SYDNEY GRAMMAR SCHOOL

NAME

MATHS MASTER

CANDIDATE NUMBER

2023 Annual

Form V Mathematics Advanced

Tuesday 29th August 2023

8:40am

General Instructions

- Working time — 2 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.
- Remove the staple before you commence writing

Total Marks: 80

Section I (8 marks) Questions 1 – 8

- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (72 marks) Questions 9 – 14

- Relevant mathematical reasoning and calculations are required.
- Answer the questions in this paper in the spaces provided.

Collection

- Write your candidate number on every booklet and the multiple choice sheet
- Place everything inside this cover booklet.

Checklist

- Reference sheet
- Multiple-choice answer sheet
- Candidature: 177 pupils

Writer: FMW

	Marks
Multiple Choice	
Question 9	
Question 10	
Question 11	
Question 12	
Question 13	
Question 14	
TOTAL	

Section I

QUESTION ONE (1 mark)

Which of the following is the length of the interval joining the points $(4, 7)$ and $(6, -3)$?

(A) $2\sqrt{26}$

(B) $2\sqrt{29}$

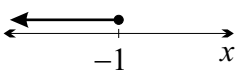
(C) 104

(D) 116

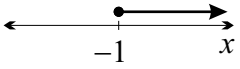
QUESTION TWO (1 mark)

Which of the following is the correct graph of the solution of the inequation $5 - 2x \leq 3$?

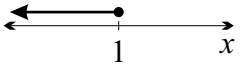
(A)



(B)



(C)



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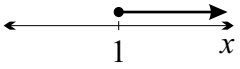


QUESTION THREE (1 mark)

What is the value of $\log_5 9$ correct to 3 significant figures?

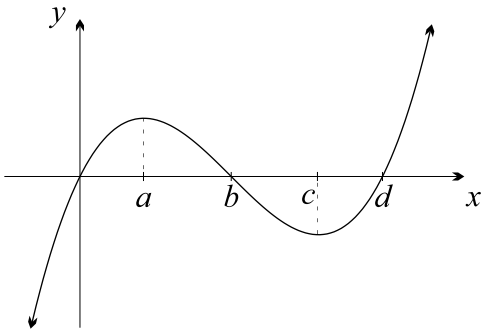
(A) 0.73

(B) 0.732

(C) 1.36

(D) 1.37

QUESTION FOUR (1 mark)



The graph of the curve $y = f(x)$ is shown above. For what values of x will the tangent to the curve have a negative gradient?

(A) $x < a$

(B) $a < x < b$

(C) $a < x < c$

(D) $x > c$

QUESTION FIVE (1 mark)

For the curve, $y = \log_2 x$, which of the following is NOT a true statement?

(A) The curve passes through the point $(1, 0)$.

(B) The gradient of the curve is always positive.

(C) The range of the curve is $y > 0$.

(D) The domain of the curve is $x > 0$.

QUESTION SIX (1 mark)

For the equation $\sin\theta = \dfrac{1}{2}$, which of the following is NOT a correct solution?

(A) $-30°$

(B) $30°$

(C) $-\dfrac{7\pi}{6}$

(D) $\dfrac{\pi}{6}$

QUESTION SEVEN (1 mark)

Which of the following is the derivative of $\dfrac{1}{\sqrt{e^{6x}}}$?

(A) $\dfrac{1}{\sqrt{6e^{6x}}}$

(B) $\dfrac{-3}{e^{3x}}$

(C) $-3e^{-3x-1}$

(D) $\dfrac{-1}{\sqrt{6e^{6x-1}}}$

QUESTION EIGHT (1 mark)

For what values of r does the graph of $x^2 + 4x + y^2 - 2y + 5 = r^2$ intersect the x-axis but not the y-axis?

(A) $r < 1$

(B) $1 < r < 2$

(C) $2 < r < \sqrt{5}$

(D) $r > \sqrt{5}$

End of Section I

The paper continues in the next section

CANDIDATE NUMBER

Section II

QUESTION NINE (12 marks) **Marks**

(a) Consider the line with equation $3x - 2y + 18 = 0$.

(i) Find its gradient. 1

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(ii) Write down the equation of the line which passes through the origin and is perpendicular to the line $3x - 2y + 18 = 0$. 1

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(b) Write down the centre and radius of the circle with equation $(x-1)^2 + y^2 = 4$. 1

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(c) Convert 29° to radians, leaving your answer in exact form. 1

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(d) A fair coin is tossed five times. What is the probability of tossing 5 heads? 1

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(e) Find the exact value of $\tan \dfrac{5\pi}{6}$. [1]

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(f) Differentiate the following functions with respect to x:

(i) $f(x) = e^{5x-1}$ [1]

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(ii) $y = (3x-4)^5$ [1]

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QUESTION NINE (Continued)

(g) An arc AB subtends an angle of 1.1 radians at the centre of a circle, centre O, of radius length 4 centimetres. Find the area of sector AOB. [1]

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(h) Find the equation of the tangent to the curve $y = \dfrac{18}{x^2}$ at the point $(3, 2)$. Give your answer in general form. [3]

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Extra writing space (Use this page only for answers for Question 9.)

If you use this space, clearly indicate which question you are answering.

CANDIDATE NUMBER

QUESTION TEN (11 marks) **Marks**

(a) (i) Factorise $2x^2 + x - 6$. 1

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(ii) Hence, simplify: 2

$$\frac{3}{x^2 - 4} + \frac{8}{2x^2 + x - 6}$$

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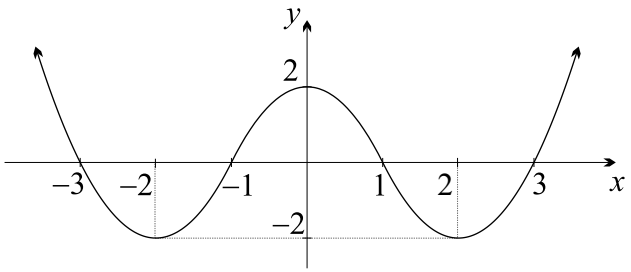
QUESTION TEN (Continued)

(b)



The diagram above shows the function $y = f(x)$.

(i) Is the function ODD or EVEN? Explain your answer. [1]

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(ii) Sketch the graph of $y = f(x-3)$, clearly showing the coordinates of any intercepts with the axes. [1]

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QUESTION TEN (Continued)

(iii) Sketch the graph of $y = -f(x) - 2$, clearly indicating any intercepts with the axes. [3]

QUESTION TEN (Continued)

(c) Find the values of p and q if $(2\sqrt{3}-1)(5-3\sqrt{3}) = p + q\sqrt{3}$. [2]

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(d) Express $\log_a x^2 y$ in terms of $\log_a x$ and $\log_a y$. [1]

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CANDIDATE NUMBER

QUESTION ELEVEN (11 marks) **Marks**

(a) Find the equation of the normal to the curve $y = 2\sqrt{x}$, at the point on the curve where $x = 9$. [3]

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QUESTION ELEVEN (Continued)

(b) (i) Sketch the curve $y = f(x)$ where: 2

$$f(x) = \begin{cases} -1, & \text{for } x < 0, \\ x^2, & \text{for } x \geq 0\,. \end{cases}$$

(ii) Is this function differentiable at $x = 0$? Explain your answer. 1

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(iii) Is this function *one-to-one* or *many-to-one*? Explain your answer. 1

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QUESTION ELEVEN (Continued)

(c) A triangle has area $12\,\text{cm}^2$. Two of the sides have lengths $6\,\text{cm}$ and $8\,\text{cm}$, and θ is the included angle. Show that $\sin\theta = \dfrac{1}{2}$ and calculate the length of the missing side, giving all possible answers, correct to 2 decimal places. [4]

Extra writing space (Use this page only for answers for Question 11.)

If you use this space, clearly indicate which question you are answering.

CANDIDATE NUMBER

QUESTION TWELVE (13 marks) **Marks**

(a) Evaluate $\lim\limits_{h\to 0} \dfrac{h^2 - 4h}{h}$. 1

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(b) Solve $\tan^2\theta = 4$ for $0 \leq \theta \leq 2\pi$. Give your answers correct to 2 decimal places. 2

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QUESTION TWELVE (Continued)

(c) (i) Sketch $y = |x - 3|$ and $y = |2x + 6|$ on the same number plane showing clearly all intercepts with the axes. [3]

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(ii) Hence, or otherwise, solve $|x - 3| = |2x + 6|$, for $-3 \leq x \leq 3$. [1]

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QUESTION TWELVE (Continued)

(d) Differentiate $y = x(2-x)^5$. [2]

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(e) Find all the points on the curve $y = \dfrac{x^2}{x+16}$, where the tangent is horizontal. [4]

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Extra writing space (Use this page only for answers for Question 12.)

If you use this space, clearly indicate which question you are answering.

CANDIDATE NUMBER

QUESTION THIRTEEN (13 marks) **Marks**

(a) A regular six-sided die is being tossed a number of times.

(i) Find the probability of tossing one five in the first three throws. 2

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(ii) Given that at least one five is tossed, find the probability that only one five is tossed in the first three throws. 2

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QUESTION THIRTEEN (Continued)

(b) The number of insects of a certain species was recorded at a particular site at weekly intervals. The number of insects present after t weeks is denoted by N.

(i) If $N = 1400e^{0.6t}$ find the rate at which the insects are increasing after 7 weeks. Give your answer correct to the nearest 100 insects. 2

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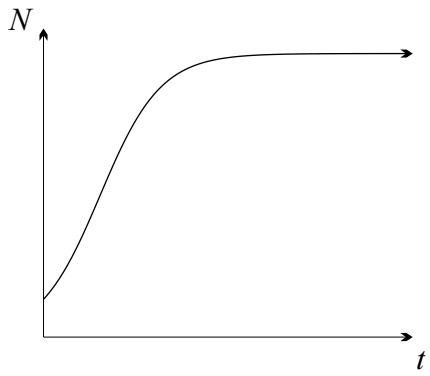
(ii) The number of insects at a different site is also recorded weekly and is modelled by the function $N = \dfrac{1500}{2 + 13e^{-0.6t}}$. The graph of this function is shown below.



(α) Use the graph to explain why the population growth is not exponential. 1

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QUESTION THIRTEEN (Continued)

(β) According to the new model, after how many weeks were 700 insects present? Give your answer correct to the nearest week. [3]

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(γ) As t increases, N approaches a certain value. What is this value? [1]

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(δ) Find $\dfrac{dN}{dt}$, the rate at which N is changing with respect to t. [2]

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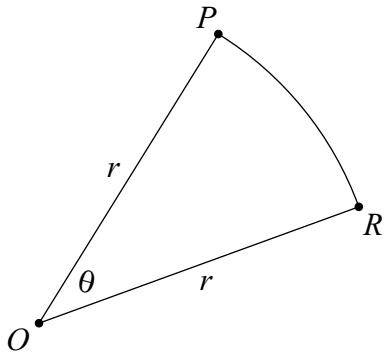
Extra writing space (Use this page only for answers for Question 13.)

If you use this space, clearly indicate which question you are answering.

CANDIDATE NUMBER

QUESTION FOURTEEN (12 marks) **Marks**

(a) 3



The diagram above shows the sector of a circle with centre O, radius r and angle at the centre θ, where θ is in radians. If the perimeter of sector OPR is 8 centimetres, find a simplified expression for the area of sector OPR in terms of θ.

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QUESTION FOURTEEN (Continued)

(b) (i) Express $2 - \cos^2 x - 4\sin x$ in the form $(\sin x + a)^2 + b$, where a and b are integers. [2]

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(ii) Hence find the maximum and minimum values of $2 - \cos^2 x - 4\sin x$. [2]

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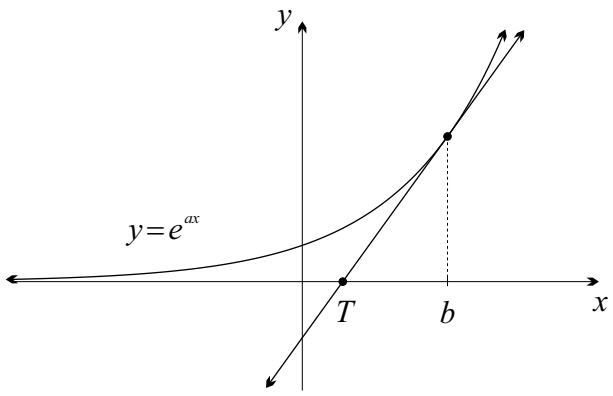
QUESTION FOURTEEN (Continued)

(c)



The diagram above shows the curve $y = e^{ax}$. The tangent to the curve at the point where $x = b$, meets the x-axis at T. Note that a is a positive real number.

(i) Find the coordinates of T in terms of a and b. 3

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QUESTION FOURTEEN (Continued)

(ii) Hence, explain why any straight line which passes through a given point $(p, 0)$, and which cuts the curve $y = e^{ax}$ twice, has gradient greater than ae^{ap+1}. 2

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END OF PAPER

SYDNEY GRAMMAR SCHOOL

CANDIDATE NUMBER

2023 Annual

Form V Mathematics Advanced

Tuesday 29th August 2023

- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

B L A N K P A G E

SYDNEY GRAMMAR SCHOOL

Solutions

NAME

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Section I (8 marks) Questions 1 – 8

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Writer: FMW

	Marks
Multiple Choice	
Question 9	
Question 10	
Question 11	
Question 12	
Question 13	
Question 14	
TOTAL	

Section I

QUESTION ONE (1 mark)

Which of the following is the length of the interval joining the points $(4, 7)$ and $(6, -3)$?

(A) $2\sqrt{26}$

(B) $2\sqrt{29}$

(C) 104

(D) 116

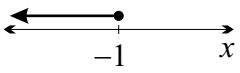
$d^2 = (4-6)^2 + (7+3)^2$
$= 4 + 100$
$= 104$
$d = \sqrt{104}$
$= 2\sqrt{26}$

QUESTION TWO (1 mark)

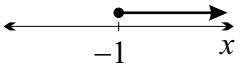
Which of the following is the correct graph of the solution of the inequation $5 - 2x \leq 3$?

$-2x \leq -2$
$x \geqslant 1$

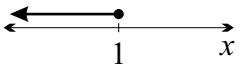
(A)



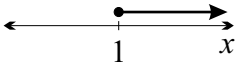
(B)



(C)



(D)



QUESTION THREE (1 mark)

What is the value of $\log_5 9$ correct to 3 significant figures?

(A) 0.73

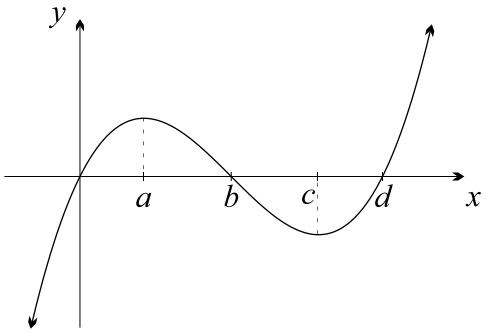
(B) 0.732

(C) 1.36

(D) 1.37

$\frac{\log 9}{\log 5} = 1.3652\ldots$

QUESTION FOUR (1 mark)



The graph of the curve $y = f(x)$ is shown above. For what values of x will the tangent to the curve have a negative gradient?

(A) $x < a$

(B) $a < x < b$

(C) $a < x < c$

(D) $x > c$

QUESTION FIVE (1 mark)

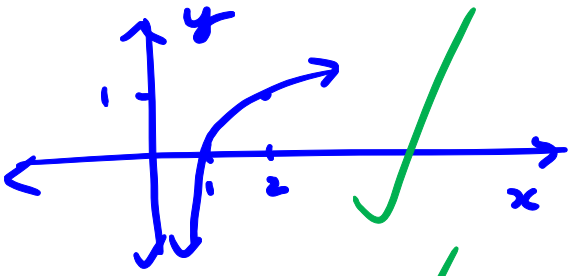
For the curve, $y = \log_2 x$, which of the following is NOT a true statement?

(A) The curve passes through the point $(1, 0)$.

(B) The gradient of the curve is always positive.

(C) The range of the curve is $y > 0$.

(D) The domain of the curve is $x > 0$.



QUESTION SIX (1 mark)

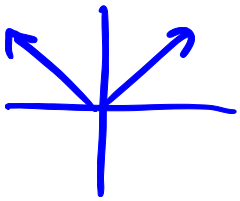
For the equation $\sin\theta = \frac{1}{2}$, which of the following is NOT a correct solution?

(A) $-30°$

(B) $30°$

(C) $-\frac{7\pi}{6}$

(D) $\frac{\pi}{6}$



QUESTION SEVEN (1 mark)

Which of the following is the derivative of $\frac{1}{\sqrt{e^{6x}}}$?

$= \frac{1}{(e^{6x})^{\frac{1}{2}}}$

$= e^{-3x}$

$\frac{d}{dx}(e^{-3x}) = -3e^{-3x}$

$= -\frac{3}{e^{3x}}$

(A) $\frac{1}{\sqrt{6e^{6x}}}$

(B) $\frac{-3}{e^{3x}}$

(C) $-3e^{-3x-1}$

(D) $\frac{-1}{\sqrt{6e^{6x-1}}}$

QUESTION EIGHT (1 mark)

For what values of r does the graph of $x^2 + 4x + y^2 - 2y + 5 = r^2$ intersect the x-axis but not the y-axis?

$x^2+4x+4 + y^2-2y+1 = r^2$

$(x+2)^2 + (y-1)^2 = r^2$

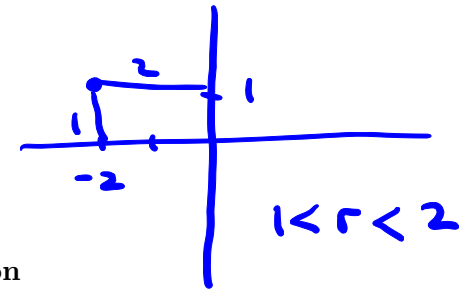
circle centre $(-2, 1)$

(A) $r < 1$

(B) $1 < r < 2$

(C) $2 < r < \sqrt{5}$

(D) $r > \sqrt{5}$



End of Section I

The paper continues in the next section

/8

CANDIDATE NUMBER

Section II

QUESTION NINE (12 marks) **Marks**

(a) Consider the line with equation $3x - 2y + 18 = 0$.

(i) Find its gradient. [1]

$2y = 3x + 18$

$y = \frac{3}{2}x + 9 \qquad m_1 = \frac{3}{2}$ ✓

(ii) Write down the equation of the line which passes through the origin and is perpendicular to the line $3x - 2y + 18 = 0$. [1]

$m_2 = -\frac{2}{3}$ so $y = -\frac{2}{3}x$ ✓

(b) Write down the centre and radius of the circle with equation $(x-1)^2 + y^2 = 4$. [1]

centre $(1, 0)$

radius 2 ✓

(c) Convert $29°$ to radians, leaving your answer in exact form. [1]

$180° = \pi$

$1° = \frac{\pi}{180}$

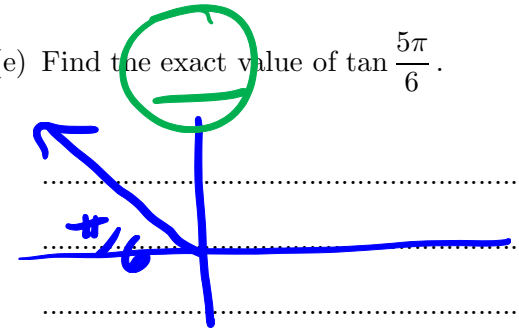
$29° = \frac{29\pi}{180}$ ✓

(d) A fair coin is tossed five times. What is the probability of tossing 5 heads? [1]

$P(E) = \left(\frac{1}{2}\right)^5$ ✓

$= \frac{1}{32}$

(e) Find the exact value of $\tan \frac{5\pi}{6}$. [1]



$= -\tan \frac{\pi}{6}$

$= -\frac{1}{\sqrt{3}}$ ✓

(f) Differentiate the following functions with respect to x:

(i) $f(x) = e^{5x-1}$ [1]

$f'(x) = 5e^{5x-1}$ ✓

(ii) $y = (3x-4)^5$ [1]

$y' = 5(3x-4)^4 \times 3$

$= 15(3x-4)^4$ ✓

QUESTION NINE (Continued)

(g) An arc AB subtends an angle of 1.1 radians at the centre of a circle, centre O, of radius length 4 centimetres. Find the area of sector AOB. [1]

$A = \frac{1}{2}(4)^2 \times 1.1$

$= 8.8 \text{ cm}^2$ ✓

(h) Find the equation of the tangent to the curve $y = \frac{18}{x^2}$ at the point $(3, 2)$. Give your answer in general form. [3]

$y = 18x^{-2}$

$y' = -36x^{-3}$ ✓

$= -\frac{36}{x^3}$

at $x = 3$

$y' = -\frac{36}{3^3}$

$= -\frac{4}{3}$

tangent : $y - 2 = -\frac{4}{3}(x - 3)$ ✓

$3y - 6 = -4x + 12$

$4x + 3y - 18 = 0$ ✓

answer in general form

Extra writing space (Use this page only for answers for Question 9.)

If you use this space, clearly indicate which question you are answering.

CANDIDATE NUMBER

QUESTION TEN (11 marks) **Marks**

(a) (i) Factorise $2x^2 + x - 6$. 1

$2x^2 + x - 6 = 2x^2 + 4x - 3x - 6$

$= 2x(x+2) - 3(x+2)$

$= (2x-3)(x+2)$

p: -12
s: +1
f: 4, -3

(ii) Hence, simplify: 2

$$\frac{3}{x^2-4} + \frac{8}{2x^2+x-6}$$

$= \frac{3}{(x-2)(x+2)} + \frac{8}{(2x-3)(x+2)}$

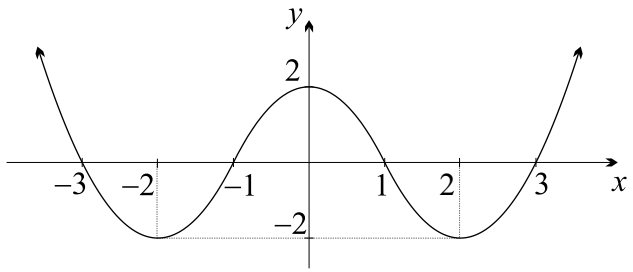
$= \frac{3(2x-3) + 8(x-2)}{(x-2)(x+2)(2x-3)}$

$= \frac{6x - 9 + 8x - 16}{(x-2)(x+2)(2x-3)}$

$= \frac{14x - 25}{(x-2)(x+2)(2x-3)}$

QUESTION TEN (Continued)

(b)



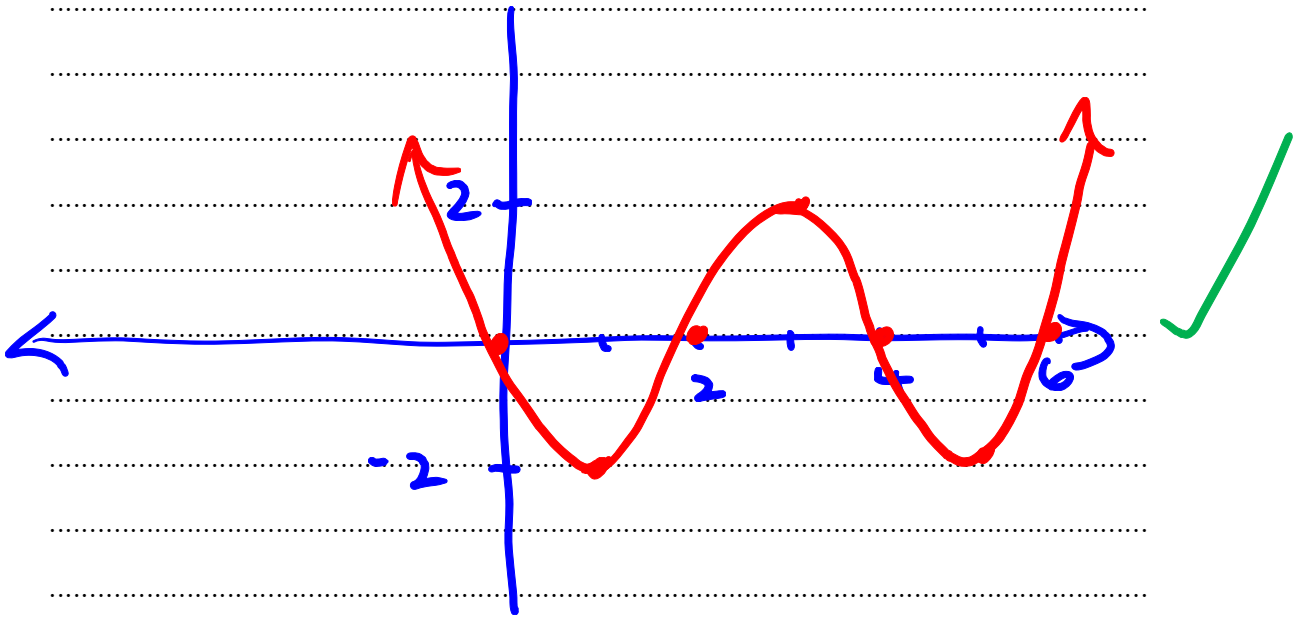
The diagram above shows the function $y = f(x)$.

(i) Is the function ODD or EVEN? Explain your answer. [1]

the function is even as the y-axis is an axis of symmetry

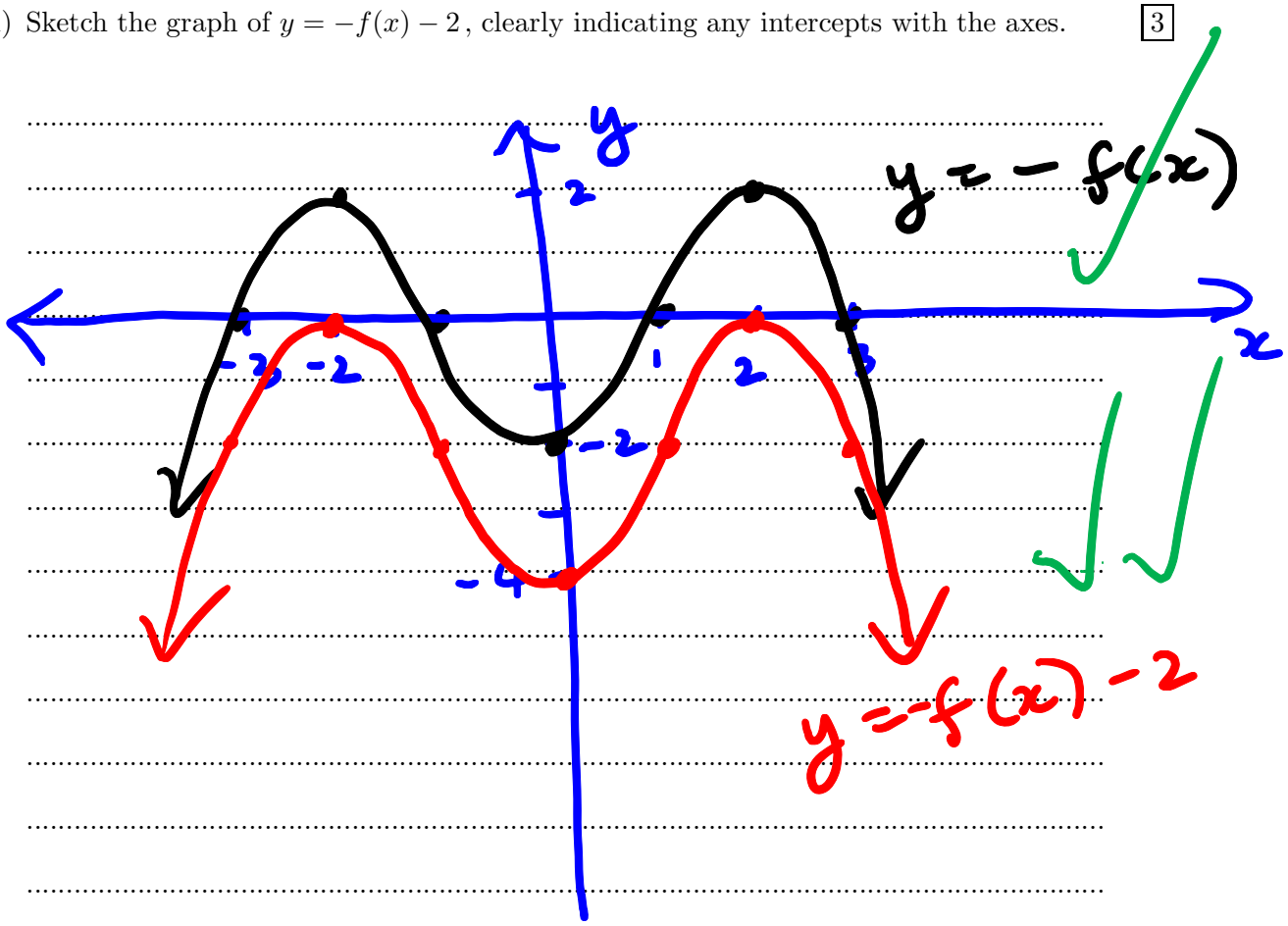
(ii) Sketch the graph of $y = f(x-3)$, clearly showing the coordinates of any intercepts with the axes. [1]

→ 3 units to the right



QUESTION TEN (Continued)

(iii) Sketch the graph of $y = -f(x) - 2$, clearly indicating any intercepts with the axes. [3]



QUESTION TEN (Continued)

(c) Find the values of p and q if $(2\sqrt{3}-1)(5-3\sqrt{3}) = p+q\sqrt{3}$. [2]

LHS $= 10\sqrt{3} - 6\times 3 - 5 + 3\sqrt{3}$ ✓

$= 13\sqrt{3} - 23$

$p = -23$, $q = 13$ ✓

(d) Express $\log_a x^2 y$ in terms of $\log_a x$ and $\log_a y$. [1]

$= \log_a x^2 + \log_a y$

$= 2\log_a x + \log_a y$ ✓

CANDIDATE NUMBER

QUESTION ELEVEN (11 marks) **Marks**

(a) Find the equation of the normal to the curve $y = 2\sqrt{x}$, at the point on the curve where $x = 9$. 3

$y = 2x^{\frac{1}{2}}$

$y' = \frac{1}{2} \times 2x^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{x}}$ ✓

if $x = 9$, $y = 2 \times \sqrt{9}$

$= 6$

$y' = \frac{1}{\sqrt{9}}$

$= \frac{1}{3}$ ✓ either

gradient of normal $= -3$

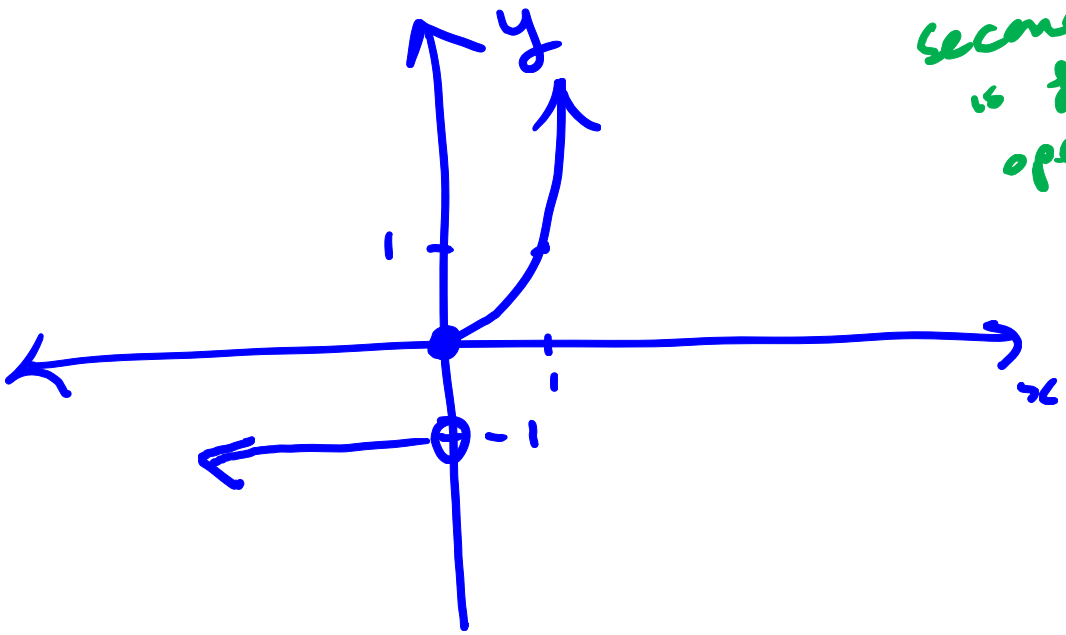
tangent : $y - 6 = -3(x - 9)$

$y = -3x + 33$ ✓

QUESTION ELEVEN (Continued)

(b) (i) Sketch the curve $y = f(x)$ where: 2

$$f(x) = \begin{cases} -1, & \text{for } x < 0, \\ x^2, & \text{for } x \geq 0\,. \end{cases}$$



second mark is for open/closed circles

(ii) Is this function differentiable at $x = 0$? Explain your answer. 1

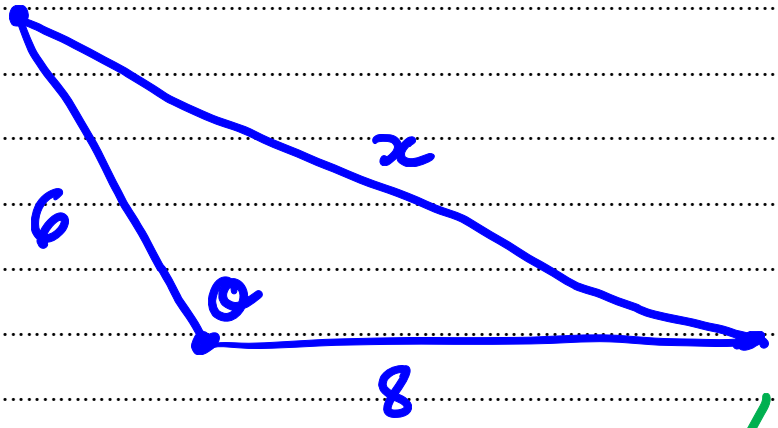
NO as it is not continuous at $x = 0$

(iii) Is this function *one-to-one* or *many-to-one*? Explain your answer. 1

The function is many-to-one. Infinitely many x-values, $x < 0$, have $y = -1$.

QUESTION ELEVEN (Continued)

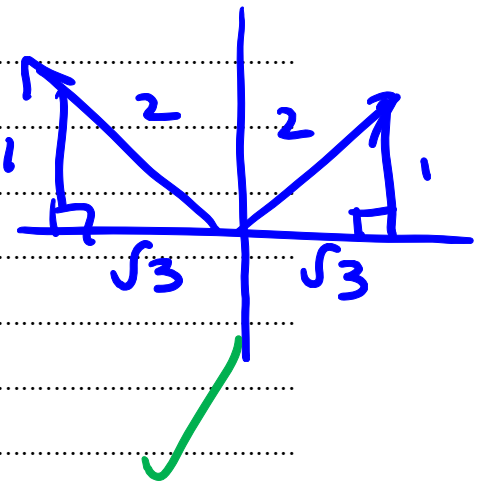
(c) A triangle has area $12\,\text{cm}^2$. Two of the sides have lengths 6 cm and 8 cm, and θ is the included angle. Show that $\sin\theta = \frac{1}{2}$ and calculate the length of the missing side, giving all possible answers, correct to 2 decimal places. [4]



$\frac{1}{2} \times 6 \times 8 \sin\theta = 12$

$\sin\theta = \frac{12}{24}$

$\sin\theta = \frac{1}{2}$



$\cos\theta = -\frac{\sqrt{3}}{2}$ or $\frac{\sqrt{3}}{2}$

$x^2 = 6^2 + 8^2 - 2(6)(8)\left(-\frac{\sqrt{3}}{2}\right)$

$= 183.13\ldots$

$x \doteq 13.53$ (2 d.p.)

OR $x^2 = 6^2 + 8^2 - 2(6)(8)\left(\frac{\sqrt{3}}{2}\right)$

Extra writing space (Use this page only for answers for Question 11.)

If you use this space, clearly indicate which question you are answering.

$x^2 = 16.8615...$

$x = 4.11$ (2 d.p.) ✓

note: if $\sin\theta = \frac{1}{2}$

$\theta = 30°$ or $150°$

and continue with cos rule

CANDIDATE NUMBER

QUESTION TWELVE (13 marks) **Marks**

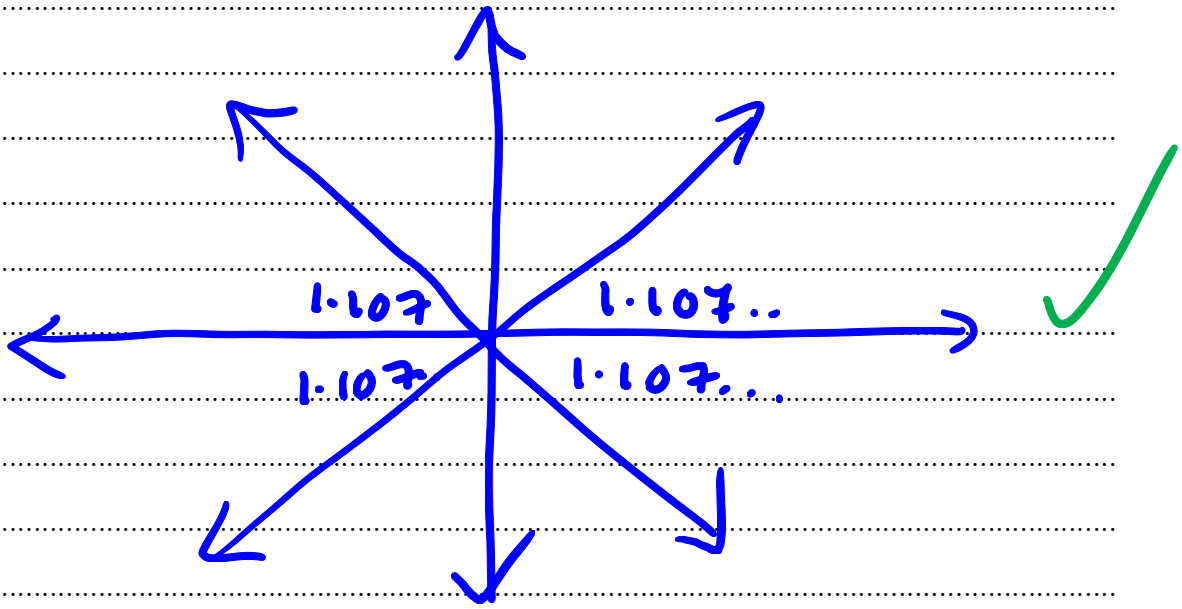
(a) Evaluate $\lim_{h\to 0} \dfrac{h^2 - 4h}{h}$. 1

$= \lim_{h \to 0} (h - 4)$ ✓

$= -4$

(b) Solve $\tan^2\theta = 4$ for $0 \le \theta \le 2\pi$. Give your answers correct to 2 decimal places. 2

$\tan\theta = 2$ or $\tan\theta = -2$



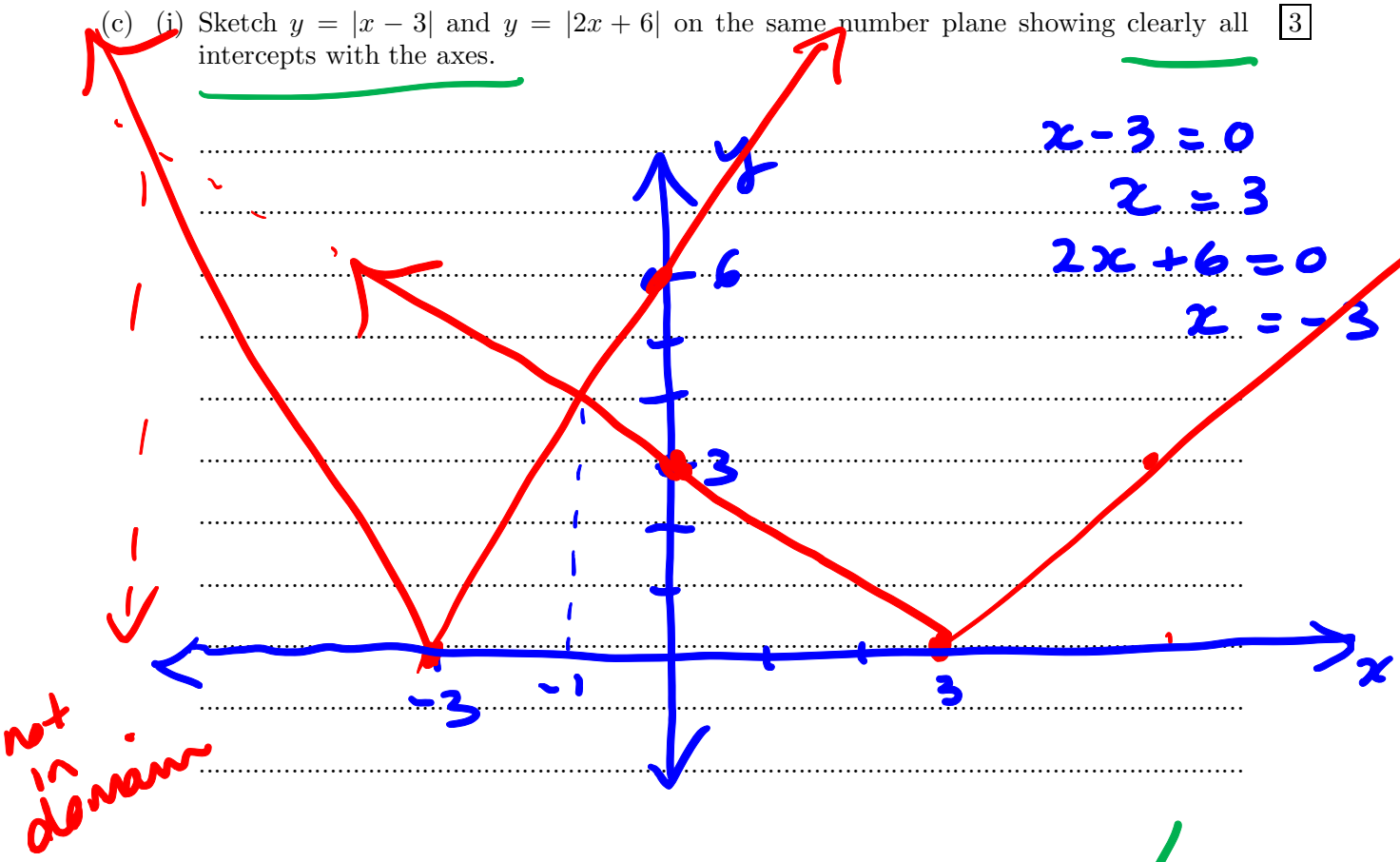
✓

$\theta \doteq 1.11, 2.03, 4.25, 5.18$ ✓

answers must be in radians

QUESTION TWELVE (Continued)

(c) (i) Sketch $y = |x - 3|$ and $y = |2x + 6|$ on the same number plane showing clearly all intercepts with the axes. [3]



(ii) Hence, or otherwise, solve $|x - 3| = |2x + 6|$, for $-3 \leq x \leq 3$. [1]

$x = -1$ from graph

OR $2x + 6 = -(x - 3)$

$2x + 6 = -x + 3$

$3x = -3$

$x = -1$

QUESTION TWELVE (Continued)

(d) Differentiate $y = x(2-x)^5$. [2]

$u = x \qquad v = (2-x)^5$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 5(2-x)^4 \times -1$

$\frac{dv}{dx} = -5(2-x)^4$

$\frac{dy}{dx} = x \times -5(2-x)^4 + (2-x)^5 \times 1$

$= (2-x)^4(-5x + (2-x))$

$= (2-x)^4(-6x + 2)$

$= 2(2-x)^4(1-3x)$

(e) Find all the points on the curve $y = \frac{x^2}{x+16}$, where the tangent is horizontal. [4]

$u = x^2 \qquad v = x + 16$

$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{(x+16)(2x) - x^2 \times 1}{(x+16)^2}$

$= \frac{2x^2 + 32x - x^2}{(x+16)^2}$

$= \frac{x^2 + 32x}{(x+16)^2}$

Extra writing space (Use this page only for answers for Question 12.)

If you use this space, clearly indicate which question you are answering.

$$\frac{dy}{dx} = \frac{x(x+32)}{(x+16)^2}$$

tangent is horizontal when

$\frac{dy}{dx} = 0$, $x = 0$ or $x = -32$ (x) ✓

$y = 0$ $y = \frac{(-32)^2}{-32+16}$

(y) ✓ $= -64$

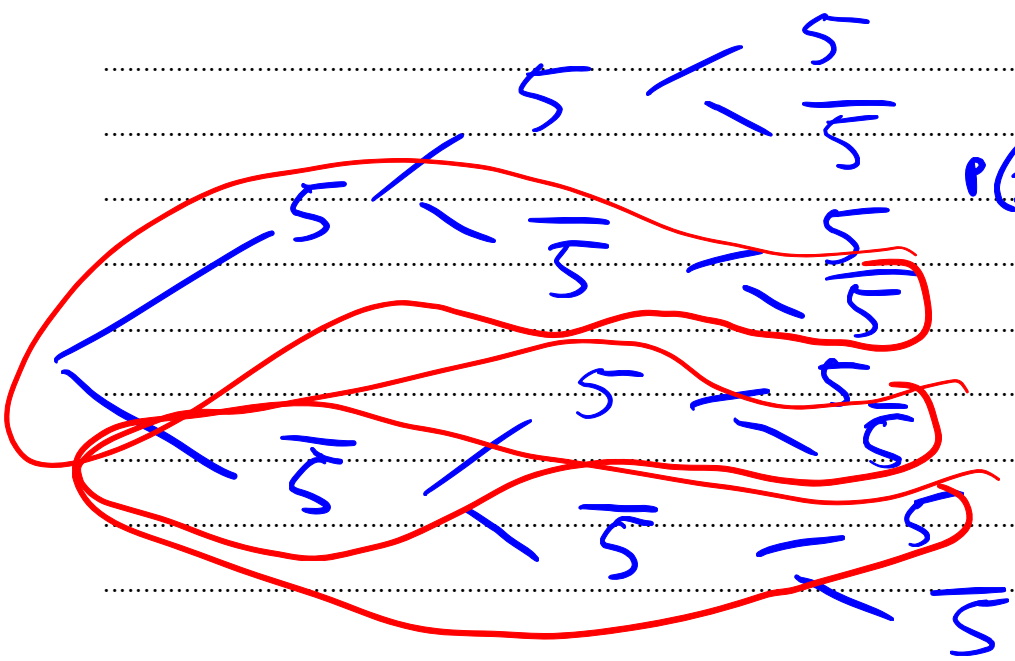
points are $(0,0)$ and $(-32,-64)$

CANDIDATE NUMBER

QUESTION THIRTEEN (13 marks) **Marks**

(a) A regular six-sided die is being tossed a number of times.

(i) Find the probability of tossing one five in the first three throws. [2]



$P(E) = \frac{1}{6} \times \left(\frac{5}{6}\right)^2 \times 3$

$= \frac{25}{72}$

(ii) Given that at least one five is tossed, find the probability that only one five is tossed in the first three throws. [2]

$P(\text{at least one 5}) = 1 - P(\text{no 5's})$

$= 1 - \left(\frac{5}{6}\right)^3$

$= \frac{91}{216}$

$P(E) = \frac{\frac{25}{72}}{\frac{91}{216}}$

$= \frac{75}{91}$

QUESTION THIRTEEN (Continued)

(b) The number of insects of a certain species was recorded at a particular site at weekly intervals. The number of insects present after t weeks is denoted by N.

(i) If $N = 1400e^{0.6t}$ find the rate at which the insects are increasing after 7 weeks. Give your answer correct to the nearest 100 insects. 2

$\frac{dN}{dt} = 0.6 \times 1400 e^{0.6t}$ ✓

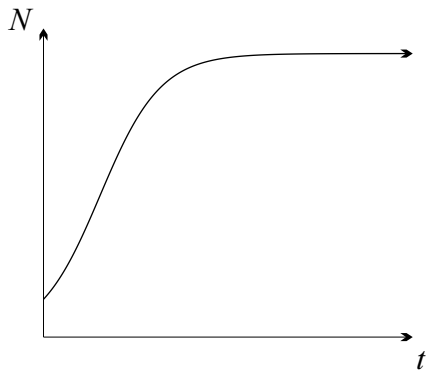
$\frac{dN}{dt} = 840 e^{0.6t}$

at $t = 7$, $\frac{dN}{dt} = 840 e^{0.6 \times 7}$

$= 56016.5\ldots$

$\doteq 56000$ insects / week ✓

(ii) The number of insects at a different site is also recorded weekly and is modelled by the function $N = \dfrac{1500}{2 + 13e^{-0.6t}}$. The graph of this function is shown below.



(α) Use the graph to explain why the population growth is not exponential. 1

Rather than increase as $t \to \infty$, the graph flattens out. ✓

QUESTION THIRTEEN (Continued)

(β) According to the new model, after how many weeks were 700 insects present? Give your answer correct to the nearest week. [3]

$$700 = \frac{1500}{2+13e^{-0.6t}}$$

$$700(2+13e^{-0.6t}) = 1500$$

$$1400 + 9100e^{-0.6t} = 1500$$

$$9100e^{-0.6t} = 100$$

$$e^{-0.6t} = \frac{1}{91}$$

$$-0.6t = \log_e\left(\frac{1}{91}\right)$$

$$t = \frac{\log_e\left(\frac{1}{91}\right)}{-0.6t}$$

$$= 7.518\ldots$$

$$\doteq 8 \text{ weeks}$$

(γ) As t increases, N approaches a certain value. What is this value? [1]

as $t \to \infty$, $13e^{-0.6t} \to 0$

and $N \to \frac{1500}{2} = 750$

(δ) Find $\frac{dN}{dt}$, the rate at which N is changing with respect to t. [2]

$$N = 1500(2+13e^{-0.6t})^{-1}$$

$$\frac{dN}{dt} = -1500(2+13e^{-0.6t})^{-2} \times (-0.6 \times 13e^{-0.6t})$$

$$= \frac{11700e^{-0.6t}}{(2+13e^{-0.6t})^2}$$

Extra writing space (Use this page only for answers for Question 13.)

If you use this space, clearly indicate which question you are answering.

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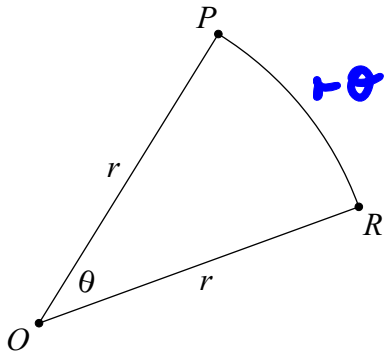
QUESTION FOURTEEN (12 marks) **Marks**

(a) 3



The diagram above shows the sector of a circle with centre O, radius r and angle at the centre θ, where θ is in radians. If the perimeter of sector OPR is 8 centimetres, find a simplified expression for the area of sector OPR in terms of θ.

$r + r + r\theta = 8$ ✓

$2r + r\theta = 8$

$r(2+\theta) = 8$

$r = \frac{8}{2+\theta}$ ✓

$A = \frac{1}{2}r^2\theta$

$= \frac{1}{2}\left(\frac{8}{2+\theta}\right)^2 \times \theta$

$= \frac{32\theta}{(2+\theta)^2}$ ✓

QUESTION FOURTEEN (Continued)

(b) (i) Express $2 - \cos^2 x - 4\sin x$ in the form $(\sin x + a)^2 + b$, where a and b are integers. [2]

$2 - \cos^2 x - 4\sin x = 2 - (1 - \sin^2 x) - 4\sin x$ ✓

$= 1 + \sin^2 x - 4\sin x$

$= \sin^2 x - 4\sin x + 4 - 3$ ✓

$= (\sin x - 2)^2 - 3$

note : $a = -2$, $b = -3$

(ii) Hence find the maximum and minimum values of $2 - \cos^2 x - 4\sin x$. [2]

$-1 \leq \sin x \leq 1$

max. value $(\sin x - 2)^2 - 3 = (-1 - 2)^2 - 3$ ✓

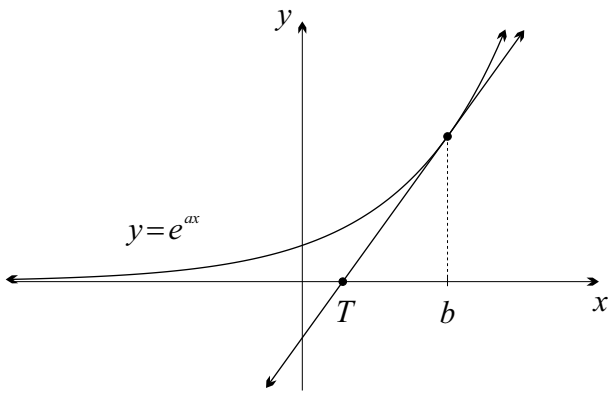
$= 6$

min. value $(\sin x - 2)^2 - 3 = (1 - 2)^2 - 3$ ✓

$= -2$

QUESTION FOURTEEN (Continued)

(c)



The diagram above shows the curve $y = e^{ax}$. The tangent to the curve at the point where $x = b$, meets the x-axis at T. Note that a is a positive real number.

(i) Find the coordinates of T in terms of a and b. 3

$y' = ae^{ax}$

If $x=b$, $y = e^{ab}$, $y' = ae^{ab}$ ✓

tangent is $y - e^{ab} = ae^{ab}(x-b)$

If $y=0$, $-e^{ab} = ae^{ab}(x-b)$ ✓

$-1 = ax - ab$

$ax = ab - 1$

$x = \frac{ab-1}{a}$

$= b - \frac{1}{a}$ ✓

T is the point

$\left(b - \frac{1}{a}, 0\right)$

or $\left(\frac{ab-1}{a}, 0\right)$

or $\left(\frac{abe^{ab} - abe^{ab}}{ae^{ab}}, 0\right)$

QUESTION FOURTEEN (Continued)

(ii) Hence, explain why any straight line which passes through a given point $(p, 0)$, and which cuts the curve $y = e^{ax}$ twice, has gradient greater than ae^{ap+1}. [2]

if $p = b - \frac{1}{a}$ ✓

$b = p + \frac{1}{a}$

the tangent has gradient

$ae^{ab} = ae^{a(p+\frac{1}{a})}$

$= ae^{ap+1}$

✓ any straight line which cuts the curve twice will be steeper than the tangent in (i) with gradient $ae^{ab} = ae^{ap+1}$ and will therefore have gradient $> ae^{ap+1}$

need argument

END OF PAPER

SYDNEY GRAMMAR SCHOOL

CANDIDATE NUMBER

2023 Annual

Form V Mathematics Advanced

Tuesday 29th August 2023

- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

B L A N K P A G E