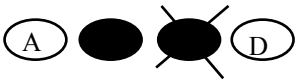
Student Number

Name:_____ Teacher: _____

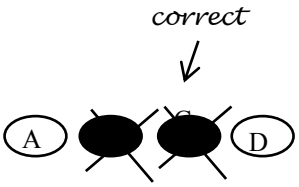
Multiple Choice – Mathematics Advanced

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.



If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer



Start Here →

1. A **B** C D
2. A B C **D**
3. **A** B C D
4. A B C **D**
5. A **B** C D
6. A B **C** D
7. A B **C** D
8. **A** B C D
9. A **B** C D
10. A **B** C D

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10

---

1 Which of the following is equivalent to $\left(2\sqrt{5}-3\right)^2$?

$4\times5-12\sqrt{5}+9$
$=29-12\sqrt{5}$

A. 11

B. $29-12\sqrt{5}$

C. $11-12\sqrt{5}$

D. 29

2 Which expression is equivalent to $\log_a(m)-\log_a(2m)$?

$\log_a\left(\frac{m}{2m}\right)$
$=\log_a\left(\frac{1}{2}\right)$

A. $-\log_a(m)$

B. $\log_a(-m)$

C. $\log_a\left(\frac{1}{m}\right)$

D. $\log_a\left(\frac{1}{2}\right)$

3 Which of the following functions is a many-to-one function?

A. $f(x)=\sqrt{8-x^2}$

B. $f(x)=\sqrt{8-x}$

C. $f(x)=x^3+8$

D. $f(x)=\dfrac{8}{x}$

4 What is the correct set of solutions to $3x^2+3x\leq 18$?

A. $x\in\{(-\infty,-2]\cup[3,\infty)\}$

B. $x\in[-2,3]$

C. $x\in\{(-\infty,-6]\cup[1,\infty)\}$

D. $x\in[-3,2]$

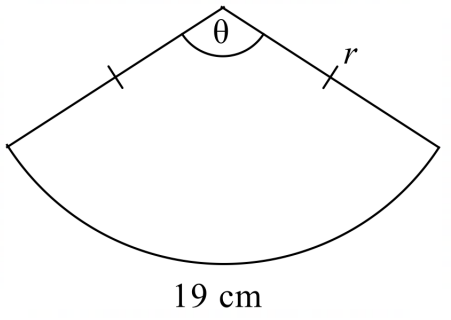
$x^2+x-6\leq 0$

$(x+3)(x-2)\leq 0$

$-3\leq x\leq 2$

5 The following sector has an arc length of 19 cm and an area of 76 cm^2.

What is the exact length of the radius r?



$\ell = r\theta$

$A = \frac{1}{2}r^2\theta$

$76 = \frac{1}{2} \times 19r$

$\therefore 8 = r$

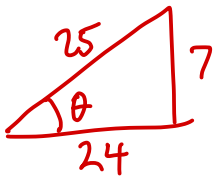
A. $8\sqrt{2}$ cm

B. 8 cm

C. 4 cm

D. $2\sqrt{2}$ cm

6 If $\cot\theta = \frac{24}{7}$ and $180^\circ < \theta \leq 270^\circ$, what is the exact value of $\sin\theta$?



A. $\frac{24}{25}$

B. $\frac{7}{25}$

C. $-\frac{7}{25}$

D. $-\frac{25}{24}$

7 The number of hours (h) taken to clean an apartment block is inversely proportional to the square of the number of cleaners (C). If it takes 6 cleaners 12.5 hours on average to clean an apartment block, which of the following equations represents this situation?

A. $h = \dfrac{C^2}{75}$

B. $C = \dfrac{\sqrt{75}}{h^2}$

C. $h = \dfrac{450}{C^2}$

D. $h = \dfrac{450}{C^2}$

$h \propto \frac{1}{C^2} = \frac{k}{C^2}$

when $h = 12.5$, $C = 6$

$\therefore 12.5 = \frac{k}{36}$

$\therefore k = 450$

8 What is the value of $\lim\limits_{x \to -3} \dfrac{2x+6}{x^2-9}$?

A. $-\dfrac{1}{3}$

B. 0

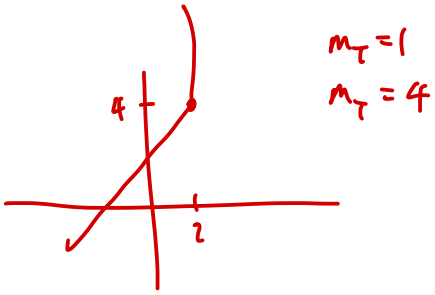
C. $\dfrac{2}{3}$

D. 1

$\frac{2(x+3)}{(x+3)(x-3)} = \frac{2}{-6}$

$= \frac{-1}{3}$

9 Which statement is true of the function $f(x)=\begin{cases} x+2, & x<2 \\ x^2, & x\geq 2 \end{cases}$ at the point $x=2$?



A. Continuous and differentiable

B. Continuous but not differentiable

C. Not continuous but differentiable

D. Neither continuous nor differentiable

10 Let $g(x)=x^3+3$ and $f(x)=\sqrt{x}$. What is the simplified expression for $f'(g(x))$?

A. $x\sqrt{3}$

B. $\dfrac{1}{2\sqrt{x^3+3}}$

C. $\sqrt{x^3+3}$

D. $x\sqrt{x}+3$

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Student Number

Mathematics Advanced
Section II Answer Booklet 1

1

Section II

74 marks
Attempt Questions 11–35
Allow about 1 hour and 45 minutes for this section

Booklet 1 – Attempt Questions 11–27 (45 marks)
Booklet 2 – Attempt Questions 28–35 (29 marks)

Instructions

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Please turn over

Question 11 (2 marks)

Solve $|5-3x|=6$. **2**

$5-3x=6$ or $-(5-3x)=6$

$3x=-1$ $3x-5=6$

$x=\frac{-1}{3}$ $x=\frac{11}{3}$

$x=\frac{-1}{3}, \frac{11}{3}$

Question 12 (2 marks)

State the domain and range of $f(x)=3-\frac{2}{x+4}$. **2**

D: $x\in\mathbb{R}, x\neq -4$

R: $y\in\mathbb{R}, y\neq 3$

Question 13 (1 mark)

Convert $\frac{23\pi}{12}$ radians into degrees. **1**

$\frac{23\pi}{12}\times\frac{180^\circ}{\pi}=345^\circ$

Question 14 (2 marks)

Solve the equation $\sqrt{3}\operatorname{cosec} x = 2$ for $-2\pi \le x \le \pi$. **2**

$\sin x = \frac{\sqrt{3}}{2}$

Reference : $x = \frac{\pi}{3}$

Quadrants ✓ | ✓

$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{-4\pi}{3}, \frac{-5\pi}{3}$

Question 15 (3 marks)

Differentiate the following:

(a) $y = 5 - 7x^2 + 3x^{-4}$ **1**

$y' = -14x - 12x^{-5}$

(b) $y = \frac{5x^2}{x+2}$ **2**

$u = 5x^2 \qquad v = x+2$

$u' = 10x \qquad v' = 1$

$\therefore y' = \frac{10x(x+2) - 5x^2}{(x+2)^2}$

$= \frac{10x^2 + 20x - 5x^2}{(x+2)^2}$

$y' = \frac{5x^2 + 20x}{(x+2)^2}$

Question 16 (5 marks)

A small manufacturer makes and sells power generator units with the following finances:

- Fixed costs of $600 per week;
- Each unit costs $15 to produce;
- The selling price of each unit is $30.

(a) Write a linear equation to represent the Cost function (C) for the number of power generator units (u) produced. **1**

$C = 600 + 15u$

(b) Find the number of units to be sold for the manufacturer to break even. **1**

$R = 30u$

Break even when $C = R$

$30u = 600 + 15u$

$u = 40$

(c) The manufacturer sells 24 units in a particular week. **1**
Determine the amount of profit or loss incurred during this week.

$\text{Profit} = R - C$

$= 30u - (600 + 15u)$

$= 15u - 600$

When $u = 24$

$\text{Profit} = 15 \times 24 - 600$

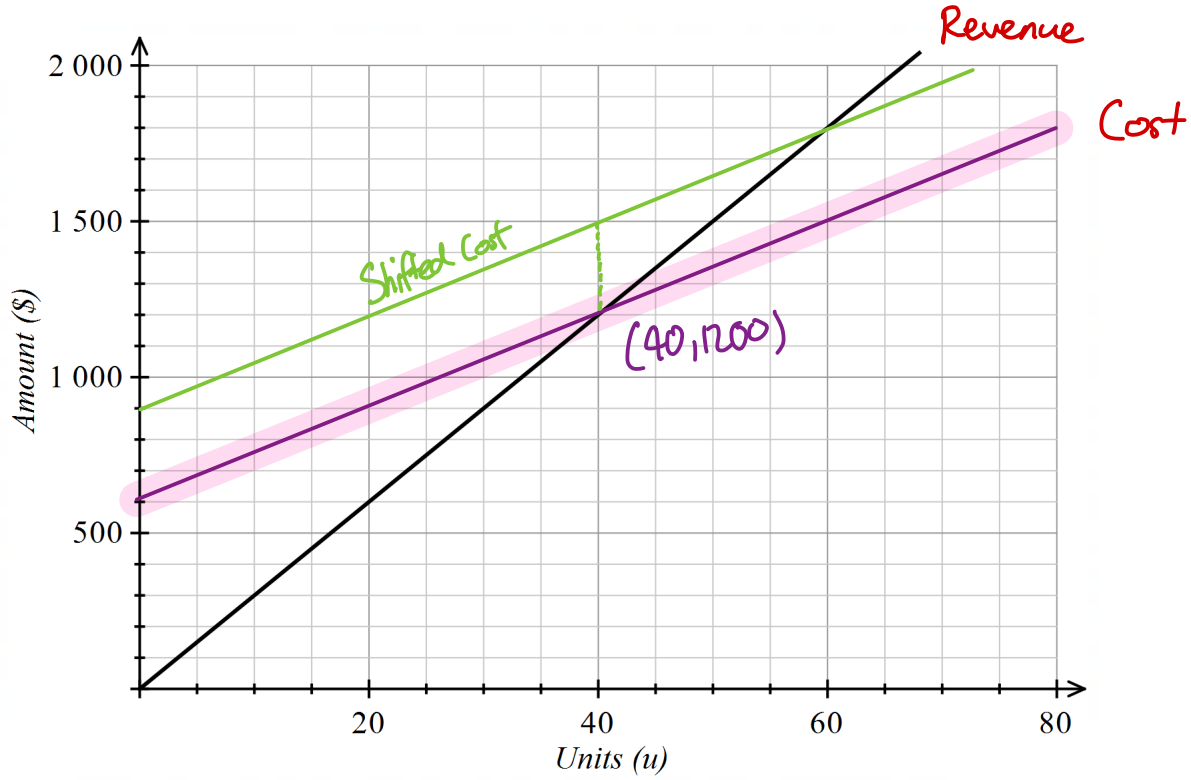
$= -240$

$\therefore$ Loss of $240

Question 16 continues on page 13

(d) The graph of the Revenue function (R) is provided in the diagram below. **1**

Draw the graph of the Cost function on the same axes, clearly labelling the break-even point.



(e) The manufacturer needs to increase their fixed costs to \$900. **1**

Write a new equation for the Revenue function so that the break-even point remains at the same number of units sold.

New cost = $15 \times 40 + 900 = 1500$

Revenue required: $R_2 = \frac{1500}{40}x$

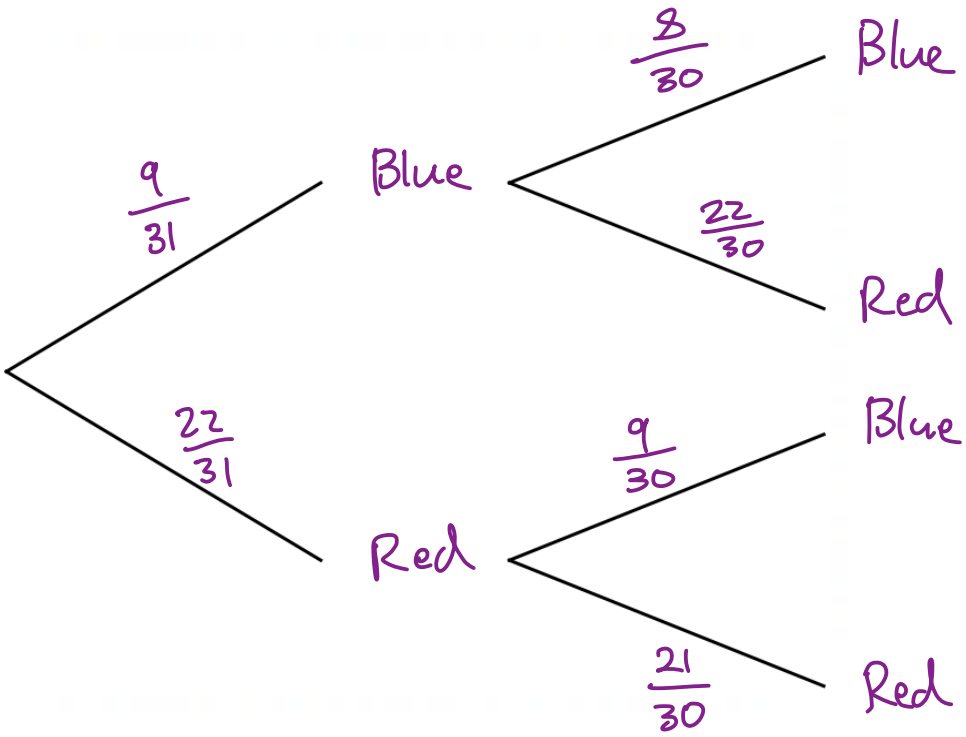
$R_2 = \frac{75}{2}x$

End of Question 16

Question 17 (3 marks)

A bag contains 9 blue marbles and 22 red marbles. Two marbles are drawn at random without replacement and their colours are noted.

(a) Complete the tree diagram to represent the possible outcomes and their respective probabilities. **1**



(b) Find the probability that at least one red marble is drawn. **2**

$P(\text{At least one Red}) = 1 - P(\text{no Red})$

$= 1 - \left(\frac{9}{31} \times \frac{8}{30}\right)$

$= \frac{143}{155}$

Question 18 (2 marks)

Find the minimum value of the quadratic function $k(x) = 2x^2 - 5x + 18$. **2**

Axis: $x_s = \frac{-(-5)}{2 \times 2} = \frac{5}{4}$

$\therefore$ Vertex at $k\left(\frac{5}{4}\right) = 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right) + 18$

$= \frac{25}{8} - \frac{25}{4} + 18$

$= \frac{119}{8}$

Minimum value is $\frac{119}{8}$

Question 19 (2 marks)

Differentiate $f(x) = 3x - 4x^2$ using first principles, given $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. **2**

$f(x+h) = 3(x+h) - 4(x+h)^2$

$= 3x + 3h - 4x^2 - 8xh - 4h^2$

$\therefore f'(x) = \lim_{h \to 0} \frac{(3x + 3h - 4x^2 - 8xh - 4h^2) - (3x - 4x^2)}{h}$

$= \lim_{h \to 0} \frac{3h - 8xh - 4h^2}{h}$

$= \lim_{h \to 0} 3 - 8x - 4h$

$\therefore f'(x) = 3 - 8x$

Question 20 (4 marks)

Consider the function $f(x) = 3\cos\left(\frac{x}{2}\right)$. $f(x) = 3\cos\left(\frac{1}{2}x\right)$

(a) State the period and amplitude of $f(x)$. **1**

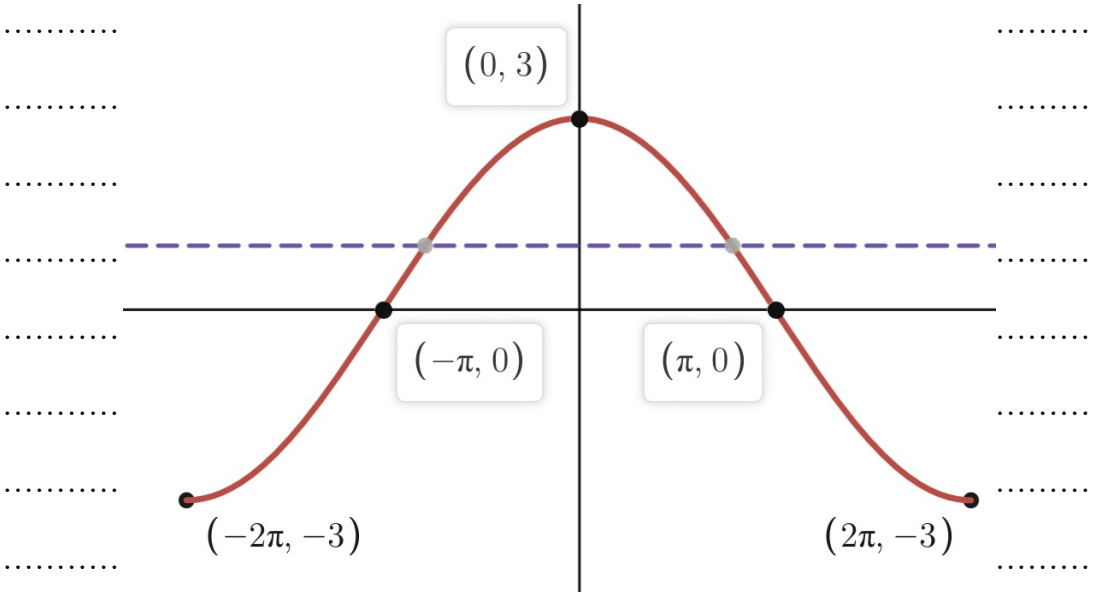
Period: $T = \frac{2\pi}{n} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

Amplitude: $A = 3$

Period: 4π

Amplitude: $A = 3$

(b) Sketch the graph $y = f(x)$ for the domain $[-2\pi, 2\pi]$. **2**



(c) Hence, or otherwise, state how many values of x satisfy the equation $f(x) = 1$ in this domain. **1**

From graph, $f(x) = 1$ occurs at two points.

$\therefore$ 2 solutions

Question 21 (2 marks)

Let $f(x)=\sqrt{1-x^2}$. Determine whether $f\left(f(x)\right)$ is odd, even, or neither. Show all working. **2**

$f(f(x)) = \sqrt{1-(\sqrt{1-x^2})^2}$

$= \sqrt{1-(1-x^2)}$

$= \sqrt{x^2}$

$= |x|$

$\therefore f(f(-x)) = |-x|$

$= |-1| \times |x|$

$= |x|$

$= f(f(x))$ $\therefore$ Even as $f(f(x)) = f(f(-x))$

Question 22 (2 marks)

Show that $\cos\theta + \tan\theta \sin\theta = \sec\theta$. **2**

LHS $= \cos\theta + \frac{\sin\theta}{\cos\theta} \times \sin\theta$

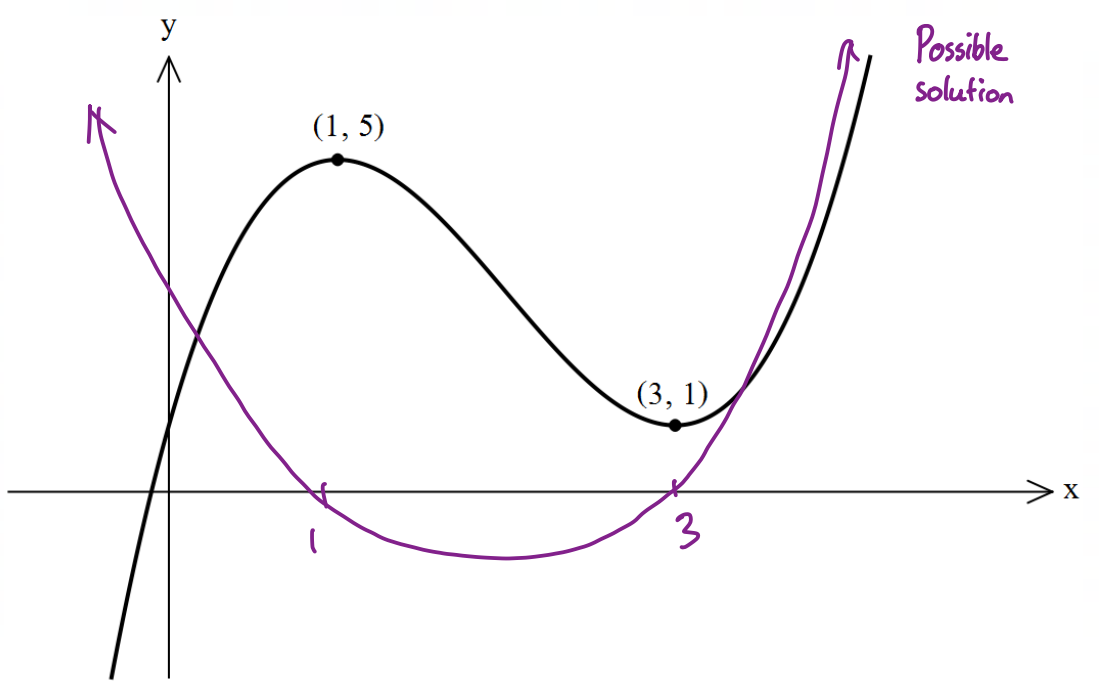
$= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta}$

$= \frac{1}{\cos\theta}$

$= \sec\theta$ = RHS

Question 23 (3 marks)

The graph of a function $y = f(x)$ passes through the points $(1,5)$ and $(3,1)$ as shown below.



(a) For what values of x is $f(x)$ decreasing? **1**

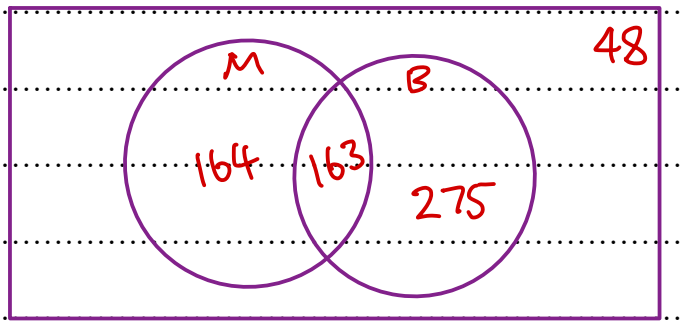
$1 < x < 3$

(b) On the number plane above, sketch the gradient function $y = f'(x)$, indicating all necessary features. **2**

Question 24 (4 marks)

At a recent classical music concert, 650 people heard a Mozart symphony and a Bach concerto. Of these, 327 liked the Mozart piece, 438 liked the Bach piece, and 48 did not like either piece.

(a) Draw a Venn diagram to represent this situation. **1**



(b) A person is selected at random. What is the probability that the person liked the Bach piece? **1**

$P(\text{Bach}) = \frac{438}{650} = \frac{219}{325}$

(c) Two people are now selected at random. What is the probability that both people liked only the Mozart piece? **2**

$P(\text{MM only}) = \frac{164}{650} \times \frac{163}{649}$

$= \frac{26732}{421850}$ either

$= 0.06\ldots$

Question 25 (3 marks)

(a) Differentiate $f(x) = x^6(1-x)^3$. **2**

let $u = x^6$ $\quad v = (1-x)^3$

$u' = 6x^5$ $\quad v' = -3(1-x)^2$

$\therefore f'(x) = -3x^6(1-x)^2 + 6x^5(1-x)^3$

(b) Hence find the values of x such that $f'(x) = 0$. **1**

$f'(x) = -3x^5(1-x)^2[x - 2(1-x)] = 0$

$= -3x^5(1-x)^2(3x-2) = 0$

$\therefore x = 0, 1, \frac{2}{3}$

Question 26 (2 marks)

If $a = \log_{\frac{1}{x}}\left(\frac{1}{N}\right)$ such that $x > 0$ and $N > 0$, show that $a = \log_x(N)$. **2**

$a = \log_{(\frac{1}{x})}(\frac{1}{N})$

$\therefore \left(\frac{1}{x}\right)^a = \frac{1}{N}$

$\frac{1}{x^a} = \frac{1}{N}$

$x^a = N$

$\therefore a = \log_x(N)$ as required

Alternatively

Using the change of base rule:

$a = \frac{\log_x(\frac{1}{N})}{\log_x(\frac{1}{x})}$

$= \frac{-\log_x(N)}{-\log_x(x)}$

$= \log_x(N)$ as required

Question 27 (3 marks)

The Richter scale is used to rate the magnitude (M) of an earthquake based on the amount of energy (E) released by the earthquake, measured in Joules. They are related by the equation

$$\log_{10}(E) = a + 1.5M$$

where a is a real constant.

The San Francisco earthquake in 1906 registered 8.2 on the Richter scale and released 5.011×10^{16} Joules of energy.

(a) Use this information to determine the value of a, correct to 1 decimal place. **1**

$\log_{10}(5.011 \times 10^{16}) = a + 1.5 \times 8.2$

$a = 4.4$ (1dp)

(b) Hence, determine the amount of energy released by an earthquake that registered 4.7 on the Richter scale. **2**

Give your answer in scientific notation to 3 significant figures.

$\log_{10}(E) = 4.4 + 1.5 \times 4.7$

$= 11.45$

$\therefore E = 10^{11.45}$

$E = 2.82 \times 10^{11}$ J

End of Booklet 1

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Student Number

Mathematics Advanced
Section II Answer Booklet 2

2

Booklet 2 – Attempt Questions 28–35 (29 marks)

Instructions

- Write your student number at the top of this page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided on pages at the end of each booklet. If you use this space, clearly indicate which question you are answering.

Please turn over

Question 28 (3 marks)

Consider the circle with equation $x^2 + y^2 - 4y + 6x = 12$.

(a) Find the centre and radius of the circle. **2**

$x^2+6x+9+y^2-4y+4 = 12+13$

$(x+3)^2+(y-2)^2 = 25$

$\therefore$ Centre : $(-3, 2)$

Radius : $r = 5$

(b) Show that the point $(2,4)$ lies outside of the circle. **1**

$d^2 = (2-(-3))^2 + (2-4)^2$

$= 25 + 4$

$d^2 = 29$

$\therefore d = \sqrt{29} > 5$ $\therefore$ Point lies outside of circle.

Question 29 (3 marks)

Solve $\sin^2 x - 3\cos x - 3 = 0$ in the domain $[0, 2\pi]$. **3**

$(1-\cos^2 x) - 3\cos x - 3 = 0$

$\cos^2 x + 3\cos x + 2 = 0$

$(\cos x + 1)(\cos x + 2) = 0$

$\therefore \cos x = -1$ or $\cos x = -2$ (reject)

$\therefore x = \pi$

Question 30 (4 marks)

A parabola with equation $y = f(x)$ has a vertex at $(-1, 3)$ and passes through the origin.

(a) Write an equation for $f(x)$. **2**

$y = a(x+1)^2 + 3$

when $x = 0$, $y = 0$

$\therefore a = -3$

$\therefore y = -3(x+1)^2 + 3$

(b) Prove that the gradients of the tangents to the parabola at the x-intercepts are equal in magnitude yet opposite in sign. **2**

Intercepts : At $y = 0$; $(x+1)^2 = 1$

$x + 1 = \pm 1$

$x = 0, -2$

$y' = -6(x+1)$

At $x = 0$, $y' = -6$

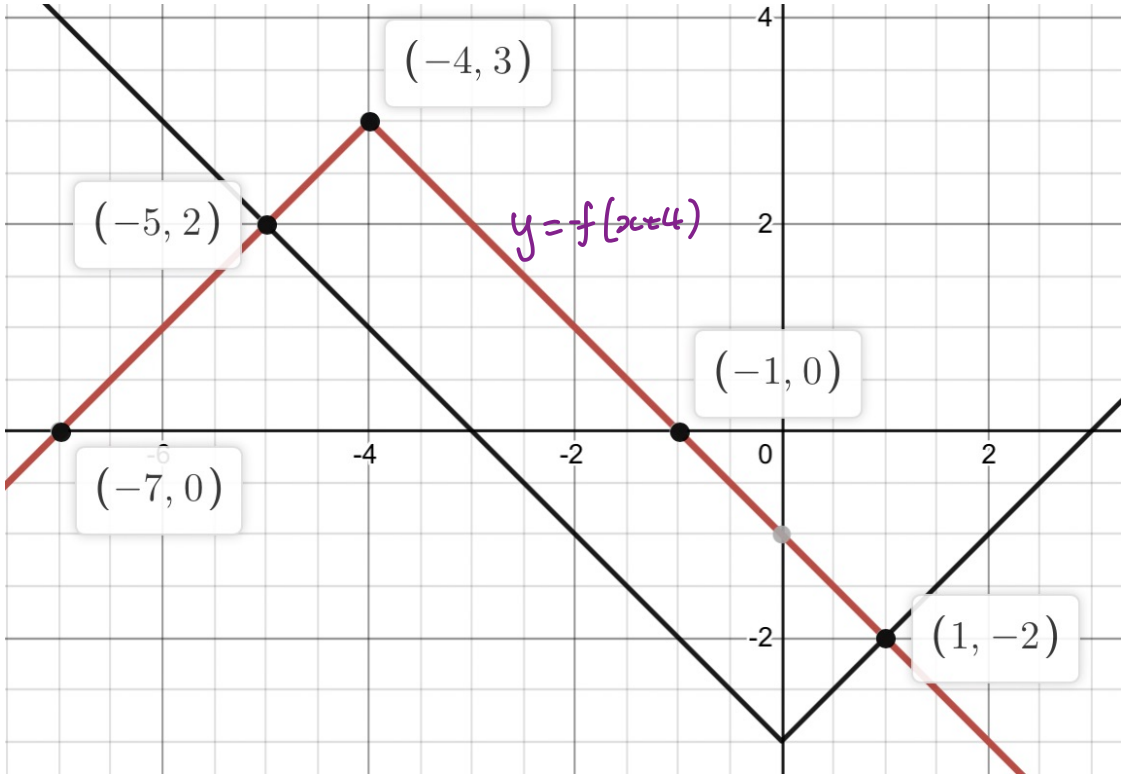
$x = -2$, $y' = 6$

$\therefore$ Gradients of tangents at x-intercepts are equal in value yet opposite in sign.

Question 31 (4 marks)

Consider the function $f(x)=|x|-3$.

(a) On the same axes, sketch the graphs of $y=f(x)$ and $y=-f(x+4)$, showing all points of intersection. **3**



(b) Hence, or otherwise, solve $f(x)+f(x+4)\geq 0$. **1**

$f(x) \geq -f(x+4)$

From graph, $x \geq 1$ or $x \leq -5$

Question 32 (4 marks)

Consider the function $f(x) = -\sqrt{x-4}$.

(a) Determine the equation of the normal to the curve at the point where $x = 8$. **3**
Write your answer in gradient-intercept form.

$f(x) = -(x-4)^{\frac{1}{2}}$

At $x = 8$, $f(8) = -2$

$f'(x) = -\frac{1}{2}(x-4)^{\frac{-1}{2}}$

At $x = 8$

$f'(x) = \frac{-1}{2}(8-4)^{\frac{-1}{2}}$

$= \frac{-1}{4}$

$\therefore m_T = \frac{-1}{4} \Rightarrow m_N = 4$

Equation: $y - (-2) = 4(x-8)$

$\therefore y = 4x - 34$

(b) Hence, find the angle of inclination of the normal with the positive x - axis. **1**
Give your answer to the nearest degree.

Let θ be angle of inclination.

$m = \tan\theta$

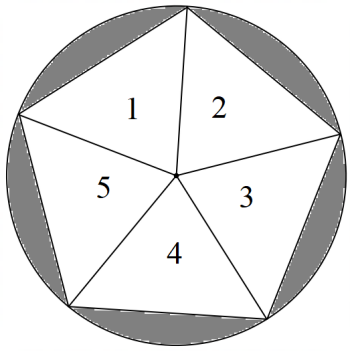
$\therefore 4 = \tan\theta$

$\therefore \theta = 76^\circ$ (taking acute angle)

Question 33 (4 marks)

A circular dartboard has a regular pentagon inscribed in it such that the vertices of the pentagon lie on the circumference of the circular game board. **4**

The pentagon is divided into five identical triangles labelled 1 to 5 as shown in the diagram. The segments between the circumference of the dartboard and the edges of the pentagon are shaded.



A player throws a single dart at the board.

The player wins if the dart lands in the shaded section next to an odd-numbered triangle, or inside an even numbered triangle.

Assuming the dart is equally likely to land at any point on the board, find the probability that the player wins the game.

Let radius be r.

Let θ be angle size of each sector.

$\therefore \theta = \frac{2\pi}{5}$

Area of each triangle : $A_\Delta = \frac{1}{2} r^2 \sin\frac{2\pi}{5}$

Area of each sector : $A_s = \frac{1}{2} r^2 \left(\frac{2\pi}{5}\right) = \frac{\pi r^2}{5}$

Area of each segment : $A_{seg} = \frac{1}{2} r^2 \left(\frac{2\pi}{5} - \sin\frac{2\pi}{5}\right)$

Desired winning area : $A_{win} = 2A + 3A_{seg}$

$= r^2 \sin\left(\frac{2\pi}{5}\right) + \frac{3}{2} r^2 \left(\frac{2\pi}{5} - \sin\frac{2\pi}{5}\right)$

$\therefore P(\text{Win}) = \frac{r^2\left(\sin\frac{2\pi}{5} + \frac{3}{2}\left(\frac{2\pi}{5} - \sin\frac{2\pi}{5}\right)\right)}{\pi r^2} = \frac{1.409\ldots}{\pi}$

$= 44.86\%$ (or equivalent)

Question 34 (3 marks)

Prove that the quadratic function $f(x) = x^2 + (m-3)x - m$ has real roots for all real values of m. **3**

For $f(x)$ to have real roots for all values of m, we can show that

$\Delta \geq 0$

So $\Delta = (m-3)^2 - 4(-m)$

$= m^2 - 6m + 9 + 4m$

$= m^2 - 2m + 9$

$= (m^2 - 2m + 1) + 8$

$= (m-1)^2 + 8$

Since $(m-1)^2 \geq 0$ then $(m-1)^2 + 8 \geq 8 \geq 0$

Therefore $\Delta \geq 0$ and $f(x)$ will always have two roots for all values of m.

Alternatively, $\Delta = m^2 - 2m + 9$ and for $\Delta \geq 0$, we can show that Δ will have no solutions to $\Delta = 0$

The discriminant of Δ is

$\Delta_\Delta = (-2)^2 - 4 \times 9$

$= -32 < 0$

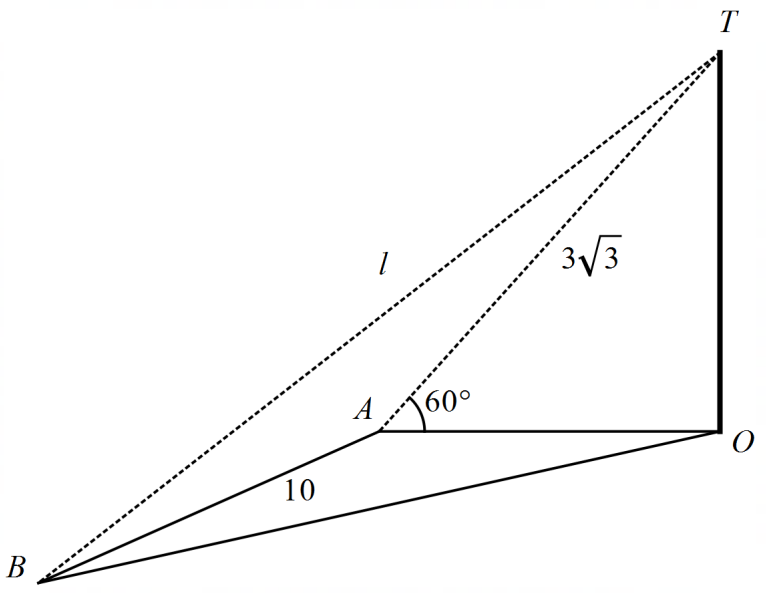
Therefore Δ is positive definite and so $\Delta \geq 0$ for all values of m.

Therefore, $f(x)$ has real roots for all values of m.

Question 35 (4 marks)

A tall pole OT is held in place vertically by ropes anchored to two pegs in the ground 10 metres apart at A and B as shown. A is due west of O and B is at a bearing of 240°T from O. **4**

The rope connecting the top of the pole T to the peg at A is $3\sqrt{3}$ metres long and makes an angle of $60°$ with the ground.



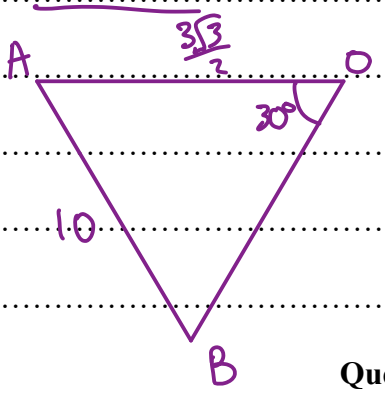
Find the length of the rope connecting T and B. Give your answer to the nearest metre.

In $\triangle AOT$

$\sin 60° = \frac{OT}{3\sqrt{3}}$ $\cos 60° = \frac{AO}{3\sqrt{3}}$

$TO = \frac{9}{2}$ $AO = \frac{3\sqrt{3}}{2}$

In $\triangle AOB$



Method 1

$\frac{\sin \angle OBA}{AO} = \frac{\sin 30°}{10}$

$\therefore \sin \angle OBA = \frac{3\sqrt{3}}{40}$

Question 35 continues on page 33

$\therefore \angle OBA = 7°28', 172°30'$ (reject)

$\therefore \angle BAO = 142°32'$

so $\frac{BO}{\sin \angle BAO} = \frac{10}{\sin 30°}$

$\therefore BO = \frac{10 \sin 142°32'}{\sin 30°}$

$= 12.17$

Method 2: In $\triangle AOB$, $AB^2 = OB^2 + OA^2 - 2 \times OB \times OA \times \cos \angle AOB$

$\therefore 100 = \frac{27}{4} + OB^2 - 2\left(\frac{3\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \times OB$

$\therefore 400 = 27 + 4 \times OB^2 - 18OB$

$\therefore 4(OB)^2 - 18(OB) - 373 = 0$

$OB = \frac{18 \pm \sqrt{18^2 + 4 \times 373 \times 4}}{8}$

$= 12.17$ as $OB > 0$

In $\triangle TOB$

$\ell^2 = BO^2 + TO^2$

$= 12.17^2 + \left(\frac{9}{2}\right)^2$

$= 168.24$

$\therefore \ell = 12.97m$

$\therefore$ Length of rope is approximately $\boxed{13m}$

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