Student Name: _____

Student Number: | | | | | | | | | |

Teachers Name: _____

ABBOTSLEIGH

2024

Higher School Certificate

Trial Examination

Mathematics Advanced

General Instructions

- Reading time – 10 minutes.
- Working time – 3 hours.
- Write using black pen only.
- NESA approved calculators may be used.
- A NESA Reference sheet is provided.
- All necessary working should be shown in every question.
- Make sure your Student Number is on the front cover of each section.
- Answer all Multiple-Choice questions on the answer sheet provided.
- In Questions 11 – 16 show relevant mathematical reasoning and/or calculations.
- Extra Writing spaces are provided on pages 15, 23, 31, 39, 45 and 55.

Total Marks – 100

Attempt all questions.

Section I — Pages 3 – 10

10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.

Section II — Pages 11 – 56

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section.

Outcomes to be assessed:

Mathematics Advanced

Preliminary:

A student

MA11-1 uses algebraic and graphical techniques to solve, and where appropriate, compare alternative solutions to problems.

MA11-2 uses the concepts of functions and relations to model, analyse and solve practical problems.

MA11-3 uses the concepts and techniques of trigonometry in the solution of equations and problems involving geometric shapes.

MA11-4 uses the concepts and techniques of periodic functions in the solutions of trigonometric equations or proof of trigonometric identities.

MA11-5 interprets the meaning of the derivative, determines the derivative of functions and applies these to solve simple practical problems.

MA11-6 manipulates and solves expressions using the logarithmic and index laws, and uses logarithms and exponential functions to solve practical problems

MA11-7 uses concepts and techniques from probability to present and interpret data and solve problems in a variety of contexts, including the use of probability distributions.

MA11-8 uses appropriate technology to investigate, organise, model and interpret information in a range of contexts.

MA11-9 provides reasoning to support conclusions which are appropriate to the context.

HSC:

A student

MA12-1 uses detailed algebraic and graphical techniques to critically construct, model and evaluate arguments in a range of familiar and unfamiliar contexts.

MA12-2 models and solves problems and makes informed decisions about financial situations using mathematical reasoning and techniques.

MA12-3 applies calculus techniques to model and solve problems.

MA12-4 applies the concepts and techniques of arithmetic and geometric sequences and series in the solution of problems.

MA12-5 applies the concepts and techniques of periodic functions in the solution of problems involving trigonometric graphs.

MA12-6 applies appropriate differentiation methods to solve problems.

MA12-7 applies the concepts and techniques of indefinite and definite integrals in the solution of problems.

MA12-8 solves problems using appropriate statistical processes.

MA12-9 chooses and uses appropriate technology effectively in a range of contexts, models and applies critical thinking to recognise appropriate times for such use.

MA12-10 constructs arguments to prove and justify results and provides reasoning to support conclusions which are appropriate to the context.

SECTION I

10 marks
Attempt Questions 1 – 10

Use the multiple-choice answer sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

(A) ○ (B) ● (C) ○ (D) ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) ● (B) ⊗ (C) ○ (D) ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

correct ←

(A) ⊗ (B) ⊗ (C) ○ (D) ○

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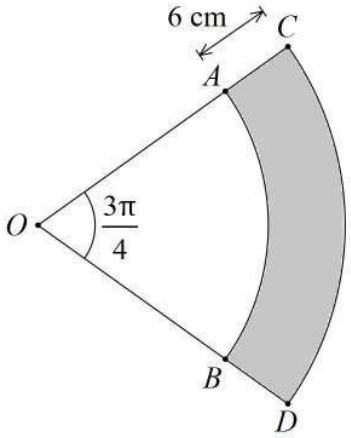
1. In the figure, OAB and OCD are sectors with centre O. It is given that the area of the shaded region is 99π cm^2.



NOT TO SCALE

If $\angle AOB = \dfrac{3\pi}{4}$ radians and $AC = 6$ cm, what is the length of OA?

A. 16 cm

B. 19 cm

C. 24 cm

D. 25 cm

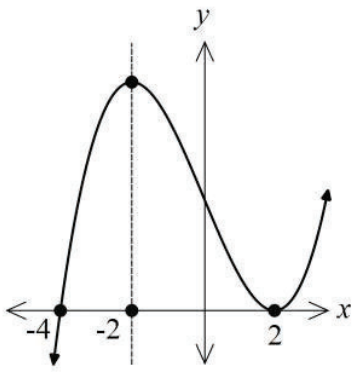
Section I continues on page 4

2. Characteristics of the cubic function $f(x)$ are given.

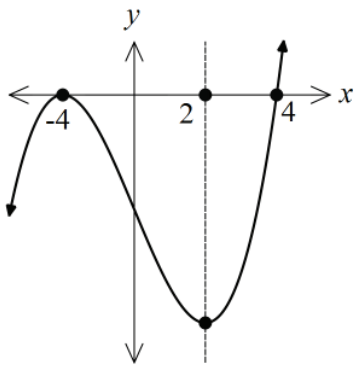
- $(x+4)$ is a factor of $f(x)$.
- $x=2$ is a repeated root of $f(x)$.
- $f'(-2)=0$
- $f'(x)>0$ where the graph with equation $y=f(x)$ crosses the y-axis.

Which of the following could be the graph of $y=f(x)$?

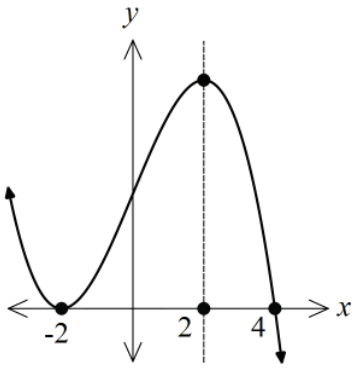
A.



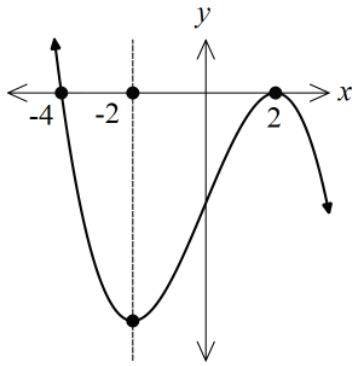
B.



C.



D.



3. Using the trapezoidal rule with four intervals, what is the approximate area under the curve $y=\sqrt{3x+4}$ between $x=0$ and $x=4$?

A. $3+\sqrt{7}+\sqrt{10}+\sqrt{13}$

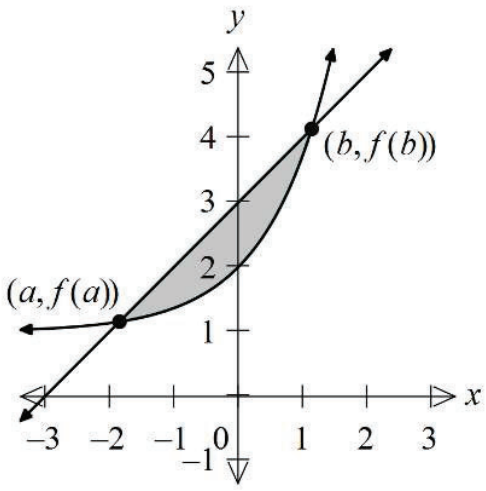
B. $3+2\left(\sqrt{7}+\sqrt{10}+\sqrt{13}\right)$

C. $6+2\left(\sqrt{7}+\sqrt{10}+\sqrt{13}\right)$

D. $6+\sqrt{7}+\sqrt{10}+\sqrt{13}$

Section I continues on page 5

4. The region enclosed by $y = e^x + 1$ and $y = x + 3$ is shaded in the diagram. The points of intersection of the two curves are shown.



Which integral represents the area of the shaded region?

A. $\int_a^b \left(e^x - x + 4\right) dx$

B. $\int_a^b \left(x + 4 - e^x\right) dx$

C. $\int_a^b \left(e^x - x - 2\right) dx$

D. $\int_a^b \left(x + 2 - e^x\right) dx$

5. Differentiate the function $f(x) = \ln\left(\sqrt[3]{x^3 + 3}\right)$ to find $f'(x)$.

A. $f'(x) = \dfrac{3x^2}{x^3 + 3}$

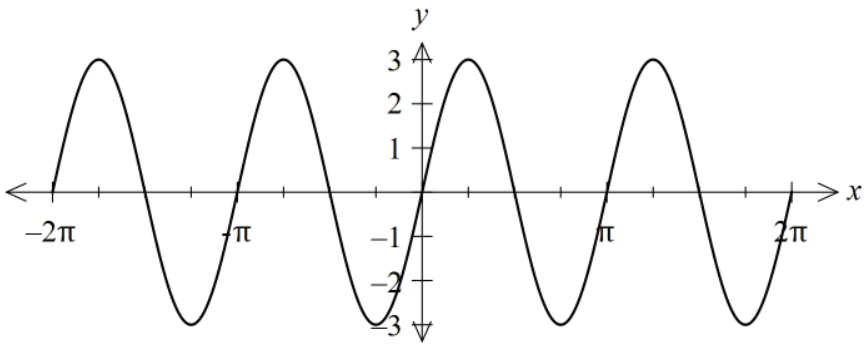
B. $f'(x) = \dfrac{3x^2}{\sqrt[3]{x^3 + 3}}$

C. $f'(x) = \dfrac{x^2}{x^3 + 3}$

D. $f'(x) = 3x^2\left(\sqrt[3]{x^3 + 3}\right)$

Section I continues on page 6

6. The graph of $y = f(x)$ is shown in the domain $[-2\pi, 2\pi]$.



Which of the following functions represents $f(x)$?

A. $y = 3\sin 2x$

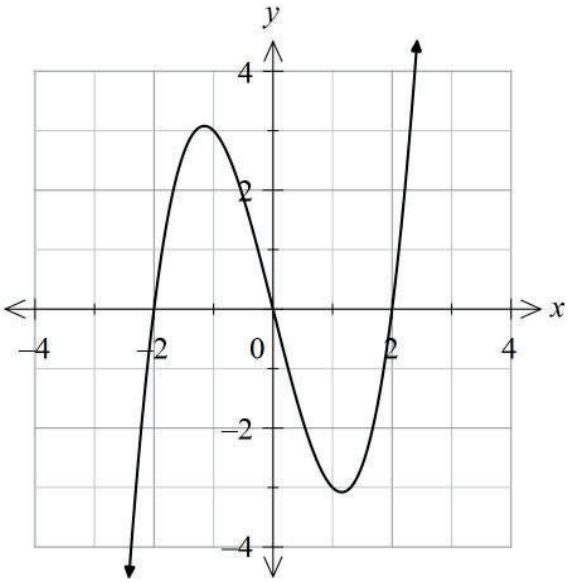
B. $y = 3\cos 2x$

C. $y = 3\sin x$

D. $y = 3\cos x$

Section I continues on page 7

7. The graph of $y = x(x-2)(x+2)$ is shown.



How many solutions does the equation $x(x-2)(x+2) = \left|\frac{x+1}{2}\right|$ have?

A. 2

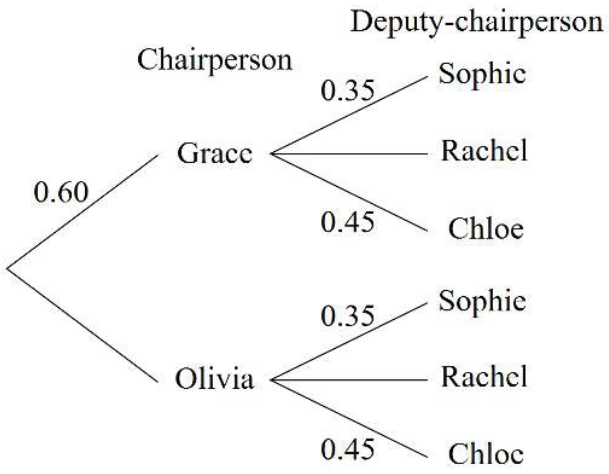
B. 3

C. 4

D. 5

Section I continues on page 8

8. The partially completed probability tree diagram shows the voting intentions of shareholders in an election for Chairperson and Deputy-chairperson of the board of directors of a large ASX listed company.



Based on these voting intentions, what is the probability that Olivia is elected Chairperson, with either Rachel or Sophie as Deputy-chairperson?

A. 0.22

B. 0.26

C. 0.33

D. 0.48

Section I continues on page 9

9. Data on the time in minutes it takes for students at Abbotsleigh to travel to school is normally distributed with a mean of 37 minutes and a standard deviation of 11 minutes.

Using the *z*-score table below, what is the approximate 90th percentile for the data?

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

A. 47 minutes

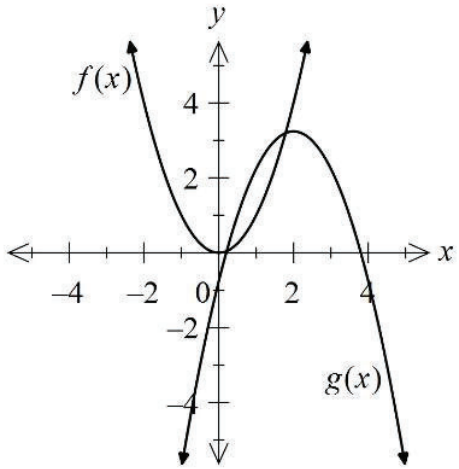
B. 49 minutes

C. 51 minutes

D. 53 minutes

Section I continues on page 10

10. The graphs of $f(x) = x^2$ and $y = g(x)$ are shown.



How many stationary points does the graph of $y = g(f(x))$ have?

A. 0

B. 1

C. 2

D. 3

End of Section I

SECTION II

Student Number: | | | | | | | | | |

Total Marks – 90
Attempt Questions 11 – 16

Answer each question in the space provided.

Your responses should include relevant mathematical reasoning and/or calculations.

Extra writing space is provided on pages 15, 23, 31, 39, 45 and 55.

Question 11 (15 Marks) **Marks**

(a) Solve $\log_2 9 = 5\log_2 3 - \log_2 x$. **2**

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(b) Given that $\sin\alpha = \frac{5}{13}$ and $\tan\alpha < 0$, find the exact value of $\cos\alpha$. **2**

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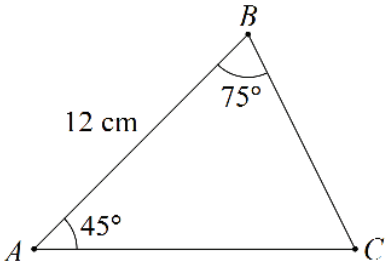
Question 11 continues on page 12

(c) In $\triangle ABC$, $AB = 12$ cm, $\angle A = 45°$ and $\angle B = 75°$.



NOT TO SCALE

(i) Show that the length of BC is $4\sqrt{6}$ cm. **2**

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(ii) Hence or otherwise, find the area of the triangle to 2 decimal places. **1**

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Question 11 continues on page 13

(d) Given that $f(x)=(1-\cos x)^5$, find $f'\left(\frac{\pi}{2}\right)$. **2**

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(e) Find the derivative of the function $f(x)=x^2e^{5x}$. Give your answer in factorised form. **2**

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Question 11 continues on page 14

(f) Solve the equation $(\sin x + 3)(2\sin x + 1) = 0$ in the domain $0 \leq x \leq 2\pi$. **2**

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(g) Evaluate $\int_1^3 \frac{27x^2 - 21}{3x^3 - 7x + 8}\, dx$, leaving your answer in exact simplified form. **2**

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End of Question 11

Extra writing space

Student Number:

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Question 12 (15 Marks)

Your responses should include relevant mathematical reasoning and/or calculations.

Question 12 (15 Marks) **Marks**

(a) Consider the geometric series

$$1+\left(\sqrt{11}-3\right)+\left(\sqrt{11}-3\right)^2+\left(\sqrt{11}-3\right)^3+\ldots$$

(i) Show with calculations why this geometric series has a limiting sum. **2**

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(ii) Determine the exact value of the limiting sum. Express your answer with a rational denominator. **2**

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Question 12 continues on page 18

(b) Tables of future value and present value interest factors for an annuity of $1 are shown below.

Future Value Interest Factors for an Annuity of $1						
Number of Periods	Rate per Period					
	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%
5	5.101	5.204	5.309	5.416	5.526	5.637
10	10.462	10.950	11.464	12.006	12.578	13.181
20	22.019	24.297	26.870	29.778	33.066	36.786
30	34.785	40.568	47.575	56.085	66.439	79.058
40	48.886	60.402	75.401	95.026	120.800	154.762

Present Value Interest Factors for an Annuity of $1						
Number of Periods	Rate per Period					
	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%
5	4.853	4.713	4.580	4.452	4.329	4.212
10	9.471	8.983	8.530	8.111	7.722	7.360
20	18.046	16.351	14.877	13.590	12.462	11.470
30	25.808	22.396	19.600	17.292	15.372	13.765
40	32.835	27.355	23.115	19.793	17.159	15.046

(i) Isabella plans to deposit $7500 at the end of each year into an annuity paying 5.0% p.a. compounding annually. What is the value of her investment after 10 years? **1**

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Question 12 continues on page 19

Question 12 (continued) **Marks**

(b) (continued)

(ii) What amount could Isabella invest now at 5.0% p.a. compounding annually, to achieve the same financial outcome as the annuity after 10 years? **1**

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(c) Find the primitive function of $\sec^2(5x) - \sin(x) + 5^x$. **3**

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Question 12 continues on page 20

(d) Determine the equation of the tangent to the curve $y = \ln\left(3x^2 + 2\right)$ at the point $\left(2,\ \ln 14\right)$. Express your answer in general form $ax + by + c = 0$. **3**

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Question 12 continues on page 21

(e) It is given that y is inversely proportional to x.

(i) When $x = 9$, $y = 3$. Determine the equation relating x in terms of y. **1**

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(ii) If the value of x is increased from 2 to 18, determine the change in the value of y. **1**

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(iii) Hence graph the relationship between x and y using the values you have found previously. **1**

End of Question 12

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Extra writing space

Student Number:

Question Number:

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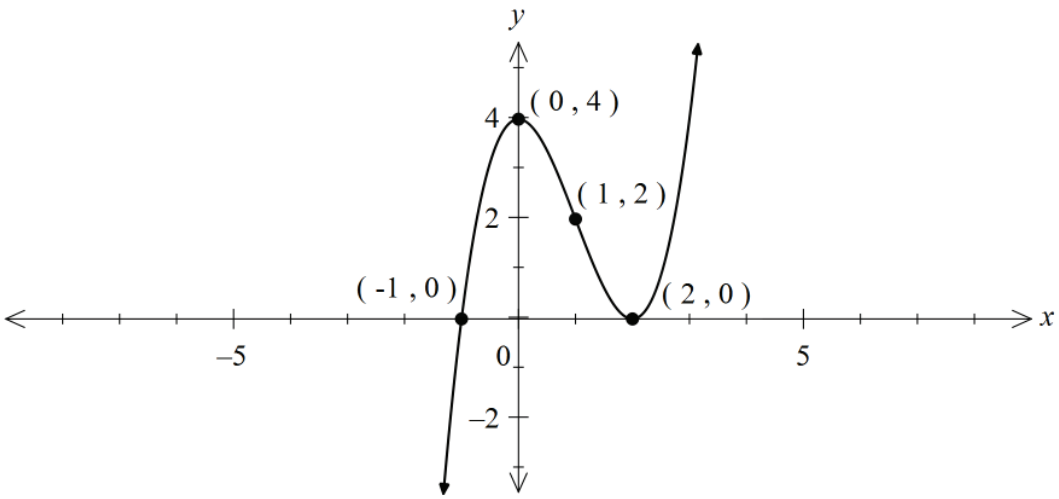
Student Number: | | | | | | | | | |

Question 13 (15 Marks)

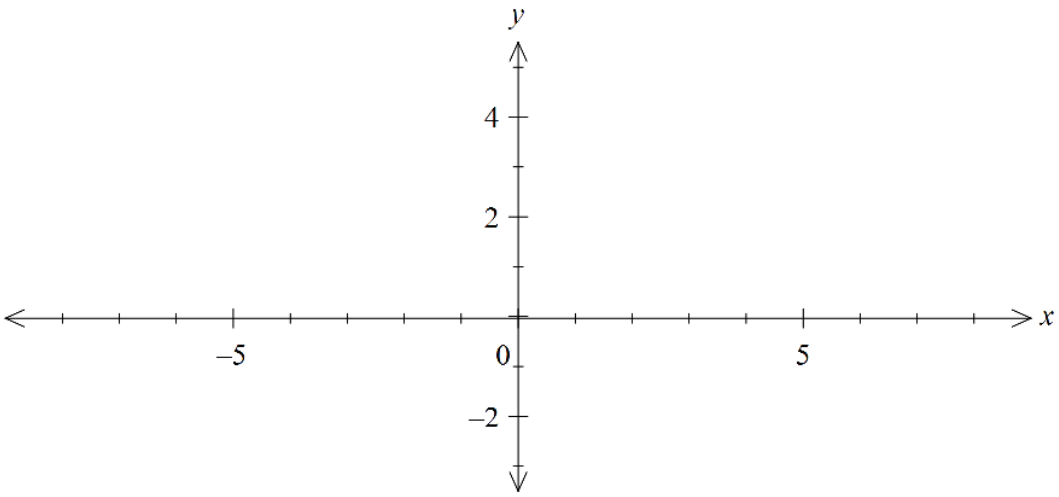
Your responses should include relevant mathematical reasoning and/or calculations.

Question 13 (15 Marks) **Marks**

(a) The diagram shows the graph of $y = f(x)$. **2**



Sketch the graph of $y = f\left(\frac{x}{2} - 1\right)$, labelling all stationary points and intercepts.



Question 13 continues on page 26

(b) Ava is managing the growth of a population of bacteria in a laboratory. The population can be modelled by the exponential equation $P(t) = 500e^{kt}$ where the initial population is 500 bacteria, k is the growth constant, and t is time in minutes.

(i) Find the value of k correct to 1 significant figure, given that there are 1660 bacteria present after 2 hours. **2**

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(ii) Determine the rate of change of the population with respect to time after 2 hours. Give your answer to the nearest whole number. **2**

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Question 13 continues on page 27

(b) (continued)

(iii) Ava is concerned about the growth of the bacteria and wants to determine if she needs to introduce a control mechanism. **3**

Interpret the nature of the population growth with reference to the derivatives $P'(t)$ and $P''(t)$, and hence determine if a control would be needed.

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Question 13 continues on page 28

(c) Lara works as a sales representative for a company that sells laboratory equipment. The number of sales she makes in a day is given by the probability distribution table.

x	0	1	2	3	4	5
$P(X = x)$	0.14	0.15	0.25	0.30	0.11	0.05

(i) Determine the expected value of the number of sales that Lara makes on a particular day. **1**

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(ii) Another sales representative, Rebecca, has an expected number of sales of 2.5 and a standard deviation in the number of sales of 1.2. **2**

The CEO of the company would like to promote a sales representative to be the leader of the sales team.

If the CEO bases her decision on sales performance, are they more likely to promote Lara or Rebecca? Justify your answer with appropriate statistical reasoning.

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Question 13 continues on page 29

(d) Find all the values of θ where $-\pi \leq \theta \leq \pi$ such that **3**

$$\cos\left(\theta - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

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End of Question 13

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Extra writing space

Student Number: | | | | | | | | | |

Question Number: | |

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Student Number: | | | | | | | | | |

Question 14 (15 Marks)

Your responses should include relevant mathematical reasoning and/or calculations.

Question 14 (15 Marks) **Marks**

(a) The first three terms of a series are 3, 9, 15, ……

(i) Calculate the fifteenth term in this series. **1**

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(ii) The sum of the first n consecutive terms of the series is 243. **2**
What is the value of n?

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Question 14 continues on page 34

(b) A bag contains eight cards that are each printed with one of the following sets of characters. **3**

aaa aab aba baa abb bab bba bbb

A student chooses one card at random.

Let N_1 be the event that the first character printed on the card is an 'a' and N_2 be the event that the second character printed on the card is also an 'a'.

Determine whether N_1 and N_2 are independent events.

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Question 14 continues on page 35

(c) Mr. Fredricksen has just retired from work as a balloon salesman. He has received a superannuation payment and has worked out that he will require \$7000 per month for the next 20 years.

Mr. Fredricksen intends to invest a set amount (P) at 6% p.a. interest compounding monthly and withdraws \$7000 from this account at the end of each month.

(i) Show that the amount A_3 remaining in the account after the third withdrawal of \$7000 is $A_3 = P(1.005)^3 - 7000\left[(1.005)^2 + 1.005 + 1\right]$ **2**

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(ii) Find the amount P that will need to be invested if he is to withdraw \$7000 each month for the next 20 years. Express your answer to the nearest cent. **2**

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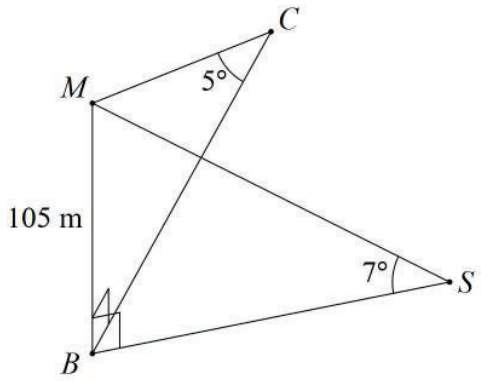
Question 14 continues on page 36

(d) Sarah and Catherine decide to return to Sydney from Schoolies on the Gold Coast aboard separate sailing vessels. Both vessels are still out to sea but are approaching Sydney harbour and can view the light from the Macquarie lighthouse that is situated at Vaucluse near South Head.



NOT TO SCALE

From Catherine's position (C), the Macquarie lighthouse (B) is at a bearing of $203°$ and the angle of elevation to the light at the top of the lighthouse (M) is $5°$.

From Sarah's position (S), the Macquarie lighthouse is at a bearing of $255°$ and the angle of elevation to the light at the top of the lighthouse is $7°$.

The light at the top of the Macquarie lighthouse is 105 metres above sea level.

(i) Show that $\angle CBS = 52°$. **2**

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Question 14 continues on page 37

(d) (continued)

(ii) Find the distance between Catherine and Sarah. Give your answer to the nearest metre. **3**

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End of Question 14

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Question 15 (15 Marks)

Your responses should include relevant mathematical reasoning and/or calculations.

Question 15 (15 Marks) **Marks**

(a) Evaluate the definite integral $\int_0^{\pi} 3\sin\left(\frac{x}{4}\right) dx$. Express your answer with a rational denominator. **3**

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Question 15 continues on page 42

(b) A particle moves along a straight line. The velocity of the particle, v, in m s^{-1}, is given by

$$v(t) = 2 - 5\sin\left(\frac{\pi t}{4}\right)$$

where t is the time, in seconds, and $0 \leq t \leq 8$.

(i) Find the particle's maximum and minimum velocity. **1**

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(ii) Hence, or otherwise, sketch the graph of the particle's velocity, labelling all important features (you do not need to find x-intercepts). **3**

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Question 15 continues on page 43

(b) (continued)

(iii) Determine how many times the particle changes its direction. **2**
Explain your answer with reference to the graph in part (ii).

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(c) A 40-centimetre length of wire is cut into three pieces. Two pieces are bent to form two congruent circles and the final piece is bent to form a square.

(i) If the radius of each congruent circle is r centimetres, show that the total area of the three shapes is given by $A = (10 - \pi r)^2 + 2\pi r^2$ **2**

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Question 15 continues on page 44

(c) (continued)

(ii) Hence, find the length of the side of the square and the value of r that will minimise the total area of the three shapes. **4**

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Question 16 (15 Marks)

Your responses should include relevant mathematical reasoning and/or calculations.

Question 16 (15 Marks) **Marks**

(a) A jelly baby is a lolly.

A bag of jelly babies is labelled as having a net weight of 220 grams. The production process produces bags whose weights are normally distributed with a mean of 224.7 grams and a standard deviation of 2.35 grams.

(i) If 1200 bags of jelly babies are produced, how many bags would be expected to contain less than the labelled weight? **2**

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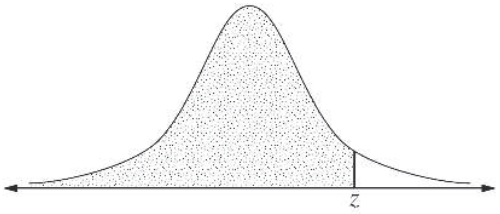
Question 15 continues on page 48

(a) (continued)

The table below gives the probability that a random variable lies below z for some positive values of z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974

The probability values given in the table are represented by the shaded area in the following diagram.



(ii) In the batch of 1200 bags of jelly babies from part (i), how many bags would be expected to have a weight of more than 231 grams, 5% over the labelled weight of 220 grams? **2**

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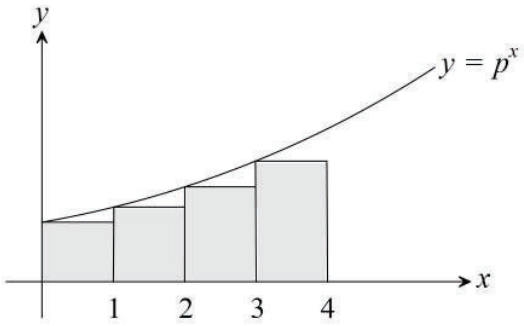
Question 16 continues on page 49

(b) The diagram shows the graph of $y = p^x$, where $p > 1$. Also shown on the diagram are 4 inner rectangles of width 1 unit.



NOT TO SCALE

(i) Show that the area of the 4 rectangles is $\frac{p^4 - 1}{p - 1}$ square units. **2**

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Question 16 continues on page 50

(b) (continued)

(ii) By finding the area under the curve $y = p^x$ between $x = 0$ and $x = 4$, and using the result in part (i), prove that $\ln p < p - 1$. **3**

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Question 16 continues on page 51

(c) The waiting time in minutes, t, for a calculator check to take place before a student is allowed to enter the gymnasium to sit a mathematics task, has a probability density function given by

$$f(t) = \begin{cases} kt\left(36 - t^2\right) & 0 \le t \le 6 \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $k = \dfrac{1}{324}$ **1**

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Question 16 continues on page 52

(c) (continued)

(ii) Determine the mode waiting time. **2**

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Question 16 continues on page 53

(c) (continued)

(iii) What is the probability that a candidate must wait for more than 3 minutes for a calculator check to take place, given that they have already waited for 1 minute. **3**

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End of Paper

Student Name: Solutions.

Student Number: | | | | | | | | | |

Teachers Name: _____

ABBOTSLEIGH

2024

Higher School Certificate

Trial Examination

Mathematics Advanced

General Instructions

- Reading time – 10 minutes.
- Working time – 3 hours.
- Write using black pen only.
- NESA approved calculators may be used.
- A NESA Reference sheet is provided.
- All necessary working should be shown in every question.
- Make sure your Student Number is on the front cover of each section.
- Answer all Multiple-Choice questions on the answer sheet provided.
- In Questions 11 – 16 show relevant mathematical reasoning and/or calculations.
- Extra Writing spaces are provided on pages 15, 23, 31, 39, 45 and 55.

Total Marks – 100

Attempt all questions.

Section I — Pages 3 – 10

10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.

Section II — Pages 11 – 56

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section.

Outcomes to be assessed:

Mathematics Advanced

Preliminary:

A student

MA11-1 uses algebraic and graphical techniques to solve, and where appropriate, compare alternative solutions to problems.

MA11-2 uses the concepts of functions and relations to model, analyse and solve practical problems.

MA11-3 uses the concepts and techniques of trigonometry in the solution of equations and problems involving geometric shapes.

MA11-4 uses the concepts and techniques of periodic functions in the solutions of trigonometric equations or proof of trigonometric identities.

MA11-5 interprets the meaning of the derivative, determines the derivative of functions and applies these to solve simple practical problems.

MA11-6 manipulates and solves expressions using the logarithmic and index laws, and uses logarithms and exponential functions to solve practical problems

MA11-7 uses concepts and techniques from probability to present and interpret data and solve problems in a variety of contexts, including the use of probability distributions.

MA11-8 uses appropriate technology to investigate, organise, model and interpret information in a range of contexts.

MA11-9 provides reasoning to support conclusions which are appropriate to the context.

HSC:

A student

MA12-1 uses detailed algebraic and graphical techniques to critically construct, model and evaluate arguments in a range of familiar and unfamiliar contexts.

MA12-2 models and solves problems and makes informed decisions about financial situations using mathematical reasoning and techniques.

MA12-3 applies calculus techniques to model and solve problems.

MA12-4 applies the concepts and techniques of arithmetic and geometric sequences and series in the solution of problems.

MA12-5 applies the concepts and techniques of periodic functions in the solution of problems involving trigonometric graphs.

MA12-6 applies appropriate differentiation methods to solve problems.

MA12-7 applies the concepts and techniques of indefinite and definite integrals in the solution of problems.

MA12-8 solves problems using appropriate statistical processes.

MA12-9 chooses and uses appropriate technology effectively in a range of contexts, models and applies critical thinking to recognise appropriate times for such use.

MA12-10 constructs arguments to prove and justify results and provides reasoning to support conclusions which are appropriate to the context.

SECTION I

10 marks
Attempt Questions 1 – 10

Use the multiple-choice answer sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

(A) ○ (B) ● (C) ○ (D) ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) ● (B) ⊗ (C) ○ (D) ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

correct ←

(A) ⊗ (B) ⊗ (C) ○ (D) ○

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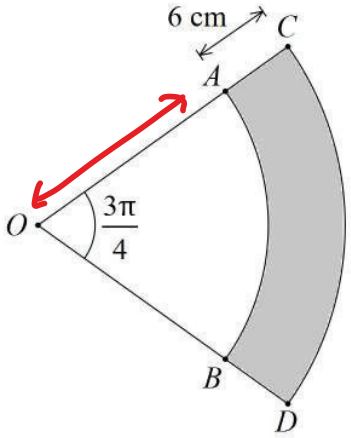
1. In the figure, OAB and OCD are sectors with centre O. It is given that the area of the shaded region is 99π cm^2.



NOT TO SCALE

$99\pi = \frac{1}{2}(r+6)^2 \times \frac{3\pi}{4} - \frac{1}{2}r^2 \times \frac{3\pi}{4}$

$99\pi = \frac{3\pi}{8}(r+6)^2 - \frac{3\pi}{8}r^2$

$264 = r^2 + 12r + 36 - r^2$

$264 = 12r + 36$

$r = 19.$

If $\angle AOB = \frac{3\pi}{4}$ radians and $AC = 6$ cm, what is the length of OA?

A. 16 cm

B. 19 cm

C. 24 cm

D. 25 cm

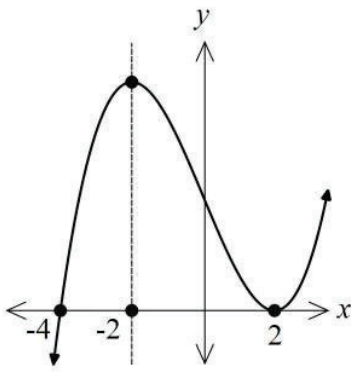
Section I continues on page 4

2. Characteristics of the cubic function $f(x)$ are given.

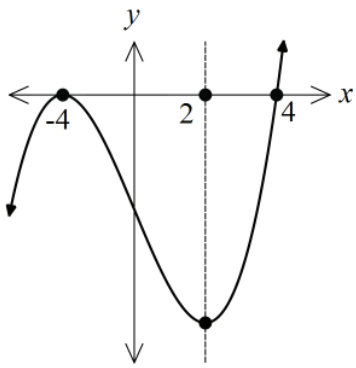
- $(x+4)$ is a factor of $f(x)$. → x int at -4
- $x=2$ is a repeated root of $f(x)$. → tangent to x-axis at x = 2
- $f'(-2)=0$ → stationary at x = -2
- $f'(x)>0$ where the graph with equation $y=f(x)$ crosses the y-axis. → increasing at y int

Which of the following could be the graph of $y=f(x)$?

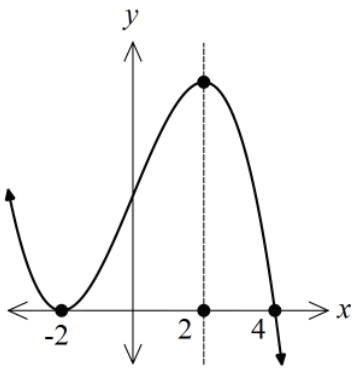
A.



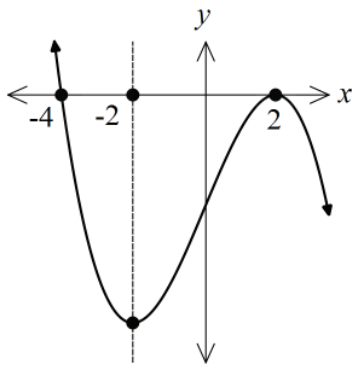
B.



C.



D.



3. Using the trapezoidal rule with four intervals, what is the approximate area under the curve $y=\sqrt{3x+4}$ between $x=0$ and $x=4$?

x	0	1	2	3	4
y	2	$\sqrt{7}$	$\sqrt{10}$	$\sqrt{13}$	4

$A \approx \frac{1}{2}\left(2+4+2\left(\sqrt{7}+\sqrt{10}+\sqrt{13}\right)\right)$

$\approx \frac{1}{2}\left(6+2\sqrt{7}+2\sqrt{10}+2\sqrt{13}\right)$

$\approx 3+\sqrt{7}+\sqrt{10}+\sqrt{13}$

A. $3+\sqrt{7}+\sqrt{10}+\sqrt{13}$

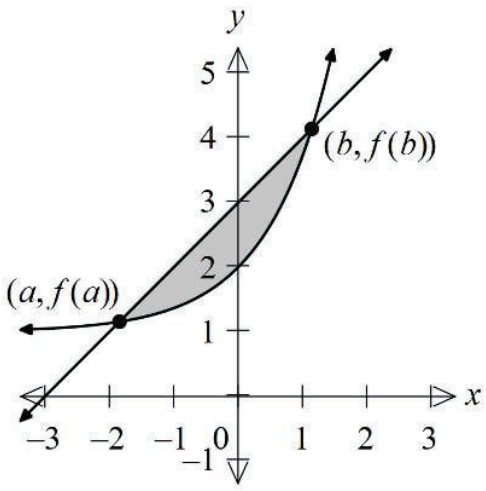
B. $3+2\left(\sqrt{7}+\sqrt{10}+\sqrt{13}\right)$

C. $6+2\left(\sqrt{7}+\sqrt{10}+\sqrt{13}\right)$

D. $6+\sqrt{7}+\sqrt{10}+\sqrt{13}$

Section I continues on page 5

4. The region enclosed by $y = e^x + 1$ and $y = x + 3$ is shaded in the diagram. The points of intersection of the two curves are shown.



Which integral represents the area of the shaded region?

A. $\int_a^b \left(e^x - x + 4\right) dx$

B. $\int_a^b \left(x + 4 - e^x\right) dx$

C. $\int_a^b \left(e^x - x - 2\right) dx$

D. $\int_a^b \left(x + 2 - e^x\right) dx$

$A = \int_a^b x + 3 - (e^x + 1)\, dx$

$= \int_a^b x + 2 - e^x\, dx$

5. Differentiate the function $f(x) = \ln\left(\sqrt[3]{x^3 + 3}\right)$ to find $f'(x)$.

A. $f'(x) = \dfrac{3x^2}{x^3 + 3}$

B. $f'(x) = \dfrac{3x^2}{\sqrt[3]{x^3 + 3}}$

C. $f'(x) = \dfrac{x^2}{x^3 + 3}$

D. $f'(x) = 3x^2\left(\sqrt[3]{x^3 + 3}\right)$

$f(x) = \ln\left((x^3 + 3)^{\frac{1}{3}}\right)$

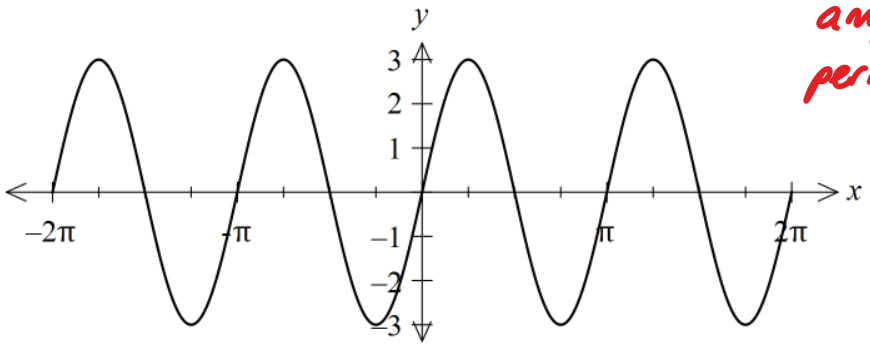
$f'(x) = \dfrac{\frac{1}{3}(x^3 + 3)^{-\frac{2}{3}} \times 3x^2}{(x^3 + 3)^{\frac{1}{3}}}$

$= \dfrac{x^2}{(x^3 + 3)^{\frac{2}{3}} \times (x^3 + 3)^{\frac{1}{3}}}$

$= \dfrac{x^2}{x^3 + 3}$

Section I continues on page 6

6. The graph of $y = f(x)$ is shown in the domain $[-2\pi, 2\pi]$.



amp = 3
period = π
$\therefore \frac{2\pi}{n} = \pi$
$n = 2$

Which of the following functions represents $f(x)$?

A. $y = 3\sin 2x$

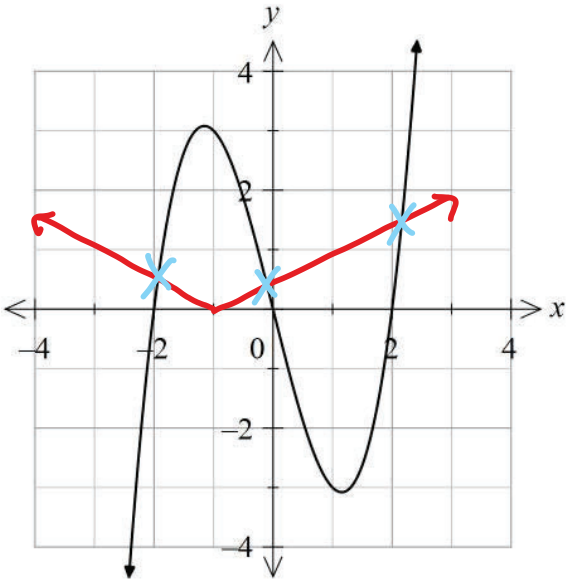
B. $y = 3\cos 2x$

C. $y = 3\sin x$

D. $y = 3\cos x$

Section I continues on page 7

7. The graph of $y = x(x-2)(x+2)$ is shown.



How many solutions does the equation $x(x-2)(x+2) = \left|\frac{x+1}{2}\right|$ have?

$y = \left|\frac{x+1}{2}\right|$

$y = \left|\frac{1}{2}x + \frac{1}{2}\right|$

A. 2

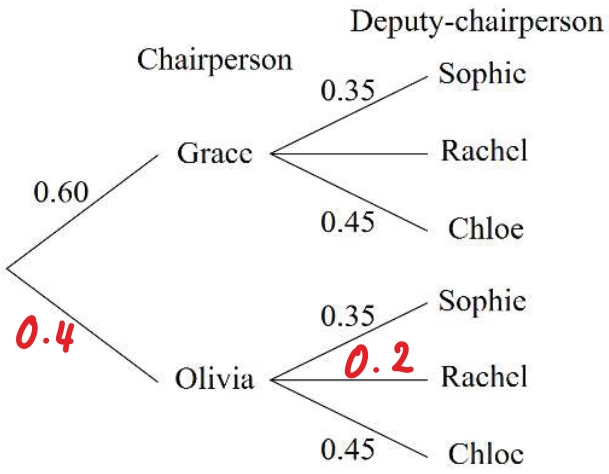
B. 3

C. 4

D. 5

Section I continues on page 8

8. The partially completed probability tree diagram shows the voting intentions of shareholders in an election for Chairperson and Deputy-chairperson of the board of directors of a large ASX listed company.



Based on these voting intentions, what is the probability that Olivia is elected Chairperson, with either Rachel or Sophie as Deputy-chairperson?

A. 0.22

B. 0.26

C. 0.33

D. 0.48

P(OR or OS) = 0.4×0.2 + 0.4×0.35
= 0.22

Section I continues on page 9

9. Data on the time in minutes it takes for students at Abbotsleigh to travel to school is normally distributed with a mean of 37 minutes and a standard deviation of 11 minutes.

Using the z-score table below, what is the approximate 90th percentile for the data?

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

A. 47 minutes

B. 49 minutes

C. 51 minutes

D. 53 minutes

$\mu = 37, \sigma = 11$

$P(X \leq ?) \approx 0.9$

$\therefore \Phi(?) \approx 0.9$

from table, $z \approx 1.28$

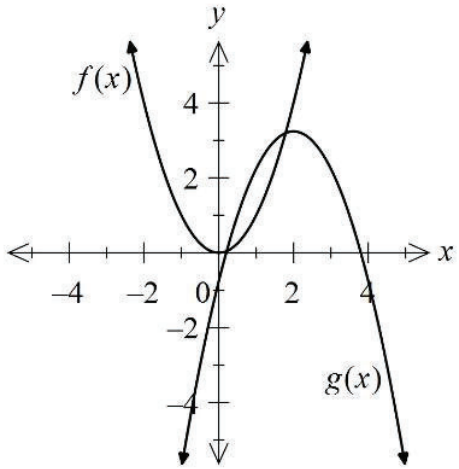
$x = z\sigma + \mu$

$x = 1.28 \times 11 + 37$

$= 51.08$

Section I continues on page 10

10. The graphs of $f(x)=x^2$ and $y=g(x)$ are shown.



How many stationary points does the graph of $y=g(f(x))$ have?

A. 0

B. 1

C. 2

D. 3

$g(x)=-a(x-2)^2+k$, $f(x)=x^2$

$g(f(x))=-a(x^2-2)^2+k$

$g'(f(x))=-2a(x^2-2)\times 2x$

$=-4ax(x^2-2)$

Stat pts, let $g'(f(x))=0$:

$0=-4ax(x^2-2)$

$\therefore$ 3 solutions.

End of Section I

SECTION II

Student Number: | | | | | | | | | |

Total Marks – 90
Attempt Questions 11 – 16

Answer each question in the space provided.

Your responses should include relevant mathematical reasoning and/or calculations.

Extra writing space is provided on pages 15, 23, 31, 39, 45 and 55.

Question 11 (15 Marks) **Marks**

(a) Solve $\log_2 9 = 5\log_2 3 - \log_2 x$. **2**

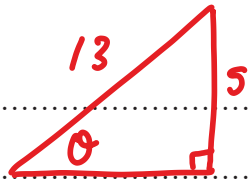
$\log_2 9 = \log_2 3^5 - \log_2 x$ (1)

$\log_2 9 = \log_2 \frac{243}{x}$

$9 = \frac{243}{x}$

$x = 27$ (1)

(b) Given that $\sin\alpha = \frac{5}{13}$ and $\tan\alpha < 0$, find the exact value of $\cos\alpha$. **2**



$\cos\theta = \frac{12}{13}$

(1)

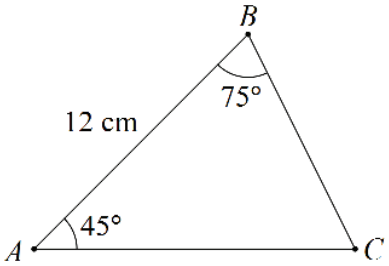
$\sin\alpha > 0$, $\tan\alpha < 0$

$\therefore \alpha$ is in the 2nd quadrant

$\therefore \cos\alpha = -\frac{12}{13}$ (1)

Question 11 continues on page 12

(c) In $\triangle ABC$, $AB = 12$ cm, $\angle A = 45°$ and $\angle B = 75°$.



NOT TO SCALE

(i) Show that the length of BC is $4\sqrt{6}$ cm. **2**

$\frac{BC}{\sin 45°} = \frac{12}{\sin 60°}$ (1)

$BC = \frac{12 \sin 45}{\sin 60}$

$BC = \frac{12 \times \frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2}}$

$= 6\sqrt{2} \times \frac{2}{\sqrt{3}}$

$= \frac{12\sqrt{2}}{\sqrt{3}}$

$= \frac{12\sqrt{6}}{3}$

$= 4\sqrt{6}$ as req. (1) with sufficient working.

(ii) Hence or otherwise, find the area of the triangle to 2 decimal places. **1**

$A = \frac{1}{2}ab\sin C$

$A = \frac{1}{2} \times 12 \times 4\sqrt{6} \times \sin 75$ (1)

≈ 56.78 cm^2 (2 dp).

Question 11 continues on page 13

(d) Given that $f(x)=(1-\cos x)^5$, find $f'\left(\frac{\pi}{2}\right)$. **2**

$f'(x) = 5(1-\cos x)^4 \times \sin x$ (1)

$= 5\sin x(1-\cos x)^4$

$f'\left(\frac{\pi}{2}\right) = 5\sin\frac{\pi}{2}\left(1-\cos\frac{\pi}{2}\right)^4$

$= 5$ (1)

(e) Find the derivative of the function $f(x)=x^2e^{5x}$. Give your answer in factorised form. **2**

$u = x^2 \quad v = e^{5x}$

$u' = 2x \quad v' = 5e^{5x}$ } either (1)

$f'(x) = 2xe^{5x} + 5x^2e^{5x}$

$= xe^{5x}(2+5x)$ (1)

Question 11 continues on page 14

(f) Solve the equation $(\sin x+3)(2\sin x+1)=0$ in the domain $0\le x\le 2\pi$. **2**

$\sin x + 3 = 0$

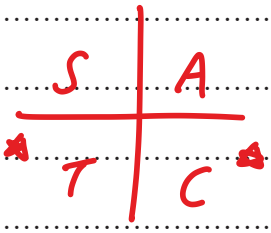
$\sin x = -3$

no solutions as $-1 \le \sin x \le 1$ (1)

$2\sin x + 1 = 0$

$\sin x = -\frac{1}{2}$

$x = \pi + \frac{\pi}{6}, \quad 2\pi - \frac{\pi}{6}$



$x = \frac{7\pi}{6}, \frac{11\pi}{6}$ (1)

(g) Evaluate $\int_1^3 \frac{27x^2-21}{3x^3-7x+8}\,dx$, leaving your answer in exact simplified form. **2**

$= 3\int_1^3 \frac{9x^2-7}{3x^3-7x+8}\,dx$

$= 3\left[\ln|3x^3-7x+8|\right]_1^3$ (1)

$= 3\left[\ln|3(3)^3-7(3)+8| - \ln|3(1)^3-7(1)+8|\right]$

$= 3[\ln 68 - \ln 4]$

$= 3\ln 17$ either (1).

End of Question 11

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Question 12 (15 Marks)

Your responses should include relevant mathematical reasoning and/or calculations.

Question 12 (15 Marks) **Marks**

(a) Consider the geometric series

$$1+\left(\sqrt{11}-3\right)+\left(\sqrt{11}-3\right)^2+\left(\sqrt{11}-3\right)^3+\ldots$$

(i) Show with calculations why this geometric series has a limiting sum. **2**

$r = \sqrt{11} - 3$

$r = 0.31662\ldots$ (1)

Since $-1 < 0.31662\ldots < 1$, $|r| < 1$ (1)

$\therefore$ The GP has a limiting sum.

(ii) Determine the exact value of the limiting sum. Express your answer with a rational denominator. **2**

$S = \dfrac{a}{1-r}$

$= \dfrac{1}{1-(\sqrt{11}-3)}$ (1)

$= \dfrac{1}{4-\sqrt{11}} \times \dfrac{4+\sqrt{11}}{4+\sqrt{11}}$

$= \dfrac{4+\sqrt{11}}{5}$ (1)

Question 12 continues on page 18

(b) Tables of future value and present value interest factors for an annuity of $1 are shown below.

Future Value Interest Factors for an Annuity of $1						
Number of Periods	Rate per Period					
	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%
5	5.101	5.204	5.309	5.416	5.526	5.637
10	10.462	10.950	11.464	12.006	12.578	13.181
20	22.019	24.297	26.870	29.778	33.066	36.786
30	34.785	40.568	47.575	56.085	66.439	79.058
40	48.886	60.402	75.401	95.026	120.800	154.762

Present Value Interest Factors for an Annuity of $1						
Number of Periods	Rate per Period					
	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%
5	4.853	4.713	4.580	4.452	4.329	4.212
10	9.471	8.983	8.530	8.111	7.722	7.360
20	18.046	16.351	14.877	13.590	12.462	11.470
30	25.808	22.396	19.600	17.292	15.372	13.765
40	32.835	27.355	23.115	19.793	17.159	15.046

(i) Isabella plans to deposit $7500 at the end of each year into an annuity paying 5.0% p.a. compounding annually. What is the value of her investment after 10 years? **1**

FV = M x interest factor

= 7500 x 12.578

= $94 335 (1)

Question 12 continues on page 19

(b) (continued)

(ii) What amount could Isabella invest now at 5.0% p.a. compounding annually, to achieve the same financial outcome as the annuity after 10 years? **1**

PV = M x interest factor

= 7500 x 7.722

= $57915 (1)

OR

94335 ÷ (1.05)^{10} = $57913.51 (1)

(c) Find the primitive function of $\sec^2(5x) - \sin(x) + 5^x$. **3**

$\int \sec^2(5x) - \sin x + 5^x \, dx$

$= \frac{1}{5}\tan 5x + \cos x + \frac{5^x}{\ln 5} + C$ (1) (1) (1)

Question 12 continues on page 20

(d) Determine the equation of the tangent to the curve $y = \ln\left(3x^2 + 2\right)$ at the point $\left(2,\ \ln 14\right)$. Express your answer in general form $ax + by + c = 0$. **3**

$y = \ln(3x^2 + 2)$

$y' = \dfrac{6x}{3x^2 + 2}$ (1)

at $x = 2$,

$y' = \dfrac{6(2)}{3(2)^2 + 2}$

$= \dfrac{6}{7}$ (1)

$\therefore m = \dfrac{6}{7}$ at $(2, \ln 14)$

$y - \ln 14 = \dfrac{6}{7}(x - 2)$

$7y - 7\ln 14 = 6x - 12$

$6x - 7y + 7\ln 14 - 12 = 0$ (1) or equivalent

Question 12 continues on page 21

(e) It is given that y is inversely proportional to x.

(i) When $x = 9$, $y = 3$. Determine the equation relating x in terms of y. **1**

$y = \frac{k}{x}$ for inverse proportion

$3 = \frac{k}{9}$

$k = 27$ $\therefore y = \frac{27}{x}$ (1)

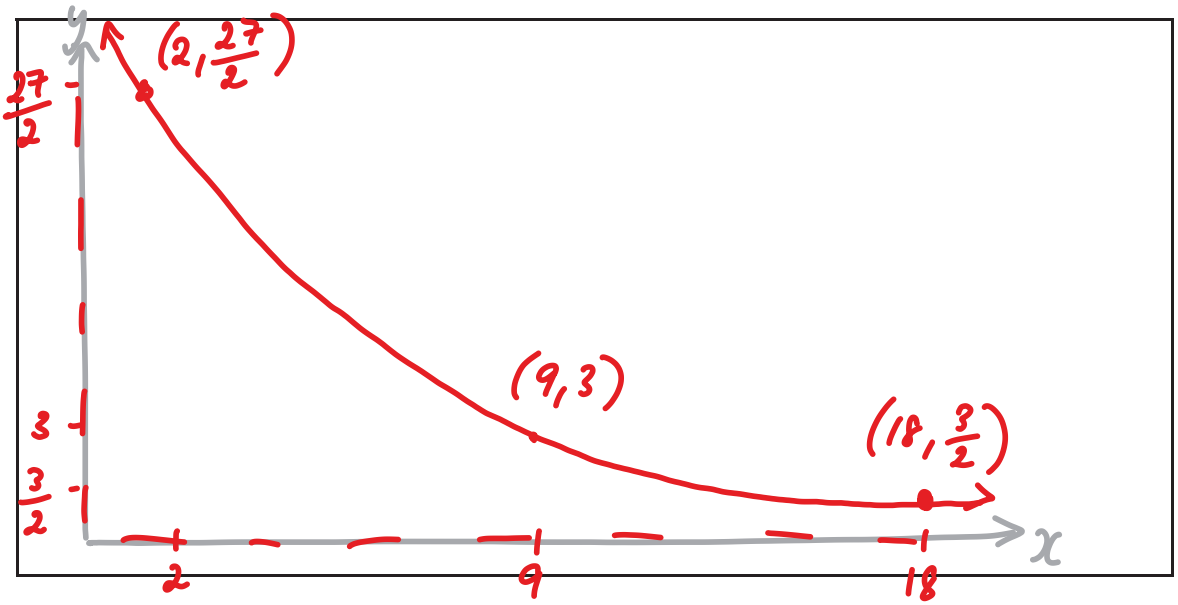
(ii) If the value of x is increased from 2 to 18, determine the change in the value of y. **1**

at $x = 2$, $y = \frac{27}{2}$

at $x = 18$, $y = \frac{3}{2}$

$\therefore$ change in $y = \frac{3}{2} - \frac{27}{2} = -12$. (1)

(iii) Hence graph the relationship between x and y using the values you have found previously. **1**



(1)

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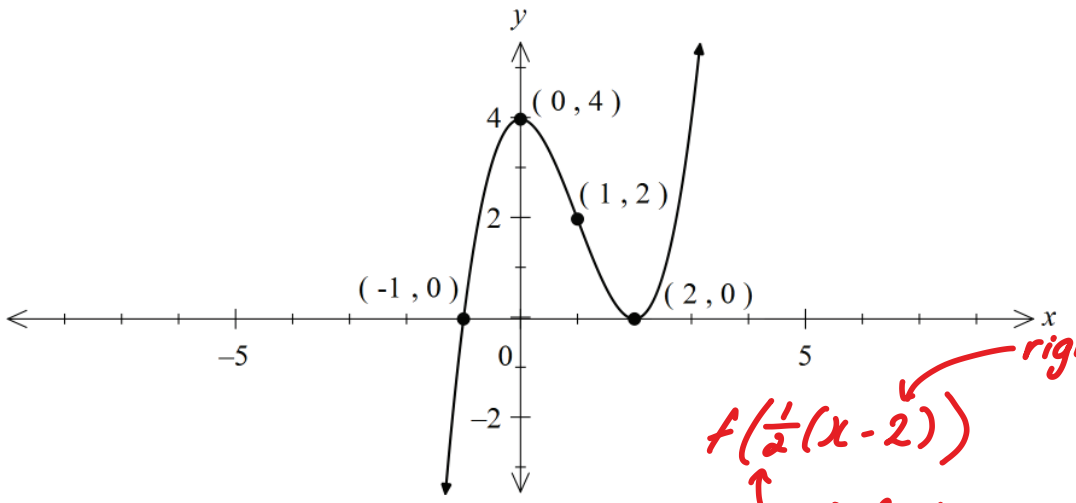
Question 13 (15 Marks)

Your responses should include relevant mathematical reasoning and/or calculations.

Question 13 (15 Marks) **Marks**

(a) The diagram shows the graph of $y = f(x)$. **2**

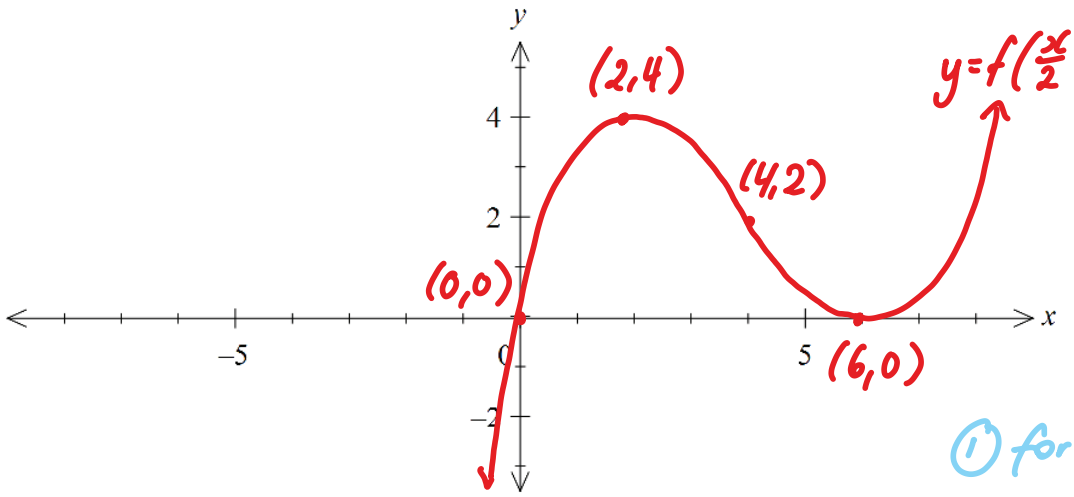


right 2 units.

$f\left(\frac{1}{2}(x-2)\right)$

HD factor 2.

Sketch the graph of $y = f\left(\frac{x}{2} - 1\right)$, labelling all stationary points and intercepts.



① for HD factor 2

① for translation 2 right

Question 13 continues on page 26

(b) Ava is managing the growth of a population of bacteria in a laboratory. The population can be modelled by the exponential equation $P(t) = 500e^{kt}$ where the initial population is 500 bacteria, k is the growth constant, and t is time in minutes.

(i) Find the value of k correct to 1 significant figure, given that there are 1660 bacteria present after 2 hours. **2**

$P(120) = 1660$:

$1660 = 500e^{120k}$

$\frac{83}{25} = e^{120k}$ (1)

$120k = \ln\frac{83}{25}$

$k = \frac{\ln\frac{83}{25}}{120}$ (1) either

≈ 0.01 (1 sig. fig).

(ii) Determine the rate of change of the population with respect to time after 2 hours. Give your answer to the nearest whole number. **2**

$P(t) = 500e^{kt}$ where $k = \frac{\ln\frac{83}{25}}{120}$

$P'(t) = 500ke^{kt}$ (1)

$P'(120) = 500ke^{k\times 120}$

$= 16.6\ldots$

$\therefore$ Increasing at approximately 17 bacteria/min at $t = 120$. (1) also accept 16 or 16.6

Question 13 continues on page 27

(b) (continued)

(iii) Ava is concerned about the growth of the bacteria and wants to determine if she needs to introduce a control mechanism. **3**

Interpret the nature of the population growth with reference to the derivatives $P'(t)$ and $P''(t)$, and hence determine if a control would be needed.

$P'(t) = 500ke^{kt}$ where $k = \frac{\ln\frac{83}{25}}{120}$ (1)

$P''(t) = 500k^2e^{kt}$

e^{kt} is >0 for all t, so $P'(t)>0$ and (1)

$P''(t)>0$ for all t.

$\therefore$ $P(t)$ and $P'(t)$ are both always increasing

$\therefore$ $P(t)$ is increasing at an increasing rate and (1)

therefore a control mechanism should be introduced.

Question 13 continues on page 28

(c) Lara works as a sales representative for a company that sells laboratory equipment. The number of sales she makes in a day is given by the probability distribution table.

x	0	1	2	3	4	5
$P(X=x)$	0.14	0.15	0.25	0.30	0.11	0.05

(i) Determine the expected value of the number of sales that Lara makes on a particular day. **1**

$E(X) = 0\times0.14 + 1\times0.15 + 2\times0.25 + 3\times0.3 + 4\times0.11 + 5\times0.05$

$= 2.24$ ✓1

(ii) Another sales representative, Rebecca, has an expected number of sales of 2.5 and a standard deviation in the number of sales of 1.2. **2**

The CEO of the company would like to promote a sales representative to be the leader of the sales team.

If the CEO bases her decision on sales performance, are they more likely to promote Lara or Rebecca? Justify your answer with appropriate statistical reasoning.

$Var(X) = 0^2\times0.14 + 1^2\times0.15 + 2^2\times0.25 + 3^2\times0.3 + 4^2\times0.11 + 5^2\times0.05 - 2.24^2$

$= 1.8424$

$\sigma = \sqrt{1.8424}$

≈ 1.36 (2 dp) ✓1

Rebecca has a higher mean and smaller SD, meaning she has higher sales on average and is more consistent. ∴ Rebecca more likely to be promoted. ✓1

Question 13 continues on page 29

(d) Find all the values of θ where $-\pi \leq \theta \leq \pi$ such that **3**

$$\cos\left(\theta - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

$-\pi \leq \theta \leq \pi \rightarrow -\pi - \frac{\pi}{6} \leq \theta - \frac{\pi}{6} \leq \pi - \frac{\pi}{6}$

$\therefore D: -\frac{7\pi}{6} \leq \theta - \frac{\pi}{6} \leq \frac{5\pi}{6}$

either (1)

acute angle $= \frac{\pi}{6}$

$\theta - \frac{\pi}{6} = -\pi - \frac{\pi}{6}, -\pi + \frac{\pi}{6}, \pi - \frac{\pi}{6}$ either (1)

$\theta - \frac{\pi}{6} = -\frac{7\pi}{6}, -\frac{5\pi}{6}, \frac{5\pi}{6}$

$\theta = -\pi, -\frac{2\pi}{3}, \pi.$ (1)

End of Question 13

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Question 14 (15 Marks)

Your responses should include relevant mathematical reasoning and/or calculations.

Question 14 (15 Marks) **Marks**

(a) The first three terms of a series are 3, 9, 15, ……

(i) Calculate the fifteenth term in this series. **1**

$a = 3, \quad d = 6, \quad n = 15$

$T_n = a + (n-1)d$

$T_{15} = 3 + 14 \times 6$ ✓

$= 87$

(ii) The sum of the first n consecutive terms of the series is 243. **2**
What is the value of n?

$243 = \frac{n}{2}(6 + 6(n-1))$ ✓

$243 = \frac{n}{2}(6n)$

$243 = 3n^2$

$n^2 = 81$

$n = 9 \quad (n > 0)$ ✓

Question 14 continues on page 34

(b) A bag contains eight cards that are each printed with one of the following sets of characters. **3**

aaa aab aba baa abb bab bba bbb

A student chooses one card at random.

Let N_1 be the event that the first character printed on the card is an 'a' and N_2 be the event that the second character printed on the card is also an 'a'.

Determine whether N_1 and N_2 are independent events.

$P(N_1) = \frac{4}{8} = \frac{1}{2}$

$P(N_2) = \frac{2}{8} = \frac{1}{4}$

$P(N_1 \cap N_2) = \frac{2}{8} = \frac{1}{4}$ (1)

$P(N_1) \times P(N_2) = \frac{1}{2} \times \frac{1}{4}$

$= \frac{1}{8}$ (1)

$P(N_1) \times P(N_2) \neq P(N_1 \cap N_2)$ (1)

$\therefore N_1$ and N_2 are not independent.

Question 14 continues on page 35

(c) Mr. Fredricksen has just retired from work as a balloon salesman. He has received a superannuation payment and has worked out that he will require \$7000 per month for the next 20 years.

Mr. Fredricksen intends to invest a set amount (P) at 6% p.a. interest compounding monthly and withdraws \$7000 from this account at the end of each month.

(i) Show that the amount A_3 remaining in the account after the third withdrawal of \$7000 is $A_3 = P(1.005)^3 - 7000\left[(1.005)^2 + 1.005 + 1\right]$ **2**

$A_1 = P(1.005) - 7000$ (1)

$A_2 = P(1.005)^2 - 7000(1.005) - 7000$

$A_3 = P(1.005)^3 - 7000(1.005)^2 - 7000(1.005) - 7000$

$= P(1.005)^3 - 7000\left[1.005^2 + 1.005 + 1\right]$ as req. (1)

(ii) Find the amount P that will need to be invested if he is to withdraw \$7000 each month for the next 20 years. Express your answer to the nearest cent. **2**

$A_{240} = P(1.005)^{240} - 7000\left[1.005^{239} + 1.005^{238} + \ldots + 1\right]$

$= P(1.005)^{240} - 7000\left[\dfrac{1.005^{240} - 1}{1.005}\right]$ (1)

$= P(1.005)^{240} - 1400000\left(1.005^{-240} - 1\right)$

let $A_{240} = 0$:

$0 = P(1.005)^{240} - 1400000\left(1.005^{-240} - 1\right)$

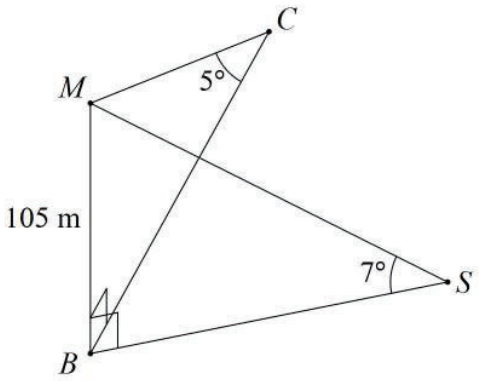
$P(1.005)^{240} = 1400000\left(1.005^{-240} - 1\right)$

$P = \dfrac{1400000\left(1.005^{-240} - 1\right)}{1.005^{240}}$

$P = \$977\,065.40$ (1)

Question 14 continues on page 36

(d) Sarah and Catherine decide to return to Sydney from Schoolies on the Gold Coast aboard separate sailing vessels. Both vessels are still out to sea but are approaching Sydney harbour and can view the light from the Macquarie lighthouse that is situated at Vaucluse near South Head.



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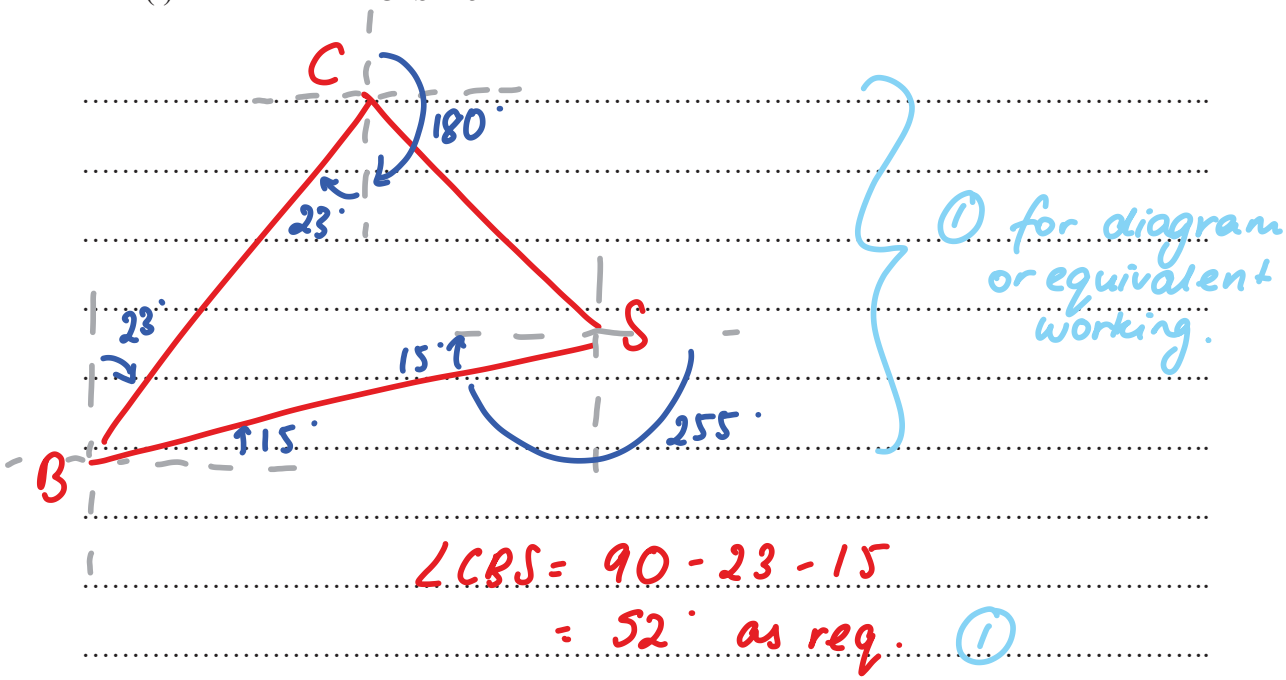
From Catherine's position (C), the Macquarie lighthouse (B) is at a bearing of $203°$ and the angle of elevation to the light at the top of the lighthouse (M) is $5°$.

From Sarah's position (S), the Macquarie lighthouse is at a bearing of $255°$ and the angle of elevation to the light at the top of the lighthouse is $7°$.

The light at the top of the Macquarie lighthouse is 105 metres above sea level.

(i) Show that $\angle CBS = 52°$. **2**



Question 14 continues on page 37

(d) (continued)

(ii) Find the distance between Catherine and Sarah. Give your answer to the nearest metre. **3**

$\tan 5^\circ = \frac{105}{BC}$

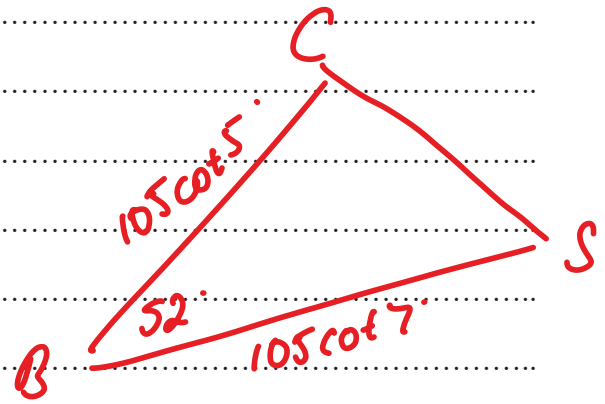
$BC = \frac{105}{\tan 5^\circ}$ — either (1)

$BC = 105\cot 5^\circ$

$\tan 7^\circ = \frac{105}{BS}$

$BS = \frac{105}{\tan 7^\circ}$

$BS = 105\cot 7^\circ$



$CS^2 = (105\cot 5^\circ)^2 + (105\cot 7^\circ)^2 - 2(105\cot 5^\circ)(105\cot 7^\circ)\cos 52^\circ$ (1)

$= \frac{105^2}{\tan^2 5} + \frac{105^2}{\tan^2 7} - \frac{2\times 105^2 \cos 52^\circ}{\tan 5^\circ \tan 7^\circ}$

$\approx 907\,933.4963\ldots$

$CS = 952.8554\ldots$

≈ 953 m (nearest metre) (1)

End of Question 14

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Question 15 (15 Marks)

Your responses should include relevant mathematical reasoning and/or calculations.

Question 15 (15 Marks) **Marks**

(a) Evaluate the definite integral $\int_0^{\pi} 3\sin\left(\frac{x}{4}\right) dx$. Express your answer with a rational denominator. **3**

$\int_0^{\pi} 3\sin\frac{x}{4}\, dx = \left[\frac{-3\cos\frac{x}{4}}{\frac{1}{4}}\right]_0^{\pi}$ either (1)

$= \left[-12\cos\frac{x}{4}\right]_0^{\pi}$

$= -12\cos\frac{\pi}{4} - (-12\cos 0)$ (1)

$= -12\times\frac{\sqrt{2}}{2} + 12$

$= 12 - 6\sqrt{2}$. (1)

Question 15 continues on page 42

(b) A particle moves along a straight line. The velocity of the particle, v, in m s^{-1}, is given by

$$v(t) = 2 - 5\sin\left(\frac{\pi t}{4}\right)$$

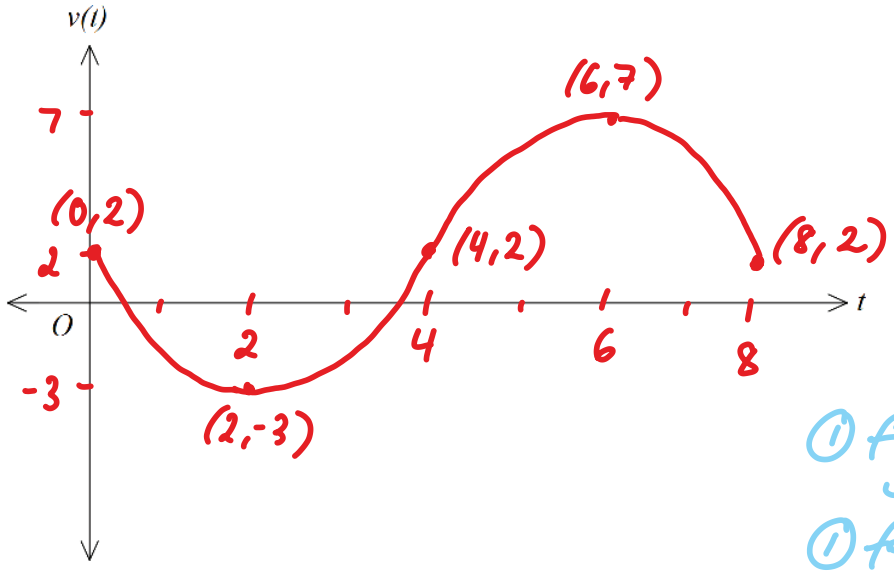
where t is the time, in seconds, and $0 \leq t \leq 8$.

(i) Find the particle's maximum and minimum velocity. **1**

max = 2 + 5 = 7 m/s (1)

min = 2 − 5 = −3 m/s

(ii) Hence, or otherwise, sketch the graph of the particle's velocity, labelling all important features (you do not need to find x-intercepts). **3**



(1) for correct shape

(1) for amplitude

(1) for period

Period = $\frac{2\pi}{\frac{\pi}{4}}$

= 8

Question 15 continues on page 43

(b) (continued)

(iii) Determine how many times the particle changes its direction. **2**

Explain your answer with reference to the graph in part (ii).

It will change direction when v(t) goes from positive to negative (or vice versa), i.e. when v(t) = 0. ①

For $0 \leq t \leq 8$, v(t) crosses the t-axis twice

∴ The particle changes direction twice. ①

(c) A 40-centimetre length of wire is cut into three pieces. Two pieces are bent to form two congruent circles and the final piece is bent to form a square.

(i) If the radius of each congruent circle is r centimetres, show that the total area of the three shapes is given by $A = (10 - \pi r)^2 + 2\pi r^2$ **2**

let the side length of the square be x:

$40 = 2\pi r + 2\pi r + 4x$ ①

$40 = 4\pi r + 4x$

$10 = \pi r + x$

$x = 10 - \pi r$

∴ $A = 2\pi r^2 + (10 - \pi r)^2$ ①

$= (10 - \pi r)^2 + 2\pi r^2$ as req.

Question 15 continues on page 44

(c) (continued)

(ii) Hence, find the length of the side of the square and the value of r that will minimise the total area of the three shapes. **4**

$A = (10 - \pi r)^2 + 2\pi r^2$

$\frac{dA}{dr} = 2(10 - \pi r) \times -\pi + 4\pi r$ (1)

$= -20\pi + 2\pi^2 r + 4\pi r$

let $\frac{dA}{dr} = 0$:

$0 = -20\pi + 2\pi^2 r + 4\pi r$

$20\pi = 2\pi^2 r + 4\pi r$

$10 = \pi r + 2r$

$10 = r(\pi + 2)$

$r = \frac{10}{\pi + 2}$ (1)

Check nature using $\frac{d^2A}{dr^2}$:

$\frac{d^2A}{dr^2} = 2\pi^2 + 4\pi$ which is > 0 for all r

$\therefore A$ is always concave up $\therefore$ min at $r = \frac{10}{\pi + 2}$ (1)

at $r = \frac{10}{\pi + 2}$, $x = 10 - \pi\left(\frac{10}{\pi + 2}\right)$

$x = \frac{10(\pi + 2) - 10\pi}{\pi + 2}$

$x = \frac{20}{\pi + 2}$ (1)

End of Question 15

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Question 16 (15 Marks)

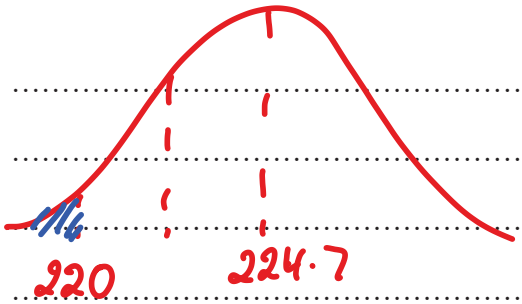
Your responses should include relevant mathematical reasoning and/or calculations.

Question 16 (15 Marks) **Marks**

(a) A jelly baby is a lolly.

A bag of jelly babies is labelled as having a net weight of 220 grams. The production process produces bags whose weights are normally distributed with a mean of 224.7 grams and a standard deviation of 2.35 grams.

(i) If 1200 bags of jelly babies are produced, how many bags would be expected to contain less than the labelled weight? **2**



$z = \frac{220 - 224.7}{2.35}$

$z = -2$

using Empirical rule,

$P(X < 220) = 0.5 \times (1 - 0.95)$

$= 0.025$ (1)

$1200 \times 0.025 = 30$

∴ We would expect 30 jelly babies to be below 220. (1)

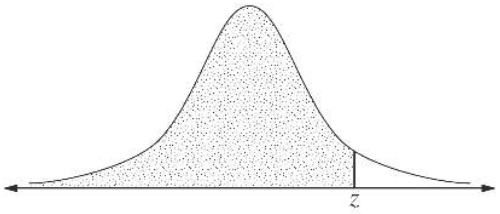
Question 15 continues on page 48

(a) (continued)

The table below gives the probability that a random variable lies below z for some positive values of z.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974

The probability values given in the table are represented by the shaded area in the following diagram.



(ii) In the batch of 1200 bags of jelly babies from part (i), how many bags would be expected to have a weight of more than 231 grams, 5% over the labelled weight of 220 grams? **2**

$z = \dfrac{231 - 224.7}{2.35}$

$z = 2.68$

$P(X > 231) = 1 - P(X \leq 231)$

$= 1 - \phi(2.68)$

$= 1 - 0.9963$

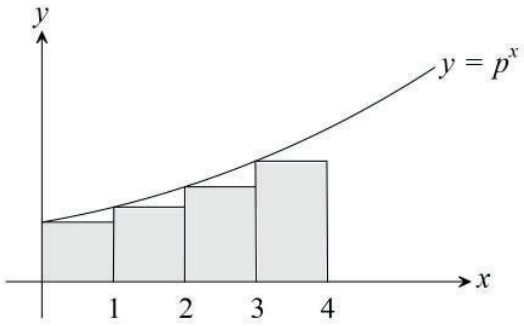
$= 0.0037$ (1)

$0.0037 \times 1200 = 4.44$

$\therefore$ We would expect 4 bags to be over 231g. (1) also accept 4.44 and 5.

Question 16 continues on page 49

(b) The diagram shows the graph of $y = p^x$, where $p > 1$. Also shown on the diagram are 4 inner rectangles of width 1 unit.



NOT TO SCALE

(i) Show that the area of the 4 rectangles is $\frac{p^4 - 1}{p - 1}$ square units. **2**

$A_{rect.} = 1 \times p^0 + 1 \times p^1 + 1 \times p^2 + 1 \times p^3$

$= 1 + p + p^2 + p^3$ (1)

GP where $a = 1$, $r = p$

$A_{rect} = \frac{1(p^4 - 1)}{p - 1}$

$= \frac{p^4 - 1}{p - 1}$ u^2 as req. (1) or equivalent algebra.

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(b) (continued)

(ii) By finding the area under the curve $y = p^x$ between $x = 0$ and $x = 4$, and using the result in part (i), prove that $\ln p < p - 1$. **3**

$A_{curve} = \int_0^4 p^x \, dx$

$= \left[\frac{p^x}{\ln p}\right]_0^4$ (1)

$= \frac{p^4}{\ln p} - \frac{p^0}{\ln p}$

$= \frac{p^4 - 1}{\ln p}$

From diagram, $A_{rect} < A_{curve}$

$\therefore \frac{p^4 - 1}{p - 1} < \frac{p^4 - 1}{\ln p}$ (1)

$\frac{1}{p-1} < \frac{1}{\ln p}$

$\ln p < p - 1$ as $p > 1$

(1)

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(c) The waiting time in minutes, t, for a calculator check to take place before a student is allowed to enter the gymnasium to sit a mathematics task, has a probability density function given by

$$f(t)=\begin{cases} kt\left(36-t^2\right) & 0\le t\le 6 \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $k=\dfrac{1}{324}$ **1**

since $f(t)$ is a PDF, $\int_0^6 f(t)\,dt = 1$

$1 = k\int_0^6 36t - t^3\,dt$

$1 = k\left[18t^2 - \frac{t^4}{4}\right]_0^6$

$1 = k\left(18(6)^2 - \frac{6^4}{4} - 0\right)$

$1 = 648k - 324k$

$1 = 324k$

$k = \frac{1}{324}$ as req. (1)

Question 16 continues on page 52

(c) (continued)

(ii) Determine the mode waiting time. **2**

Mode occurs when $f(t)$ reaches max:

$f(t) = 36kt - kt^3$

$= \frac{t}{9} - \frac{t^3}{324}$

$f'(t) = \frac{1}{9} - \frac{3t^2}{324}$

Let $f'(t) = 0$:

$0 = \frac{1}{9} - \frac{3t^2}{324}$

$\frac{3t^2}{324} = \frac{1}{9}$

$t^2 = 12$

$t = \sqrt{12} \quad (t > 0)$ (1)

$t = 2\sqrt{3}$

Check nature:

$f''(t) = -\frac{6t}{324}$

$f''(2\sqrt{3}) = -\frac{\sqrt{3}}{27}$

must show for 2/2.

$f''(2\sqrt{3}) < 0 \therefore$ max at $t = 2\sqrt{3}$

$\therefore$ Mode is $2\sqrt{3}$. (1)

Question 16 continues on page 53

(c) (continued)

(iii) What is the probability that a candidate must wait for more than 3 minutes for a calculator check to take place, given that they have already waited for 1 minute. **3**

$$P(X>3 \mid X\geq 1) = \frac{P(X>3)}{P(X\geq 1)} \quad (1)$$

$$= \frac{1-P(X\leq 3)}{1-P(X<1)}$$

$$P(X<3) = \int_0^3 \frac{t}{9} - \frac{t^3}{324}\,dt$$

$$= \left[\frac{t^2}{18} - \frac{t^4}{324}\right]_0^3$$

$$= \frac{3^2}{18} - \frac{3^4}{324} - 0$$

either (1)

$$= \frac{7}{16}$$

$$P(X<1) = \int_0^1 \frac{t}{9} - \frac{t^3}{324}\,dt$$

$$= \left[\frac{t^2}{18} - \frac{t^4}{324}\right]_0^1$$

$$= \frac{1^2}{18} - \frac{1^4}{324} - 0$$

$$= \frac{71}{1296}$$

$$\therefore P(X>3 \mid X<1) = \frac{1-\frac{7}{16}}{1-\frac{71}{1296}} = \frac{729}{1225} \quad (1)$$

End of Paper