North Sydney Girls High School

2023 **PRELIMINARY ASSESSMENT TASK 3**

Mathematics Advanced

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 80

Section I – 10 marks (pages 3 – 8)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – **70 marks** (pages 9 – 36)

- Attempt Questions 11 – 33
- Allow about 1 hour and 45 minutes for this section

NAME:_____ **TEACHER:**_____

STUDENT NUMBER:

Question	1–10	11–26	27–33	Total
Mark	/10	/42	/28	/80

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Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10

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1 Which of the following is the correct factorisation of $27x^3 - 64y^3$?

A. $(3x-4y)(9x^2+12xy+16y^2)$

B. $(3x-4y)(9x^2-12xy+16y^2)$

C. $(3x+4y)(9x^2+12xy+16y^2)$

D. $(3x+4y)(9x^2-12xy+16y^2)$

2 The first 4 terms of an arithmetic sequence are 4, 10, 16, 22, …
What is the sum of the first 20 terms?

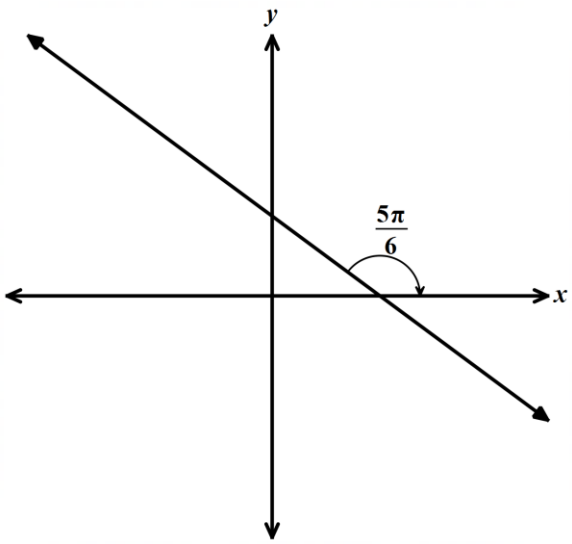
A. 1180

B. 1220

C. 1280

D. 2440

3 What is the gradient of the line shown below?



A. $\frac{1}{\sqrt{3}}$

B. $-\frac{1}{\sqrt{3}}$

C. $\sqrt{3}$

D. $-\sqrt{3}$

4 A function is given by:

$$f(x)=\begin{cases}\dfrac{8}{\sqrt{x+1}} & \text{if} \quad x\neq 3\\ m & \text{if} \quad x=3\end{cases}$$

For what value of m is the function continuous at $x=3$?

A. 4

B. –1

C. 1

D. –4

5 Which of the expressions below is the correct recurrence formula for the following sequence?

$$2,\ 5,\ 14,\ 41,...$$

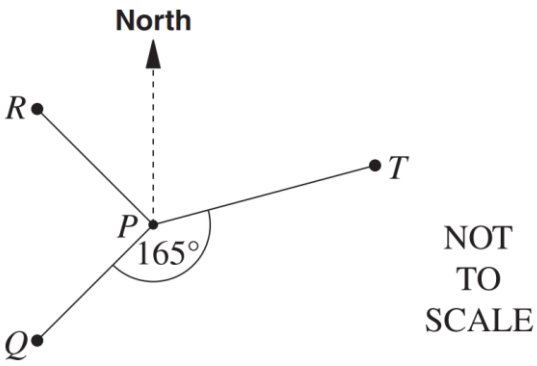
A. $T_n = 3^n$

B. $T_n = 2 \times 3^n$

C. $T_n = 2T_{n-1} \times 3, \quad T_1 = 2$

D. $T_n = 3T_{n-1} - 1, \quad T_1 = 2$

6 The diagram shows the position of Q, R and T relative to P.



In the diagram,

Q is South-west of P

R is North-west of P

$\angle QPT = 165^\circ$

What is the bearing of T from P?

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C. 105°

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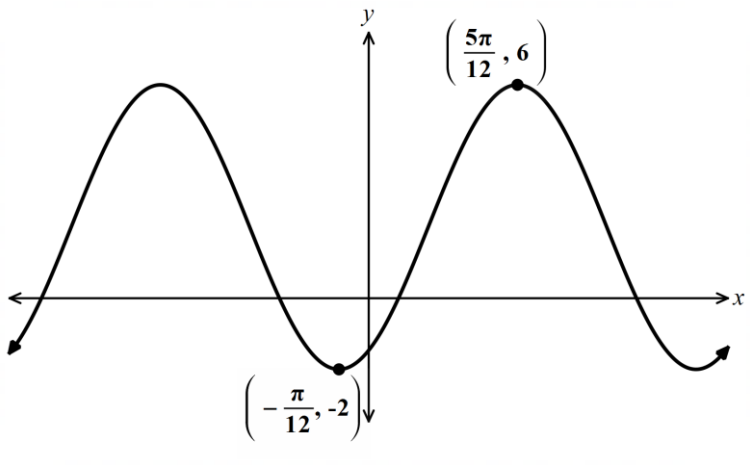


	Period	**Amplitude**
A.	$\frac{\pi}{3}$	8
B.	$\frac{2\pi}{3}$	4
C.	π	4
D.	2π	4

8 If $n \neq 0$ then which of the following does $\sqrt{\frac{2^{2n+4}+2^{2n+2}}{20}}$ simplify to?

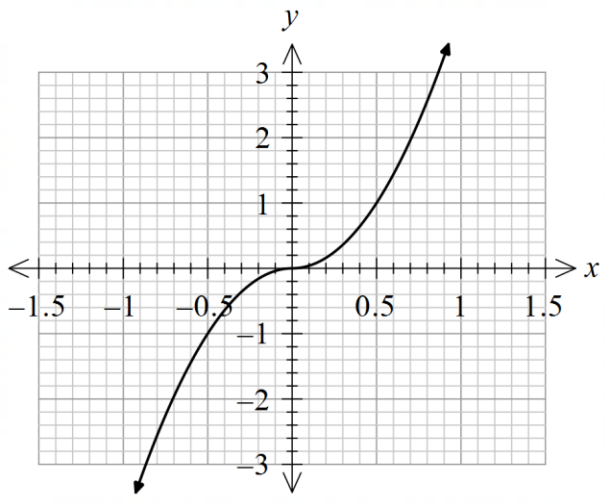
A. $\frac{2^n}{\sqrt{5}}$

B. $\frac{n}{4}$

C. $\sqrt{\frac{2^n}{5}}$

D. 2^n

9 The graph of $y = 4x|x|$ is given below.



For what values of x will the following series have a limiting sum?

$$4x|x| + 16x^4 + 64x^5|x| + \ldots$$

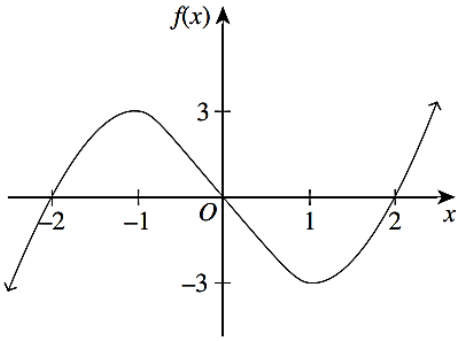
A. $-1 \le x \le 1$

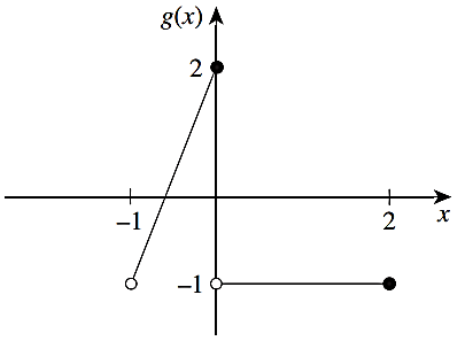
B. $x < -0.5,\ x > 0.5$

C. $-0.5 < x < 0.5$

D. $x = -0.5,\ x = 0.5$

10 The graphs of $y = f(x)$ and $y = g(x)$ are shown.





What is the domain and range of $y = f(g(x))$?

	Domain	**Range**
A.	$(-1, 2]$	$[-3, 3]$
B.	$(-1, 2]$	$[-1, 2]$
C.	$(-\infty, \infty)$	$[-3, 3)$
D.	$(-\infty, \infty)$	$(-1, 2]$

Student Number

Mathematics Advanced
Section II Answer Booklet 1

1

Section II

70 marks
Attempt Questions 11–33
Allow about 1 hour and 45 minutes for this section

Booklet 1 – Attempt Questions 11–26 (42 marks)
Booklet 2 – Attempt Questions 27–33 (28 marks)

Instructions

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Please turn over

Question 11 (2 marks)

Given that $\log_a 2 = 0.36$ and $\log_a 12 = 1.28$, evaluate $\log_a 3$. **2**

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Question 12 (2 marks)

Solve the equation $|3x - 5| = 7$. **2**

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Question 13 (2 marks)

State the domain and range of $(x-2)^2 + (y+1)^2 = 1$. Give your answer in interval notation. **2**

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Question 14 (2 marks)

Find $\lim_{x \to 2} \dfrac{2x^2 - 3x - 2}{x - 2}$. **2**

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Question 15 (2 marks)

Solve the equation $\cos^2 x = \dfrac{1}{2}$ for $0 \le x \le 2\pi$. **2**

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Question 16 (2 marks)

Determine whether the function $f(x) = \dfrac{1+\cos x}{1-\sin x}$ is odd, even or neither. **2**

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Question 17 (2 marks)

Sketch the graph of $y = \tan\left(x + \dfrac{\pi}{4}\right)$ for $0 \le x \le \pi$. **2**

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Question 18 (2 marks)

Prove $\dfrac{\csc\theta\sec\theta}{\tan\theta+\cot\theta}=1$. **2**

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Question 19 (5 marks)

Differentiate the following:

(a) $y=4+2x^2-x^{-3}$ **1**

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(b) $y=\dfrac{-x^2+10x-16}{x^2}$ **2**

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Question 19 continues on next page

Question 19 (continued)

(c) $y = (2-x)(3+2x)^5$ **2**

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Question 20 (2 marks)

Find the gradient of the tangent to the curve $y = \dfrac{3x^2}{x-1}$ at the point $\left(-1, -\dfrac{3}{2}\right)$. **2**

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Question 21 (3 marks)

Differentiate $f(x) = \frac{1}{x}$ from first principles given that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. **3**

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Question 22 (2 marks)

While writing this exam, the teacher accidentally knocked over and spilt their coffee onto their rug creating a stain. The area (in cm^2) of the coffee stain after time t seconds is given by the equation $A = 12 - \frac{12}{2t+1}$.

Find the instantaneous rate of change in the area of the coffee stain after 5 seconds. **2**

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Question 23 (2 marks)

The sum of the first *n* terms of a series is given by $S_n = 4 - n - n^2$ where $n \geq 2$. **2**

Find the sum from the 15th term to the 40th term inclusive.

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Question 24 (5 marks)

The heights above the ground of the rungs in a ladder form an arithmetic sequence.

The third rung is 70 cm above the ground and the tenth rung is 210 cm above the ground.

(a) Find the height of the first rung above the ground, and find the height difference between consecutive rungs on the ladder. **3**

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(b) If the top rung is 350 cm above the ground, how many rungs does the ladder have? **2**

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Question 25 (4 marks)

The graph of $f(x) = \sqrt{x}$ is transformed into the graph of $y = g(x)$ through the transformations given in the following order:

1. Dilate horizontally by a factor of $\frac{1}{2}$.
2. Reflect about the *x*-axis.
3. Then translate horizontally to the right by 2 units.

(a) Write down the equation of the resultant function $g(x)$. **2**

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(b) Sketch the graph of $y = g(x)$. **2**

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Question 26 **(**3 marks)

Solve $\log_{10}(x-3)+\log_{10}(x+2)=\log_{10}6$. **3**

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End of Booklet 1

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Student Number

Mathematics Advanced
Section II Answer Booklet 2

2

Booklet 2 – Attempt Questions 27–33 (28 marks)

Instructions

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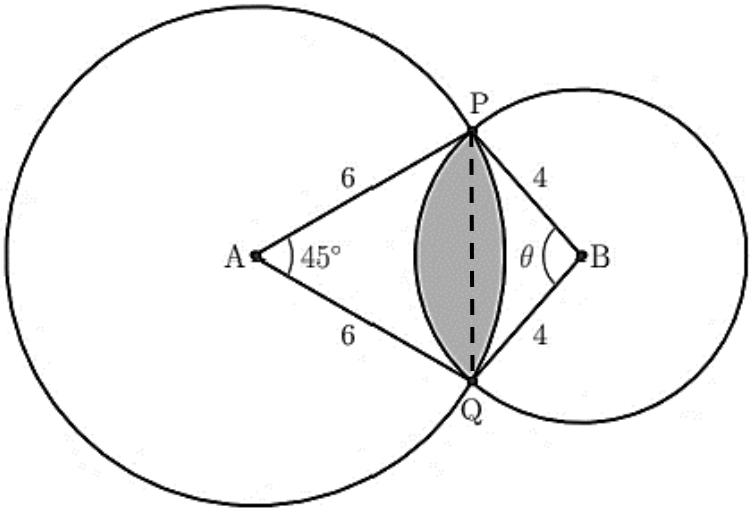
Please turn over

Question 27 (4 marks)

Two circles, with centres A and B, intersect at points P and Q. The radii of the circles are 6 cm and 4 cm and $\angle PAQ = 45^\circ$ as shown in the diagram below.



(a) Show that $PQ = 6\sqrt{2-\sqrt{2}}$ **2**

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(b) Given that $\angle PBQ = 70.1^\circ$, find the perimeter of the shaded region to the nearest cm. **2**

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Question 28 (4 marks)

Doctor Lutra is investigating the population of otters in Scotland. Based on historical data, the doctor found that the population of otters, *P*, could be modelled by $P = 7500 + A(1.09)^t$ where A is a constant and *t* is the number of years since the start of the year 2000, $t \geq 0$.

(a) At the start of the year 2000, the population of otters in Scotland was 8000. **1**

Find the value of the constant *A*.

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(b) Find the population of otters in Scotland at the start of the year 2020. **1**

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(c) Doctor Lutra estimates the peak population of otters Scotland can sustain is 15 500. **2**

Work out the year in which Doctor Lutra expects the population of otters to peak.

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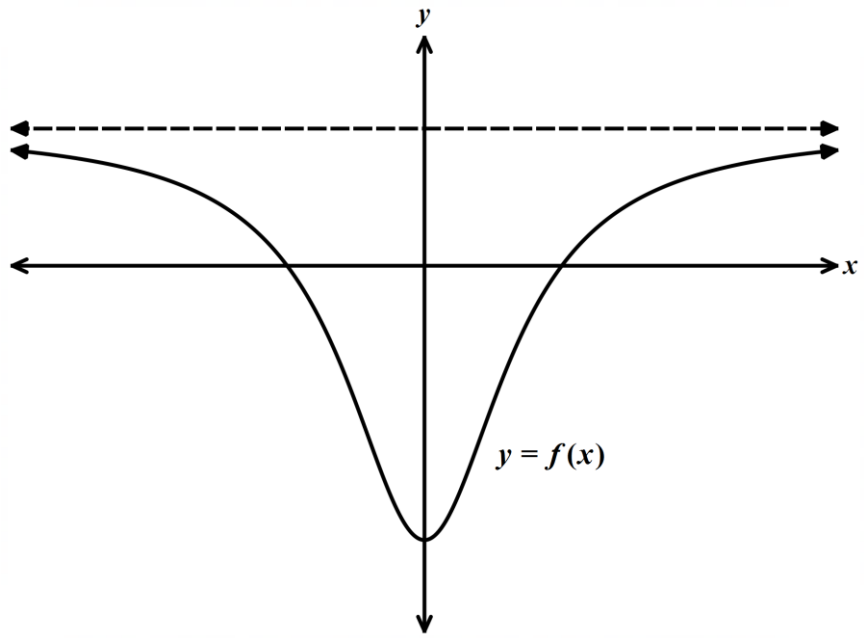
Question 29 (3 marks)

The graph of a function $y = f(x)$ is shown below.



(a) For what value of x is $f'(x) = 0$? **1**

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(b) What happens to $f'(x)$ as $x \to \infty$ and $x \to -\infty$? **1**

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(c) Sketch the graph of $y = f'(x)$ on the same number plane above showing all necessary features. **1**

Question 30 (3 marks)

Find the values of c for which the curve $y = x^2 + 2cx + 4$ and the line $y = 4x + c$ intersect at two distinct points. **3**

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Question 31 (6 marks)

The sides of a square are 16 cm in length. The midpoints of the sides of this square are joined to form a new square and four triangles. One of these triangles is shaded (diagram 1). The process is repeated twice as shown in diagrams 2 and 3. In each iteration, one of the areas of the new triangles is shaded.

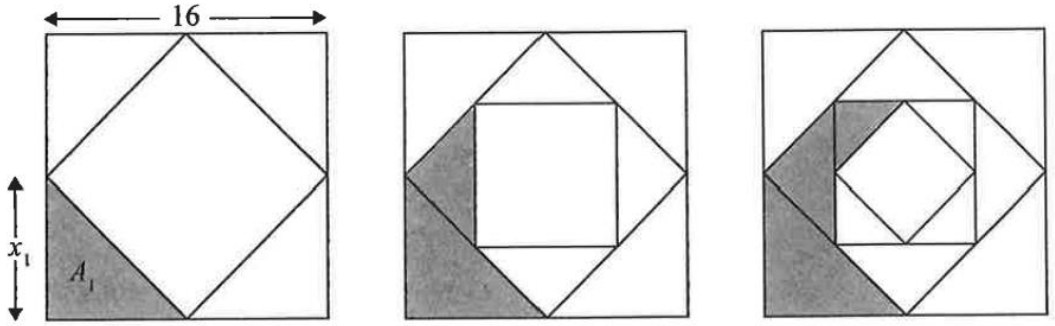

diagram 1 diagram 2 diagram 3

Let x_n denote the length of one of the equal sides of each new triangle. It is known that the lengths of the sides form a geometric progression.

Let A_n denote the area of each new triangle.

(a) Show that the expression for x_n is given by $x_n = \left(\sqrt{2}\right)^{7-n}$ **2**

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(b) Hence, find an expression for A_n. **1**

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Question 31 continues on next page

Question 31 (continued)

Now consider an initial square of side length k cm. The process described above is repeated indefinitely.

(c) By considering the pattern made by the areas of the shaded triangles, find the value of k if the total area of the shaded triangles is k cm^2. **3**

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Question 32 (4 marks)

Research shows that the stopping distance (d) in metres of a car once brakes are applied is directly proportional to the square of the speed (v) in km/h when the brakes are first applied. Assume that the driver's reaction time is neglected. The stopping distance for a car travelling at 60 km/h is 43.2 m.

(a) Show that the stopping distance for a car travelling at 70 km/h is 58.8 m. **2**

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(b) Further research shows that the driver's reaction time is an important factor in determining the stopping distance. The average driver takes 1.5 seconds to realise they need to apply the brakes.

Calculate the new stopping distance for a car travelling at 70 km/h which takes the reaction time into account. **2**

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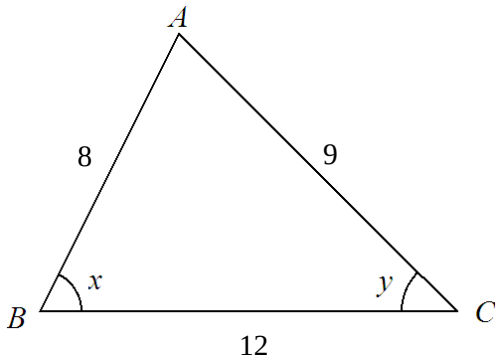
Question 33 (4 marks)

The diagram below shows ΔABC with $AC = 9$, $BC = 12$ and $AB = 8$.

$\angle ABC = x$ and $\angle ACB = y$.



(a) Express the size of $\angle BAC$ in terms of x and y. **1**

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(b) Hence, show that $\sin x + \sin y = \frac{17}{12}\sin(x + y)$ **3**

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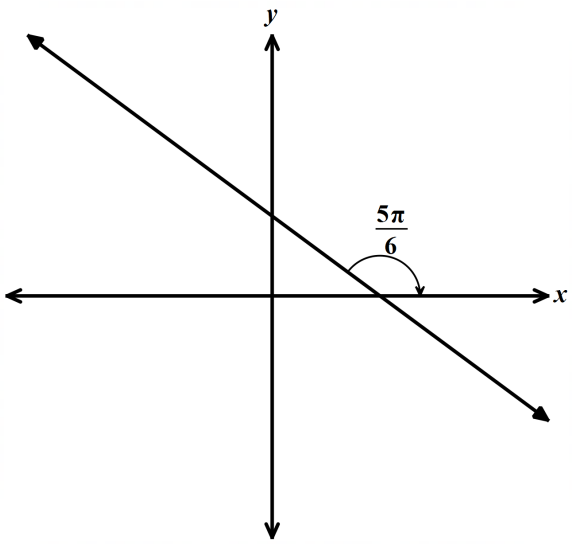

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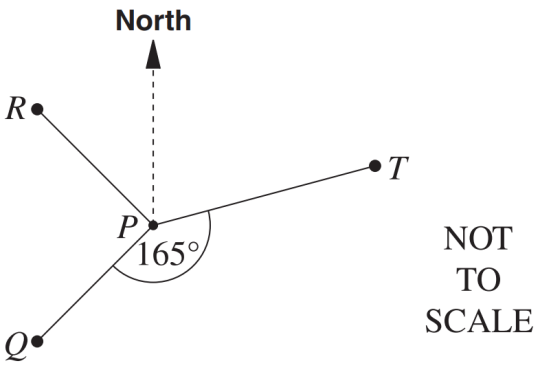

NOT TO SCALE

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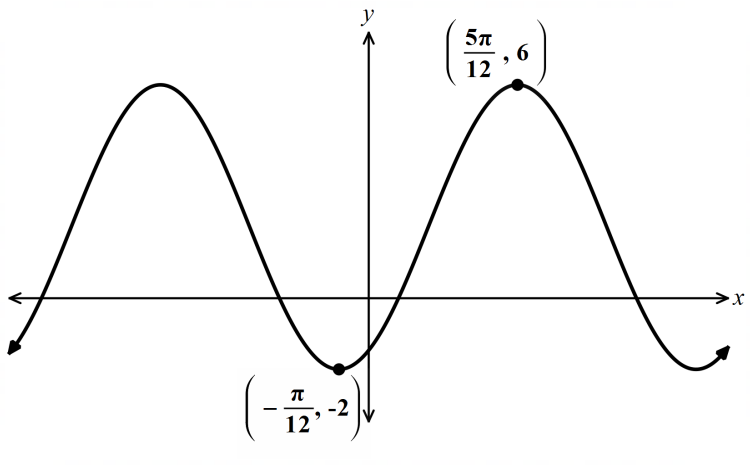

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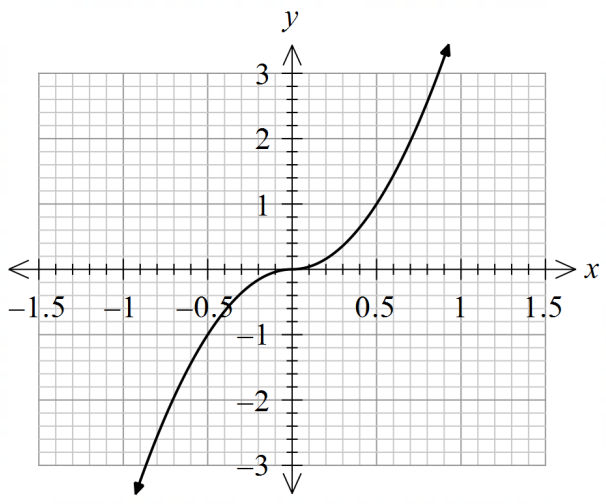

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D. $x = -0.5,\ x = 0.5$

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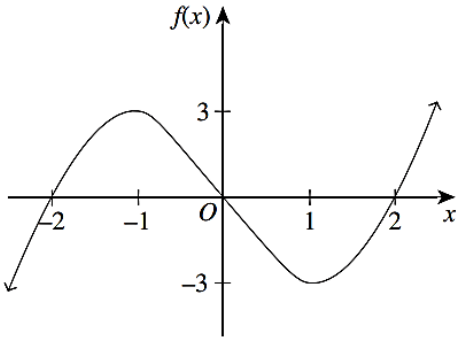

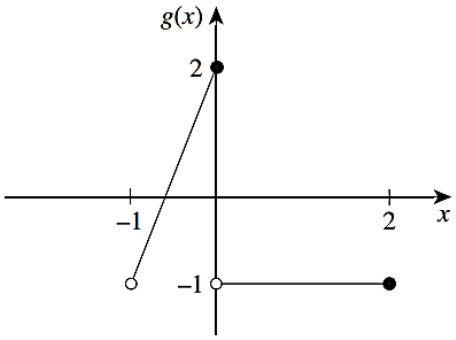

What is the domain and range of $y = f(g(x))$?

	Domain	**Range**
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B.	$(-1, 2]$	$[-1, 2]$
C.	$(-\infty, \infty)$	$[-3, 3)$
D.	$(-\infty, \infty)$	$(-1, 2]$

Student Number

Mathematics Advanced
Section II Answer Booklet 1

1

Section II

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Question 11 (2 marks)

Given that $\log_a 2 = 0.36$ and $\log_a 12 = 1.28$, evaluate $\log_a 3$. **2**

$\log_a 3 = \log_a \frac{12}{4}$

$= \log_a 12 - \log_a 4$

$= 1.28 - \log_a 2^2$

$= 1.28 - 2\log_a 2$

$= 1.28 - 2 \times 0.36$

$= 0.56$

Question 12 (2 marks)

Solve the equation $|3x - 5| = 7$. **2**

$3x - 5 = 7$ $\quad$ $3x - 5 = -7$

$3x = 12$ $\quad$ $3x = -2$

$x = 4$ $\quad$ $x = -\frac{2}{3}$

Question 13 (2 marks)

State the domain and range of $(x-2)^2 + (y+1)^2 = 1$. Give your answer in interval notation. **2**

D: $[1, 3]$

R: $[-2, 0]$

Question 14 (2 marks)

Find $\lim_{x\to 2}\dfrac{2x^2-3x-2}{x-2}$. **2**

$= \lim_{x\to 2} \dfrac{(2x+1)(x-2)}{x-2}$

$= \lim_{x\to 2} 2x+1$

$= 2\times 2+1$

$= 5$

Question 15 (2 marks)

Solve the equation $\cos^2 x = \dfrac{1}{2}$ for $0 \le x \le 2\pi$. **2**

$\cos x = \pm\sqrt{\frac{1}{2}}$

$\cos x = \pm\frac{1}{\sqrt{2}}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Question 16 (2 marks)

Determine whether the function $f(x)=\dfrac{1+\cos x}{1-\sin x}$ is odd, even or neither. **2**

$f(-x) = \dfrac{1+\cos(-x)}{1-\sin(-x)}$

$= \dfrac{1+\cos x}{1-(-\sin x)}$ since $\cos x$ is even & $\sin x$ is odd.

$= \dfrac{1+\cos x}{1+\sin x}$

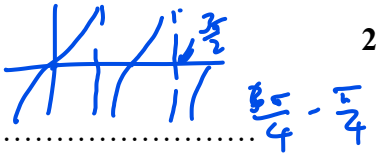
$-f(x) = -\dfrac{1+\cos x}{1-\sin x}$

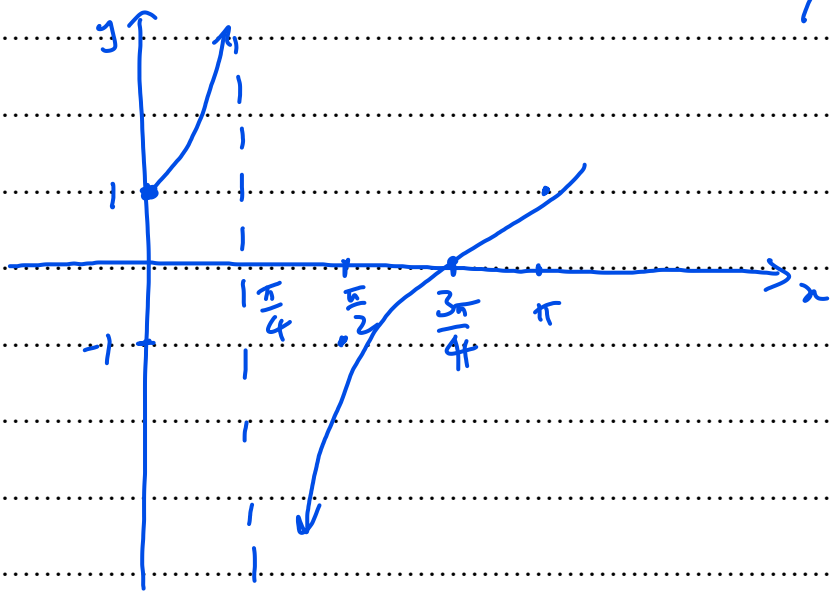
$= \dfrac{1+\cos x}{\sin x - 1}$

$\therefore f(x) \neq f(-x)$ & $-f(x) \neq f(-x)$ $\therefore f(x)$ is neither

Question 17 (2 marks)

Sketch the graph of $y=\tan\left(x+\dfrac{\pi}{4}\right)$ for $0 \le x \le \pi$. **2**





Question 18 (2 marks)

Prove $\dfrac{\csc\theta\sec\theta}{\tan\theta+\cot\theta}=1$. **2**

$$\text{LHS} = \frac{\csc\theta\sec\theta}{\tan\theta+\cot\theta}$$

$$= \frac{\frac{1}{\sin\theta}\times\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}} \quad \times \sin\theta\cos\theta \quad \times \sin\theta\cos\theta$$

$$= \frac{1}{\sin^2\theta+\cos^2\theta}$$

$$= \frac{1}{1}$$

$$= 1$$

Question 19 (5 marks)

Differentiate the following:

(a) $y = 4+2x^2-x^{-3}$ **1**

$$y' = 4x+3x^{-4}$$

(b) $y = \dfrac{-x^2+10x-16}{x^2}$ **2**

$$y = -\frac{x^2}{x^2}+\frac{10x}{x^2}-\frac{16}{x^2}$$

$$= -1+10x^{-1}-16x^{-2}$$

$$y' = 0-10x^{-2}+32x^{-3}$$

Question 19 continues on next page

Question 19 (continued)

(c) $y = (2-x)(3+2x)^5$ **2**

$y' = -1(3+2x)^5 + (2-x) \times 5(3+2x)^4 \times 2$

$= -(3+2x)^5 + 10(2-x)(3+2x)^4$

$= (3+2x)^4[10(2-x) - (3+2x)]$

$= (3+2x)^4(20 - 10x - 3 - 2x)$

$= (3+2x)^4(17 - 12x)$

Question 20 (2 marks)

Find the gradient of the tangent to the curve $y = \dfrac{3x^2}{x-1}$ at the point $\left(-1, -\dfrac{3}{2}\right)$. **2**

$y' = \dfrac{6x(x-1) - 3x^2 \times 1}{(x-1)^2}$

$= \dfrac{6x^2 - 6x - 3x^2}{(x-1)^2}$

$= \dfrac{3x^2 - 6x}{(x-1)^2}$

when $x = -1$, $y' = \dfrac{3(-1)^2 - 6(-1)}{(-1-1)^2}$

$= \dfrac{3+6}{4}$

$= \dfrac{9}{4}$

Question 21 (3 marks)

Differentiate $f(x)=\frac{1}{x}$ from first principles given that $f'(x)=\lim_{x\to\infty}\frac{f(x+h)-f(x)}{h}$. **3**

$$f'(x)=\lim_{x\to\infty}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\lim_{x\to\infty}\frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$=\lim_{x\to\infty}\frac{x-x-h}{hx(x+h)}$$

$$=\lim_{x\to\infty}\frac{-h}{hx(x+h)}$$

$$=\lim_{x\to\infty}-\frac{1}{x(x+h)}$$

$$=-\frac{1}{x(x+0)}$$

$$=-\frac{1}{x^2}$$

Question 22 (2 marks)

While writing this exam, the teacher accidentally knocked over and spilt their coffee onto their rug creating a stain. The area (in cm^2) of the coffee stain after time t seconds is given by the equation $A = 12 - \frac{12}{2t+1}$.

Find the instantaneous rate of change in the area of the coffee stain after 5 seconds. **2**

$A = 12 - \frac{12}{2t+1}$

$= 12 - 12(2t+1)^{-1}$

$\frac{dA}{dt} = -12 \times -1 \times (2t+1)^{-2} \times 2$

$= \frac{24}{(2t+1)^2}$

when $t = 5$,

$\frac{dA}{dt} = \frac{24}{(2\times 5+1)^2}$

$= \frac{24}{121}$ cm/s

Question 23 (2 marks)

The sum of the first n terms of a series is given by $S_n = 4 - n - n^2$ where $n \geq 2$. **2**

Find the sum from the 15th term to the 40th term inclusive.

Sum $= S_{40} - S_{14}$

$= (4 - 40 - 40^2) - (4 - 41 - 41^2)$

$= 82$

Question 24 (5 marks)

The heights above the ground of the rungs in a ladder form an arithmetic sequence.

The third rung is 70 cm above the ground and the tenth rung is 210 cm above the ground.

(a) Find the height of the first rung above the ground, and find the height difference between consecutive rungs on the ladder. **3**

Let the height above the ground of the first rung be 'a' & the height difference between each rung be d.

$T_3 = a + 2d$

$70 = a + 2d$ —(1)

$T_{10} = a + 9d$

$210 = a + 9d$ —(2)

(2) − (1) $210 - 70 = 7d$

$140 = 7d$

$d = 20$

Sub in (1)

$70 = a + 2 \times 20$

$a = 30$

(b) If the top rung is 350 cm above the ground, how many rungs does the ladder have? **2**

$T_n = a + (n-1)d$

$350 = 30 + (n-1) \times 20$

$320 = 20(n-1)$

$16 = n - 1$

$n = 17$

∴ the ladder has 17 rungs

Question 25 (4 marks)

The graph of $f(x) = \sqrt{x}$ is transformed into the graph of $y = g(x)$ through the transformations given in the following order:

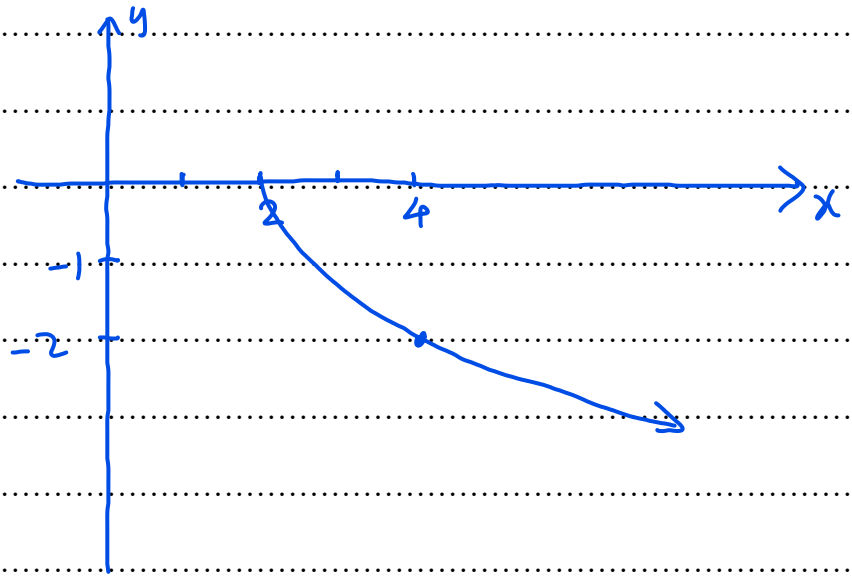
1. Dilate horizontally by a factor of $\frac{1}{2}$.
2. Reflect about the x-axis.
3. Then translate horizontally to the right by 2 units.

(a) Write down the equation of the resultant function $g(x)$. **2**

$\sqrt{x} \to \sqrt{2x} \to -\sqrt{2x} \to -\sqrt{2(x-2)}$

$g(x) = -\sqrt{2x-4}$

(b) Sketch the graph of $y = g(x)$. **2**



Question 26 (3 marks)

Solve $\log_{10}(x-3)+\log_{10}(x+2)=\log_{10}6$. **3**

$\log_{10}(x-3)(x+2) = \log_{10}6$

$(x-3)(x+2) = 6$

$x^2 - x - 6 = 6$

$x^2 - x - 12 = 0$

$(x-4)(x+3) = 0$

$x = 4, \; x = -3$

but $x-3 > 0$ & $x+2 > 0$ $\therefore$ $x = 4$ only

End of Booklet 1

Student Number

Mathematics Advanced
Section II Answer Booklet 2

2

Booklet 2 – Attempt Questions 27–33 (28 marks)

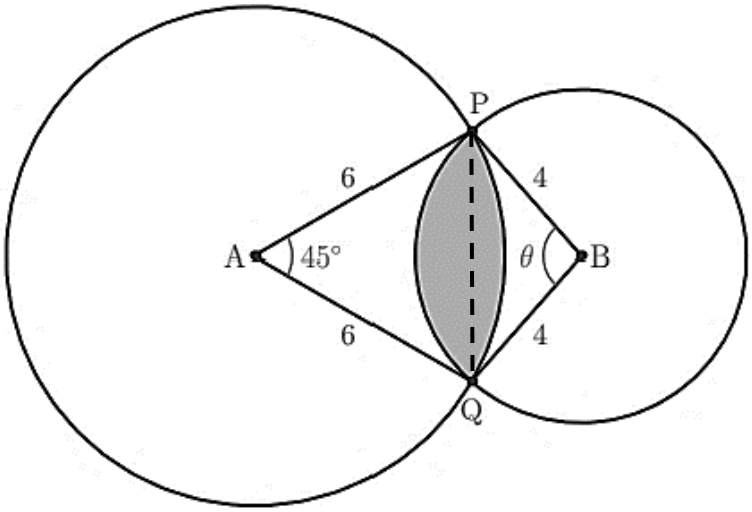
Instructions

- Write your student number at the top of this page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided on pages at the end of each booklet. If you use this space, clearly indicate which question you are answering.

Please turn over

Question 27 (4 marks)

Two circles, with centres A and B, intersect at points P and Q. The radii of the circles are 6 cm and 4 cm and $\angle PAQ = 45^\circ$ as shown in the diagram below.



(a) Show that $PQ = 6\sqrt{2-\sqrt{2}}$ **2**

$PQ^2 = 6^2 + 6^2 - 2\times6\times6\times\cos 45$

$= 72 - 72\times\frac{\sqrt{2}}{2}$

$= 72 - 36\sqrt{2}$

$= 36(2-\sqrt{2})$

$\therefore PQ = \sqrt{36(2-\sqrt{2})}$

$= 6\sqrt{2-\sqrt{2}}$

(b) Given that $\angle PBQ = 70.1^\circ$, find the perimeter of the shaded region to the nearest cm. **2**

$P = \frac{45}{360}\times 2\pi\times 6 + \frac{70.1}{360}\times 2\pi\times 4$

$= \frac{3}{2}\pi + \frac{70.1}{360}\times 8\pi$

$= 9.606...$

$= 9$ cm (nearest cm)

Question 28 (4 marks)

Doctor Lutra is investigating the population of otters in Scotland. Based on historical data, the doctor found that the population of otters, P, could be modelled by $P = 7500 + A(1.09)^t$ where A is a constant and t is the number of years since the start of the year 2000, $t \geq 0$.

(a) At the start of the year 2000, the population of otters in Scotland was 8000. **1**

Find the value of the constant A.

$t = 0, P = 8000$

$8000 = 7500 + A(1.09)^0$

$= 7500 + A \quad \therefore A = 500$

(b) Find the population of otters in Scotland at the start of the year 2020. **1**

$t = 20$

$P = 7500 + 500(1.09)^{20}$

$= 10302.205...$

$= 10302$ (nearest otter)

(c) Doctor Lutra estimates the peak population of otters Scotland can sustain is 15 500. **2**

Work out the year in which Doctor Lutra expects the population of otters to peak.

$15500 = 7500 + 500(1.09)^t$

$8000 = 500(1.09)^t$

$16 = (1.09)^t$

$\ln 16 = \ln (1.09)^t$

$\ln 16 = t \ln (1.09)$

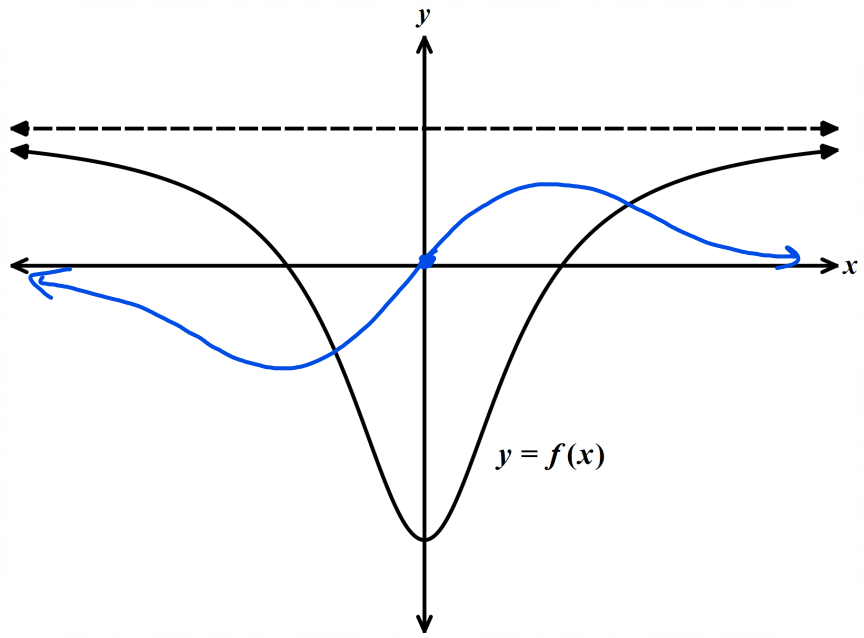
$t = \frac{\ln 16}{\ln 1.09}$

$= 32.17$

$\therefore$ in the year 2032

Question 29 (3 marks)

The graph of a function $y = f(x)$ is shown below.



(a) For what value of x is $f'(x) = 0$? **1**

$x = 0$

.....

.....

(b) What happens to $f'(x)$ as $x \to \infty$ and $x \to -\infty$? **1**

$f'(x) \to 0$

.....

.....

(c) Sketch the graph of $y = f'(x)$ on the same number plane above showing all necessary features. **1**

Question 30 (3 marks)

Find the values of c for which the curve $y = x^2 + 2cx + 4$ and the line $y = 4x + c$ intersect at two distinct points. **3**

$x^2 + 2cx + 4 = 4x + c$

$x^2 + 2cx - 4x + 4 - c = 0$

$x^2 + (2c-4)x + (4-c) = 0$

there must be 2 distinct solutions

$\therefore \Delta > 0$

$b^2 - 4ac > 0$

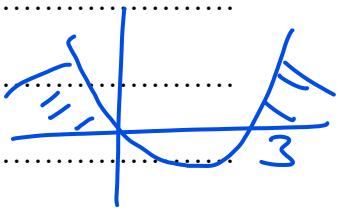
$(2c-4)^2 - 4(1)(4-c) > 0$

$4c^2 - 16c + 16 - 16 + 4c > 0$

$4c^2 - 12c > 0$

$c^2 - 3c > 0$

$c(c-3) > 0$



$c < 0, \quad c > 3$

Question 31 (6 marks)

The sides of a square are 16 cm in length. The midpoints of the sides of this square are joined to form a new square and four triangles. One of these triangles is shaded (diagram 1). The process is repeated twice as shown in diagrams 2 and 3. In each iteration, one of the areas of the new triangles is shaded.

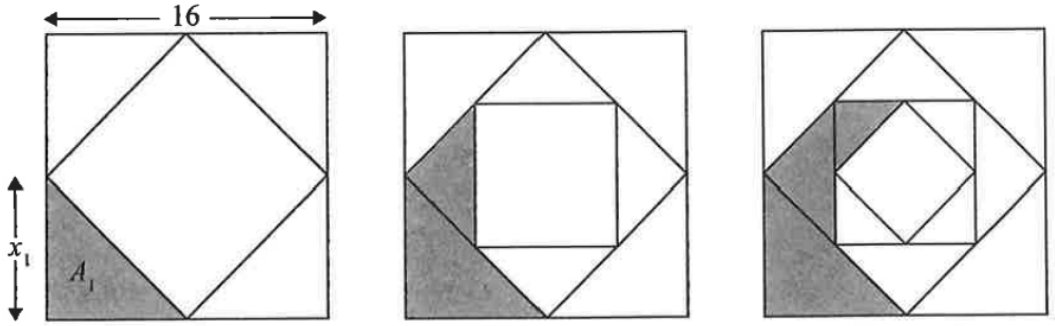


diagram 1 diagram 2 diagram 3

Let x_n denote the length of one of the equal sides of each new triangle. It is known that the lengths of the sides form a geometric progression.

Let A_n denote the area of each new triangle.

(a) Show that the expression for x_n is given by $x_n = \left(\sqrt{2}\right)^{7-n}$ **2**

$x_1 = \frac{16}{2} = 8$

$\frac{x_2}{x_1} = \frac{8}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$

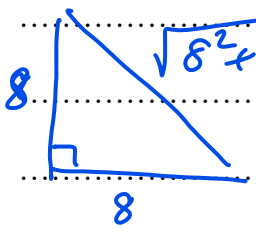
$\therefore r = \frac{1}{\sqrt{2}}$

$x_n = ar^{n-1}$

$= 8 \times \left(\frac{1}{\sqrt{2}}\right)^{n-1}$

$= (\sqrt{2})^6 \times \sqrt{2}^{1-n}$

$= \sqrt{2}^{7-n}$



$\therefore x_2 = \frac{1}{2} \times 8\sqrt{2}$

$= 4\sqrt{2}$

(b) Hence, find an expression for A_n. **1**

$A_n = \frac{1}{2}(x_n)^2$

$= \frac{1}{2} \times \left((\sqrt{2})^{7-n}\right)^2$

$= \frac{1}{2} \times \left((\sqrt{2})^2\right)^{7-n}$

$= \frac{1}{2} \times 2^{7-n}$

$= 2^{6-n}$

Question 31 continues on next page

Question 31 (continued)

Now consider an initial square of side length k cm. The process described above is repeated indefinitely.

(c) By considering the pattern made by the areas of the shaded triangles, find the value of k if the total area of the shaded triangles is k cm^2. **3**

From part (b) it can be seen each subsequent triangle is half the area of the previous one

Since $r = \frac{1}{2}$, there is a limiting sum

Area of 1st triangle $= \left(\frac{k}{2}\right)^2 \times \frac{1}{2}$

$= \frac{k^2}{8}$

$\therefore$ Total sum $= \frac{a}{1-r}$

$k = \frac{\frac{k^2}{8}}{1-\frac{1}{2}}$

$k = \frac{k^2}{4}$

$4k = k^2$

$0 = k^2 - 4k$

$0 = k(k-4)$

$k = 0, \; k = 4$

$\therefore k = 4 \quad k > 0$

Question 32 (4 marks)

Research shows that the stopping distance (d) in metres of a car once brakes are applied is directly proportional to the square of the speed (v) in km/h when the brakes are first applied. Assume that the driver's reaction time is neglected. The stopping distance for a car travelling at 60 km/h is 43.2 m.

(a) Show that the stopping distance for a car travelling at 70 km/h is 58.8 m. **2**

$d \propto v^2$

$d = kv^2$ $\quad \therefore d = 0.012 \times 70^2$

$43.2 = k \times 60^2$ $\quad = 58.8\text{ m}$

$k = 0.012$

(b) Further research shows that the driver's reaction time is an important factor in determining the stopping distance. The average driver takes 1.5 seconds to realise they need to apply the brakes.

Calculate the new stopping distance for a car travelling at 70 km/h which takes the reaction time into account. **2**

Stopping distance $= \frac{70 \times 1000}{60 \times 60} \times 1.5 + 58.8\text{ m}$

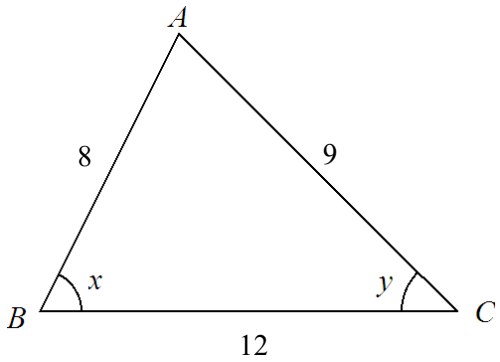
$= 87.966\ldots$

$= 87.97\text{m}$

Question 33 (4 marks)

The diagram below shows ΔABC with $AC = 9$, $BC = 12$ and $AB = 8$.

$\angle ABC = x$ and $\angle ACB = y$.



(a) Express the size of $\angle BAC$ in terms of x and y. **1**

$180 - (x+y)$

(b) Hence, show that $\sin x + \sin y = \frac{17}{12}\sin(x+y)$ **3**

$\frac{\sin(180-(x+y))}{12} = \frac{\sin x}{9} = \frac{\sin y}{8}$

$\therefore \frac{\sin(x+y)}{12} = \frac{\sin x}{9} = \frac{\sin y}{8}$

$\sin x = \frac{9}{12}\sin(x+y)$

$\sin y = \frac{8}{12}\sin(x+y)$

$\sin x + \sin y = \frac{9}{12}\sin(x+y) + \frac{8}{12}\sin(x+y)$

$= \frac{17}{12}\sin(x+y)$

End of paper