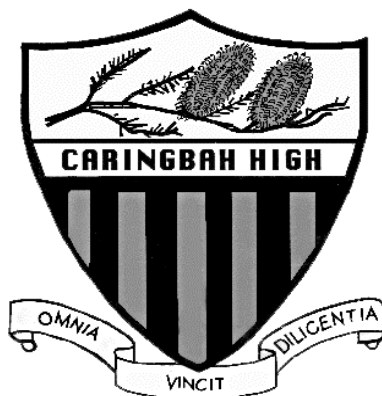




Caringbah High School

Year 12 2017

Mathematics

HSC Course

Assessment Task 4 (Trial HSC)

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A reference sheet is provided with this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 100

Section I 10 marks

Attempt Questions 1-10

Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II 90 marks

Attempt Questions 11-16

Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: _____

Class: _____

Marker's Use Only								
Section I	Section II						Total	
Q 1-10	Q11	Q12	Q13	Q14	Q15	Q16		
/10	/15	/15	/15	/15	/15	/15	/100	%

Section I

10 marks

Attempt Questions 1-10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5

1. Factorise $8x^6 - 27$
 - (A) $(2x^2 - 3)(4x^4 - 6x^2 + 9)$
 - (B) $(2x^2 + 3)(4x^4 - 6x^2 + 9)$
 - (C) $(2x^2 - 3)(4x^4 + 6x^2 + 9)$
 - (D) $(2x^2 + 3)(4x^4 + 6x^2 + 9)$

2. The quadratic equation $3x^2 + 5x - 2 = 0$ has roots α and β .
What is the value of $\alpha^2\beta + \alpha\beta^2$?
 - (A) $-\frac{10}{9}$
 - (B) $-\frac{9}{10}$
 - (C) $\frac{9}{10}$
 - (D) $\frac{10}{9}$

3. Which of the following is true about $y = 4\sin\frac{x}{2}$?
 - (A) Amplitude = 4 and period = $\frac{1}{2}$
 - (B) Amplitude = 4 and period = 2π
 - (C) Amplitude = $\frac{1}{2}$ and period = 4
 - (D) Amplitude = 4 and period = 4π

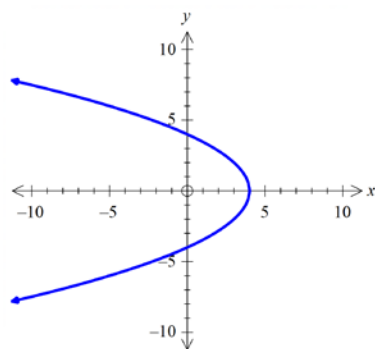
4. The equation $x^2 + y^2 + 6y = 7$ describes a circle with:
- (A) Centre (0,3) and radius 4 units
 - (B) Centre (0, -9) and radius 6 units
 - (C) Centre (0, -3) and radius 4 units
 - (D) Centre (0,3) and radius 6 units
5. The terms x , $1-x$, x^2-2 form an arithmetic series.
Which of the following are possible values for the common difference?
- (A) 1 and 9
 - (B) -1 and 9
 - (C) 1 and -9
 - (D) -1 and -9
6. What is the derivative of $\frac{e^{-x}}{x}$?
- (A) $\frac{-xe^{-x} - e^{-x}}{x^2}$
 - (B) $\frac{-xe^{-x} + e^{-x}}{x^2}$
 - (C) $\frac{e^{-x} + xe^{-x}}{x^2}$
 - (D) $\frac{e^{-x} - xe^{-x}}{x^2}$
7. What is the value of $\int_{-3}^2 |x+1| dx$?
- (A) $\frac{5}{2}$
 - (B) $\frac{11}{2}$
 - (C) $\frac{13}{2}$
 - (D) $\frac{17}{2}$

8. What is the value of $\lim_{x \rightarrow 10} \frac{x^2 - 100}{x - 10}$?

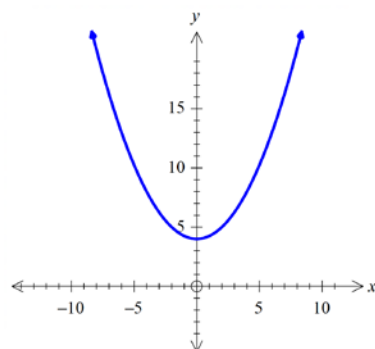
- (A) Undefined
- (B) 0
- (C) 8
- (D) 20

9. Which of the following graphs represents the parabola $y^2 = -4x + 16$?

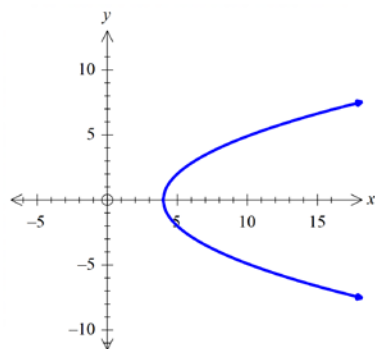
(A)



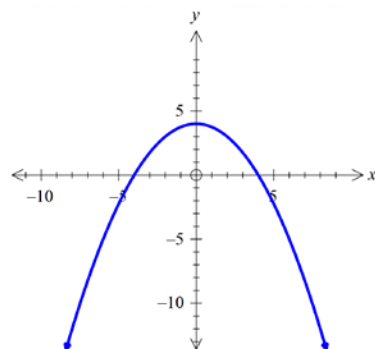
(B)



(C)



(D)



10. If $y = 10^x$, which of the following is true?

- (A) $x = \sqrt[10]{y}$
- (B) $x = \log_e y$
- (C) $\frac{dy}{dx} = 10^x$
- (D) $\frac{dy}{dx} = (\log_e 10) \times 10^x$

End of Section I

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Express $\frac{1}{\sqrt{5}+2}$ in the form $a+b\sqrt{5}$ where a and b are integers. 2

(b) Consider the table of values for $f(x)$ below. 2

x	3	3.25	3.5	3.75	4
$f(x)$	1.0	0.8	0.65	0.55	0.5

Find an approximation for the definite integral $\int_3^4 f(x) dx$ using Simpson's Rule and the above table. Give your answer correct to 3 significant figures.

(c) Find $\int_{\frac{\pi}{2}}^{\pi} \sec^2 \frac{x}{3} dx$, leaving your answer in exact form. 3

(d) Differentiate $\frac{\sin x}{e^x}$ with respect to x , leaving your answer in simplified form. 2

(e) Find the area bounded by the curve $y = x^3 - 4x$ and the x -axis. 3

(f) Solve $\log_2 x + \log_2(x+7) = 3$ for $x > 0$. 3

End of Question 11

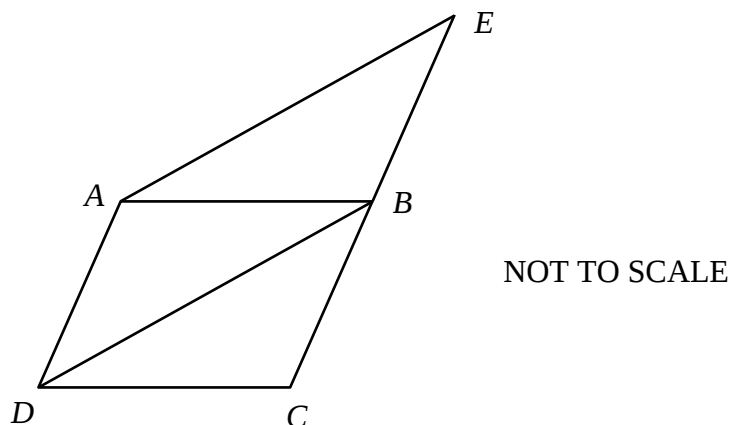
Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $|2x - 5| = 3$. 2

(b) If $\tan \theta = \frac{7}{9}$ and $\cos \theta < 0$, find the exact value of $\sin \theta$. 2

(c) For what values of x is the function $f(x) = -x^2(x + 3)$ decreasing? 2

(d) $ABCD$ is a rhombus. CB is produced to E such that $CB = BE$.



Copy the diagram into your answer booklet.

(i) Prove that $\triangle ABE \equiv \triangle DCB$. 3

(ii) Hence explain why $AE \parallel DB$. 1

(iii) What type of quadrilateral is $AEBD$? Justify your answer. 1

(e) Given that $\log_a m = 1.75$ and $\log_a n = 2.25$, find the value of:

(i) $\log_a mn$ 1

(ii) $\log_a \frac{n}{m}$ 1

(iii) $\sqrt[5]{mn^2}$ in terms of a 2

End of Question 12

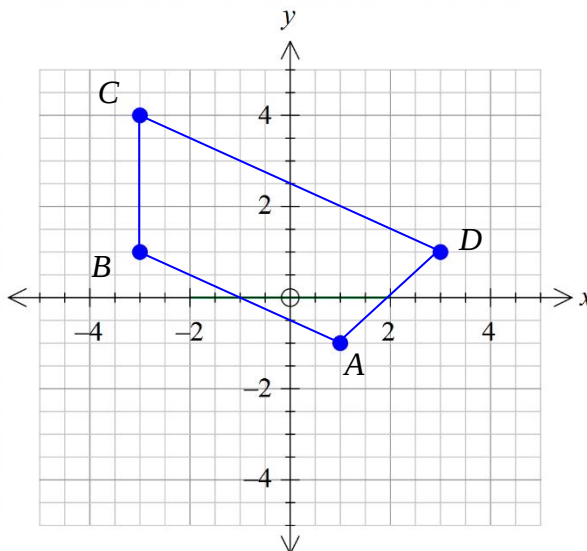
Question 13 (15 marks) Use a SEPARATE writing booklet.

- (a) The tangents of the parabola $y = x^2 + ax - 3$ at $x = 0$ and $x = 1$ are known to be perpendicular. Find the value of a . 2
- (b) Given that $\tan \theta = \frac{\sin \theta}{\cos \theta}$, find $\int \tan(2x) dx$. 2
- (c) Consider the graph of $y = x^3 + 3x^2 - 9x$.
- (i) Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. 2
- (ii) Find the coordinates of any stationary points and determine their nature. 3
- (iii) Show that there is a point of inflexion at $(-1, 11)$. 1
- (iv) Sketch the graph of $y = x^3 + 3x^2 - 9x$, labelling the above features and the y -intercept. 2
- (d) Given that $\frac{d^2y}{dx^2} = \frac{2}{x^2} + 2e^{2x}$ and that $\frac{dy}{dx} = e^2$ at $\left(1, \frac{e^2}{2}\right)$, find the expression for y in terms of x . 3

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

- (a) $A(1, -1)$, $B(-3, 1)$, $C(-3, 4)$ and $D(3, 1)$ are points on the Cartesian plane with $AB \parallel CD$.

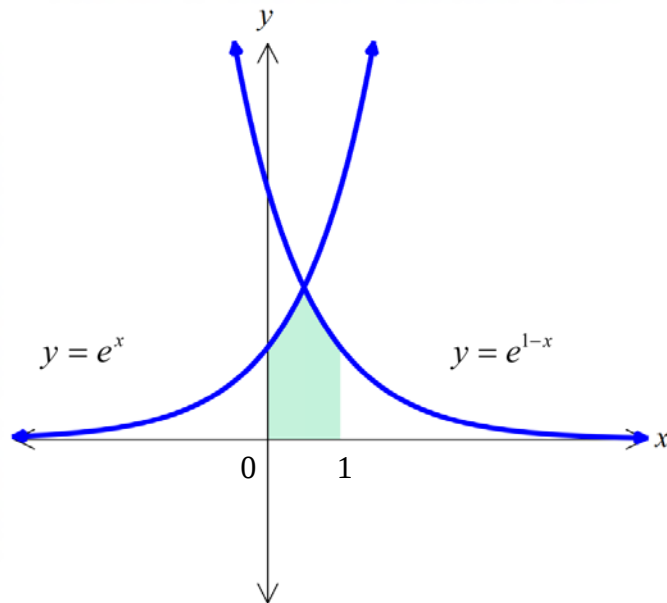


- | | | |
|-------|--|---|
| (i) | Find the distances BA and CD . | 2 |
| (ii) | Show that the equation of CD is $x + 2y - 5 = 0$. | 2 |
| (iii) | Find the perpendicular distance of A to CD . | 2 |
| (iv) | Hence, or otherwise, find the area of the quadrilateral $ABCD$. | 1 |
- (b) A and B are points on a circle with centre O . 3
- The area of the sector AOB is $3\pi \text{ cm}^2$ while the length of arc AB is $\frac{3\pi}{2} \text{ cm}$.
- Find the value of θ and r .
- (c) (i) Sketch the curve $y = \log_e x$ and shade the region bounded by the curve, the y -axis, $y = 1$ and $y = 2$. 2
- (ii) Find, in exact form, the volume of the solid of revolution formed when the above region is rotated about the y -axis. 3

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

- (a) Consider the curves $y = e^x$ and $y = e^{1-x}$ below.



- (i) Find the x -coordinate of point of intersection between the two curves. **1**
- (ii) Find the exact value of the shaded area bounded by the two curves, the coordinate axes and $x = 1$. **3**
- (b) The acceleration of a particle travelling in a straight line is given by $\frac{d^2x}{dt^2} = 8 - 6t$.
The particle is initially at the origin and travelling at 5 m/s to the right.
- (i) Find the equations for the velocity and displacement of the particle at any time t seconds. **2**
- (ii) At what time does the particle return to the origin? Find the velocity of the particle at that time. **3**

Question 15 continues on page 10

Question 15 (continued)

- (c) A car company offers a loan of \$20 000 to purchase a new car for which it charges interest at 1% per month. As a special deal, the company does not charge interest for the first 6 months; however, the monthly repayments start at the end of the first month. Wayne takes out a loan and agrees to repay the loan over 5 years by making 60 equal monthly repayments of \$ M .

Let A_n be the amount owing at the end of the n th month.

- | | | |
|-------|--|---|
| (i) | Find an expression for A_4 . | 1 |
| (ii) | Show that $A_8 = (20000 - 6M)(1.01)^2 - M(1 + 1.01)$ | 1 |
| (iii) | Find an expression for A_{60} . | 1 |
| (iv) | Find the value of M . | 3 |

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Differentiate $x^2 \log_e x$ 1

(ii) Hence, or otherwise, find $\int x \log_e x \, dx$ 2

(b) A bacteria culture of N bacteria is increasing exponentially so that $\frac{dN}{dt} = kN$

where t is time in minutes and k is a constant.

The number of bacteria increases from 100 to 400 in 2 minutes.

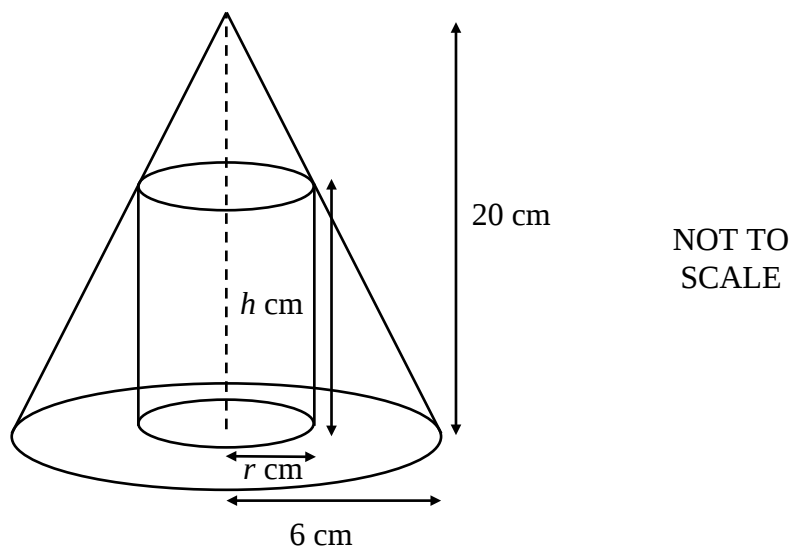
(i) Show that $N = Ae^{kt}$ is a solution to the above differential equation, where A is a constant. 1

(ii) Find the exact value of the constant k in simplest form. 2

(iii) Find the number of bacteria present after 10 minutes. 1

(iv) After how many minutes and seconds will there be 1000 bacteria? 2

(c) A cylinder of radius r cm and height h cm is inscribed in a cone with base radius 6 cm and height 20 cm, as shown in the diagram below.



(i) Show that the volume V of the cylinder is given by $V = \frac{10\pi r^2(6-r)}{3}$. 3

(ii) Hence, find the values of r and h such that the cylinder has a maximum volume. 3

End of Paper

MATHEMATICS (2U) TRIAL 2017

1. $8x^6 - 27 = (2x^2)^3 - 3^3$
 $= (2x^2 - 3)(4x^4 + 6x^2 + 9)$

(C)

2. $3x^2 + 5x - 2 = 0$

$$\begin{cases} \alpha + \beta = -5/3 \\ \alpha\beta = -2/3 \end{cases}$$

$$\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta)$$
$$= -\frac{5}{3} \times -\frac{2}{3}$$

$$= \frac{10}{9}$$

(D)

3. $y = 4 \sin(x/2)$

$$A = 4$$

$$T = \frac{2\pi}{1/2} = 4\pi$$

(D)

4. $x^2 + y^2 + 6y = 7$
 $x^2 + (y+3)^2 = 16$
 $\begin{cases} C(0, -3) \\ r = 4 \end{cases}$

(C)

5. AP: $d = T_2 - T_1 = T_3 - T_2$
 $1 - x - x = x^2 - 2 - (1 - x)$
 $1 - 2x = x^2 - 2 - 1 + x$
 $0 = x^2 + 3x - 4$

$$(x+4)(x-1) = 0$$

$$x = -4, 1$$

$$\text{i.e. } \begin{cases} -4, 5, 13 \\ 1, 0, -1 \end{cases} \text{ OR}$$

$$\therefore d = 9, -1$$

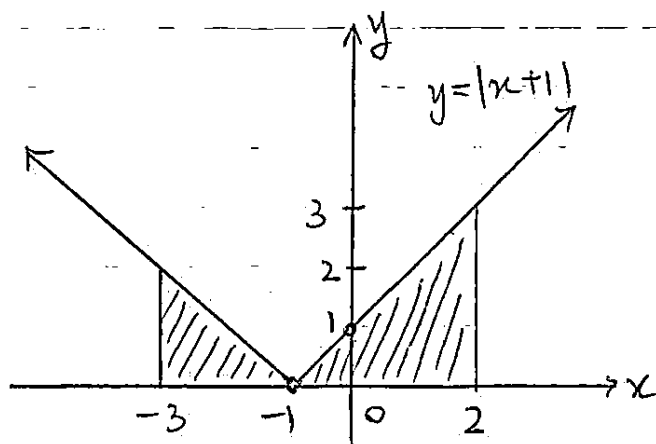
(B)

$$\begin{aligned}
 6. \quad \frac{d}{dx} \left(\frac{e^{-x}}{x} \right) &= \frac{(e^{-x})' \cdot x - e^{-x} (x)'}{x^2} \\
 &= \frac{-e^{-x} \cdot x - e^{-x} \cdot 1}{x^2} \\
 &= \frac{-xe^{-x} - e^{-x}}{x^2}
 \end{aligned}$$

(A)

$$\begin{aligned}
 7. \quad \int_{-3}^2 |x+1| dx &= \frac{1}{2} (2)(2) + \frac{1}{2} (3)(3) \\
 &= \frac{13}{2}
 \end{aligned}$$

(C)



$$\begin{aligned}
 8. \quad \lim_{x \rightarrow 10} \frac{x^2 - 100}{x - 10} &= \lim_{x \rightarrow 10} \frac{(x-10)(x+10)}{x-10} \\
 &= \lim_{x \rightarrow 10} x+10 \\
 &= 10+10 \\
 &= 20
 \end{aligned}$$

(D)

$$9. \quad y^2 = -4x + 16.$$

(A)

$$10. \quad y = 10^x$$

- $x = \log_{10} y$
- $y' = 10^x \cdot \ln 10.$

(D)

1	2	3	4	5	6	7	8	9	10
C	D	D	C	B	A	C	D	A	D

Question 11.

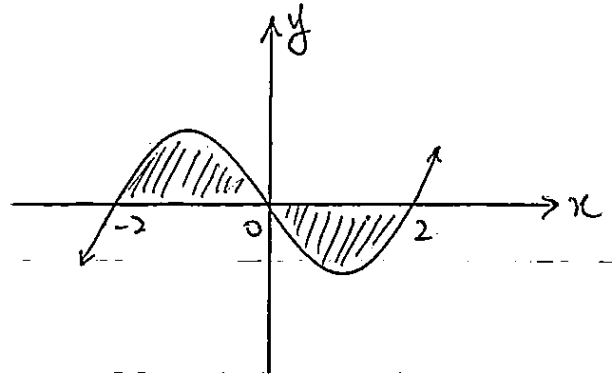
$$\begin{aligned} (a) \quad \frac{1}{\sqrt{5}+2} \times \frac{\sqrt{5}-2}{\sqrt{5}-2} &= \frac{\sqrt{5}-2}{(\sqrt{5})^2-2^2} \\ &= \frac{\sqrt{5}-2}{5-4} \\ &= \frac{\sqrt{5}-2}{1} \\ &= \sqrt{5}-2 \\ &= -2+\sqrt{5} \end{aligned}$$

$$\begin{aligned} (b) \quad \int_3^4 f(x) dx &\approx \frac{0.25}{3} [1 \cdot 0 + 4(0.8 + 0.55) + 2(0.65) + 0.5] \\ &= 0.6833 \dots \\ &= 0.683 \quad (3 \text{ sig fig}) \end{aligned}$$

$$\begin{aligned} (c) \quad \int_{\pi/2}^{\pi} \sec^2\left(\frac{x}{3}\right) dx &= \frac{1}{1/3} \left[\tan \frac{x}{3} \right]_{\pi/2}^{\pi} \\ &= 3 \left[\tan \frac{\pi}{3} - \tan \frac{\pi}{6} \right] \\ &= 3(\sqrt{3} - \frac{1}{\sqrt{3}}) \\ &= 3 \cdot \frac{3-1}{\sqrt{3}} \\ &= \frac{6}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} (d) \quad \frac{d}{dx} \left(\frac{\sin x}{e^x} \right) &= \frac{\cos x \cdot e^x - \sin x \cdot e^x}{(e^x)^2} \\ &= \frac{e^x (\cos x - \sin x)}{(e^x)^2} \\ &= \frac{\cos x - \sin x}{e^x} \end{aligned}$$

$$\begin{aligned}
 (e) \quad y &= x^3 - 4x \\
 &= x(x^2 - 4) \\
 &= x(x+2)(x-2)
 \end{aligned}$$



$$\begin{aligned}
 \text{Area} &= 2 \left| \int_0^2 x^3 - 4x \, dx \right| \\
 &= 2 \left| \left[\frac{x^4}{4} - \frac{4x^2}{2} \right]_0^2 \right| \\
 &= 2 \left| \left(\frac{2^4}{4} - 2(2)^2 - 0 \right) \right| \\
 &= 2|-4| \\
 &= 8 \text{ units}^2
 \end{aligned}$$

$$(f) \quad \log_2 x + \log_2 (x+7) = 3 \quad x > 0.$$

$$\log_2 (x(x+7)) = 3$$

$$x(x+7) = 2^3$$

$$x^2 + 7x - 8 = 0$$

$$(x-1)(x+8) = 0$$

$$x = 1, -8$$

But $x > 0$, $\therefore x = 1$ only.

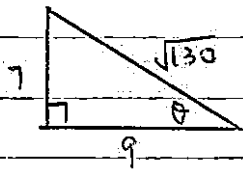
Question 12

(a) $|2x-5| = 3$

$$\begin{array}{cc} \swarrow & \searrow \\ 2x-5=3 & 2x-5=-3 \\ 2x=8 & 2x=2 \\ x=4 & x=1 \end{array}$$

$$\therefore x=1, 4.$$

(b) $\tan \theta = 7/9 > 0$, $\cos \theta < 0$. $\checkmark$



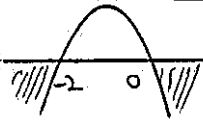
$$\sin \theta = -\frac{7}{\sqrt{130}}$$

(c) $f(x) = -x^2(x+3)$

$$= -x^3 - 3x^2$$

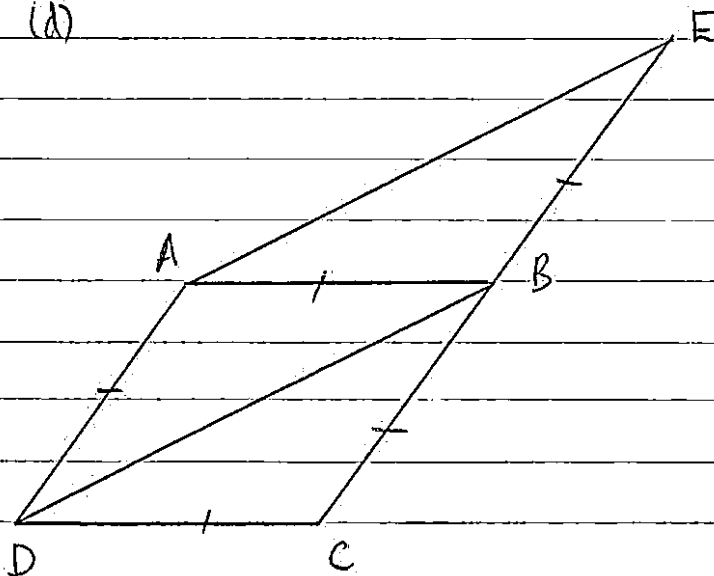
$$f'(x) = -3x^2 - 6x$$

$$= -3x(x+2) < 0 \text{ for decreasing.}$$



$$\therefore x < -2, x > 0.$$

(d)



(i) In $\triangle ABE$ & $\triangle DCB$,

$AB = DC$ (opp. sides of rhombus are equal)

$AB \parallel DC$ (opp. sides of rhombus are parallel)

$\angle ABE = \angle DCB$ (corres. $\angle$ s equal, $AB \parallel DC$)

$CB = BE$ (given)

$\therefore \triangle ABE \equiv \triangle DCB$ (SAS)

(ii) $\angle AEB = \angle DBC$ (corres. $\angle$ s in cong. $\triangle$ s are equal)

$\therefore AE \parallel DB$ (corres. $\angle$ s equal on parallel lines)

(iii) $AE = DC$ (corres. sides in cong. $\triangle$ s are equal).

$\therefore AEBD$ is a parallelogram (opp. sides are equal & parallel)

(e) $\log_a m = 1.75$, $\log_a n = 2.25$.

$$\begin{aligned} \text{(i)} \quad \log_a(mn) &= \log_a m + \log_a n \\ &= 1.75 + 2.25 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \log_a\left(\frac{n}{m}\right) &= \log_a n - \log_a m \\ &= 2.25 - 1.75 \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \sqrt[5]{mn^2} &= [a^{1.75} \times (a^{2.25})^2]^{1/5} \\ &= [a^{1.75} \times a^{4.5}]^{1/5} \\ &= [a^{6.25}]^{1/5} \\ &= a^{1.25} \\ &= a^{5/4} \end{aligned}$$

$$\begin{aligned} \log_a m &= 1.75 \\ m &= a^{1.75} \end{aligned}$$

$$\begin{aligned} \log_a n &= 2.25 \\ n &= a^{2.25} \end{aligned}$$

Question 13.

(a) $y = x^2 + ax - 3$

$$y' = 2x + a$$

$$\text{At } x=0, m_1 = 2(0) + a = a$$

$$\text{At } x=1, m_2 = 2(1) + a = 2 + a$$

$$m_1 \times m_2 = -1$$

$$a \times (2+a) = -1$$

$$a^2 + 2a + 1 = 0$$

$$(a+1)^2 = 0$$

$$\therefore a = -1.$$

(b) $\int \tan(2x) dx = \int \frac{\sin 2x}{\cos 2x} dx$

$$= -\frac{1}{2} \int \frac{-2 \sin 2x}{\cos 2x} dx$$

$$= -\frac{1}{2} \int \frac{(\cos 2x)'}{\cos 2x} dx$$

$$= -\frac{1}{2} \ln |\cos 2x| + C.$$

(c) $y = x^3 + 3x^2 - 9x$

(i) $\frac{dy}{dx} = 3x^2 + 6x - 9$

$$\frac{d^2y}{dx^2} = 6x + 6.$$

(ii) SP when $\frac{dy}{dx} = 0$

$$3(x^2 + 2x - 3) = 0$$

$$3(x-1)(x+3) = 0$$

$$x = 1, -3$$

At $x=1$, $y=1^3+3(1)^2-9(1)=-5$

$\frac{d^2y}{dx^2} = 6(1)+6 = 12 > 0 \quad \curvearrowright$

$\therefore$ min SP at $(1, -5)$

At $x=-3$, $y=(-3)^3-3(-3)^2-9(-3)=27$

$\frac{d^2y}{dx^2} = 6(-3)+6 = -12 < 0 \quad \curvearrowleft$

$\therefore$ max SP at $(-3, 27)$.

(iii) IP when $\frac{dy}{dx} = 0$

$6x+6=0$

$6x=-6$

$x=-1$

Check for concavity change

x	-1.1	-1	-0.9
$\frac{d^2y}{dx^2}$	-0.6	0	0.6

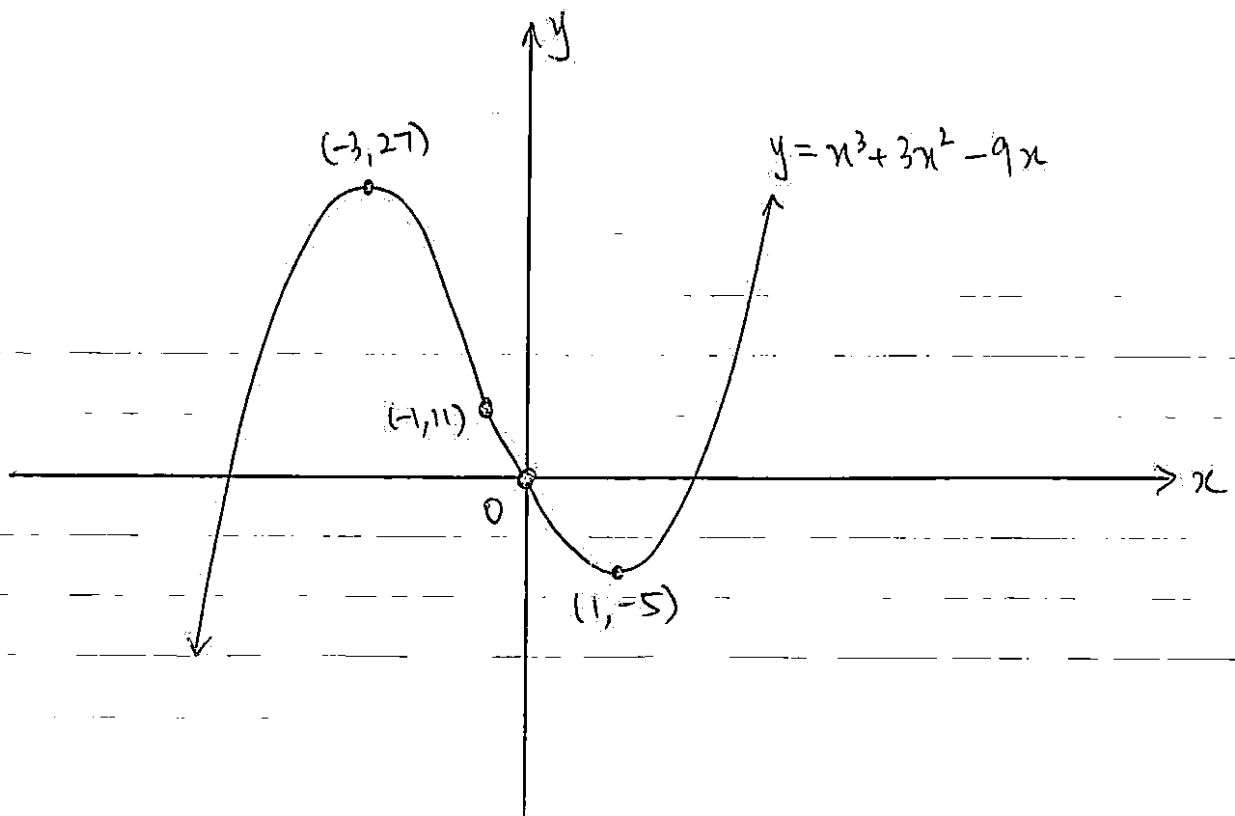
$\curvearrowleft \quad \quad \quad \curvearrowright$



At $x=-1$, $y=(-1)^3+3(-1)^2-9(-1)=11$

$\therefore$ IP at $(-1, 11)$.

(iv)



$$(A) \frac{d^2 y}{dx^2} = \frac{2}{x^2} + 2e^{2x}$$

$$= 2x^{-2} + 2e^{2x}$$

$$\frac{dy}{dx} = \int 2x^{-2} + 2e^{2x} dx$$

$$= \frac{2x^{-1}}{-1} + \frac{2e^{2x}}{2} + C_1$$

$$= -2x^{-1} + e^{2x} + C_1$$

$$\text{Sub } x=1, \frac{dy}{dx} = e^2$$

$$e^2 = -2(1)^{-1} + e^{2(1)} + C_1$$

$$= -2 + e^2 + C_1$$

$$C_1 = 2$$

$$\frac{dy}{dx} = -2x^{-1} + e^{2x} + 2$$

$$y = \int -2x^{-1} + e^{2x} + 2 dx$$

$$= -2\ln x + \frac{e^{2x}}{2} + 2x + C_2$$

$$\text{Sub } x=1, y = \frac{e^2}{2}$$

$$\frac{e^2}{2} = -2\ln(1) + \frac{e^{2(1)}}{2} + 2(1) + C_2$$

$$0 = 0 + 2 + C_2$$

$$C_2 = -2$$

$$\therefore y = -2\ln x + \frac{e^{2x}}{2} + 2x - 2.$$

Question 14.

$$\begin{aligned} \text{(a) (i)} \quad BA &= \sqrt{(-3-1)^2 + (1-1)^2} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \\ CD &= \sqrt{(3--3)^2 + (1-4)^2} \\ &= \sqrt{45} \\ &= 3\sqrt{5} \end{aligned}$$

$$\begin{aligned} A(1, -1) \\ B(-3, 1) \\ C(-3, 4) \\ D(3, 1) \end{aligned}$$

$$\text{(ii)} \quad m_{CD} = \frac{1-4}{3--3} = \frac{-3}{6} = -\frac{1}{2}$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{1}{2}(x - 3)$$

$$2y - 2 = -x + 3$$

$$CD: x + 2y - 5 = 0.$$

$$\begin{aligned} \text{(iii)} \quad d_{\perp} &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ &= \frac{|1(1) + 2(-1) - 5|}{\sqrt{1^2 + 2^2}} \\ &= \frac{|-6|}{\sqrt{5}} \\ &= \frac{6}{\sqrt{5}} \end{aligned}$$

(iv) ABCD is a trapezium ($AB \parallel CD$)

$$\text{Area} = \frac{1}{2}h(a+b)$$

$$= \frac{1}{2} \times \frac{6}{\sqrt{5}} \times (2\sqrt{5} + 3\sqrt{5})$$

$$= \frac{3}{\sqrt{5}} \times 5\sqrt{5}$$

$$= 15 \text{ units}^2$$

(b) $\text{Area} = \frac{1}{2} r^2 \theta$

$$3\pi = \frac{1}{2} r^2 \theta$$

$$6\pi = r^2 \theta \quad \text{--- (1)}$$

$$l = r\theta$$

$$\frac{3\pi}{2} = r\theta \quad \text{--- (2)}$$

Take (1) $\div$ (2)

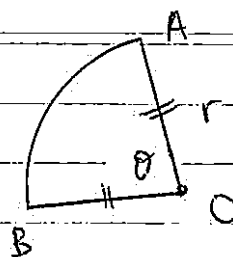
$$\frac{6\pi}{\frac{3\pi}{2}} = \frac{r^2 \theta}{r\theta}$$

$$\therefore r = 4$$

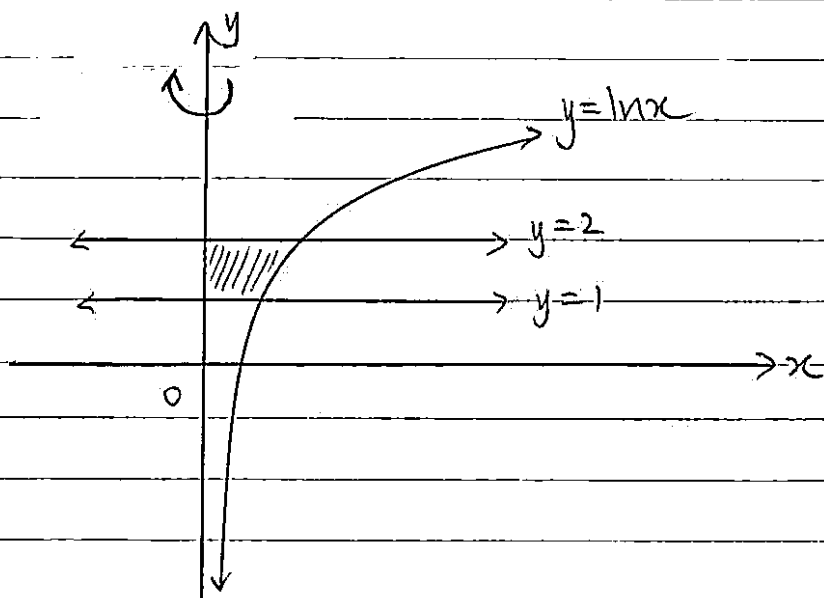
Sub $r = 4$ in (1)

$$6\pi = 16\theta$$

$$\therefore \theta = \frac{3\pi}{8}$$



(c) (i)



$$\begin{aligned} y &= \ln x \\ x &= e^y \\ x^2 &= e^{2y} \end{aligned}$$

(ii) $V = \pi \int_a^b x^2 dy$

$$= \pi \int_1^2 e^{2y} dy$$

$$= \pi \left[\frac{e^{2y}}{2} \right]_1^2$$

$$= \pi \left(\frac{e^4}{2} - \frac{e^2}{2} \right)$$

$$= \frac{\pi (e^4 - e^2)}{2}$$

Question 15.

(a) (i) $y = e^x$ — ①

$y = e^{1-x}$ — ②

POI: sub ① in ②
 $e^x = e^{1-x}$

$$x = 1-x$$

$$2x = 1$$

$$\therefore x = \frac{1}{2}$$

(ii) Area = $\int_0^{1/2} e^x dx + \int_{1/2}^1 e^{1-x} dx$

$$= 2 \int_0^{1/2} e^x dx$$

$$= 2 [e^x]_0^{1/2}$$

$$= 2(e^{1/2} - e^0)$$

$$= 2(\sqrt{e} - 1)$$

(b) (i) $\frac{d^2x}{dt^2} = 8-6t$

$$\frac{dx}{dt} = \int 8-6t dt$$

$$= 8t - \frac{6t^2}{2} + C_1$$

Sub $t=0$, $x=5$

$$5 = 8(0) - 3(0)^2 + C_1$$

$$C_1 = 5$$

$$\frac{dx}{dt} = 8t - 3t^2 + 5$$

$$x = \int 8t - 3t^2 + 5 \, dt$$

$$= \frac{8t^2}{2} - \frac{3t^3}{3} + 5t + C_2$$

Sub $t=0$, $x=0$.

$$0 = 4(0)^2 - 0^3 + 5(0) + C_2$$

$$C_2 = 0$$

$$\therefore x = 4t^2 - t^3 + 5t$$

(ii) $x = 4t^2 - t^3 + 5t = 0$

$$-t(t^2 - 4t - 5) = 0$$

$$-t(t-5)(t+1) = 0$$

$$t = 0, 5, -1$$

But $t \geq 0$, $\therefore t = 0, 5$

Returns at $t = 5$ sec.

At $t = 5$,

$$\frac{dx}{dt} = 8(5) - 3(5)^2 + 5$$

$$= -30 \text{ m/s.}$$

(c) (i) $A_1 = 20000 - M$

$$A_2 = A_1 - M$$

$$= 20000 - 2M$$

$$A_3 = 20000 - 3M$$

$$\therefore A_4 = 20000 - 4M$$

(ii) $A_6 = 20000 - 6M$

$$A_7 = A_6 \times 1.01 - M$$

$$= (20000 - 6M) \times 1.01 - M$$

$$A_8 = A_7 \times 1.01 - M$$

$$= [(20000 - 6M) \times 1.01 - M] \times 1.01 - M$$

$$= (20000 - 6M) \times 1.01^2 - M \times 1.01 - M$$

$$= (20000 - 6M) \times 1.01^2 - M(1 + 1.01),$$

$$(iii) A_9 = A_8 \times 1.01 - M$$

$$= [(20000 - 6M) \times 1.01^2 - M(1 + 1.01)] \times 1.01 - M$$

$$= (20000 - 6M) \times 1.01^3 - M(1 + 1.01) \times 1.01 - M$$

$$= (20000 - 6M) \times 1.01^3 - M(1 + 1.01 + 1.01^2)$$

$$\vdots$$

$$A_{60} = (20000 - 6M) \times 1.01^{54} - M(1 + 1.01 + 1.01^2 + \dots + 1.01^{53})$$

$$GP: a=1, r=1.01, n=54$$

$$= (20000 - 6M) \times 1.01^{54} - M \times \frac{1(1.01^{54} - 1)}{1.01 - 1}$$

$$= (20000 - 6M) \times 1.01^{54} - 100M(1.01^{54} - 1)$$

enough for
1 mark

$$(iv) A_{60} = 0$$

$$0 = (20000 - 6M) \times 1.01^{54} - 100M(1.01^{54} - 1)$$

$$= 20000 \times 1.01^{54} - 6M \times 1.01^{54} - 100M \times 1.01^{54} + 100M$$

$$= 20000 \times 1.01^{54} - M(106 \times 1.01^{54} - 100)$$

$$M(106 \times 1.01^{54} - 100) = 20000 \times 1.01^{54}$$

$$M = \frac{20000 \times 1.01^{54}}{106 \times 1.01^{54} - 100}$$

$$= 420.4448 \dots$$

$$\therefore M = \$420.44 \text{ (nrst 2)}$$

Question 16.

$$(a) (i) \frac{d}{dx} (x^2 \ln x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x}$$

$$= 2x \ln x + x$$

$$(ii) \int x \ln x \, dx$$

$$\text{From (i), } \frac{d}{dx} (x^2 \ln x) = 2x \ln x + x$$

$$x^2 \ln x = \int 2x \ln x + x \, dx$$

$$= \int 2x \ln x \, dx + \int x \, dx$$

$$= 2 \int x \ln x \, dx + \frac{x^2}{2}$$

$$2 \int x \ln x \, dx = x^2 \ln x - \frac{x^2}{2}$$

$$\therefore \int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

$$(b) (i) \frac{dN}{dt} = kN \quad \text{sub } N = Ae^{kt}$$

$$\text{LHS} = \frac{dN}{dt}$$

$$= \frac{d}{dt} (Ae^{kt})$$

$$= k \cdot Ae^{kt}$$

$$= kN$$

$$= \text{RHS}$$

$$\therefore N = Ae^{kt} \text{ is a solution.}$$

(ii) At $t=0$, $N=100$.

$$100 = Ae^{k(0)}$$

$$= A$$

$$N = 100e^{kt}$$

At $t=2$, $N=400$.

$$400 = 100e^{2k}$$

$$4 = e^{2k}$$

$$2k = \ln 4$$

$$k = \frac{1}{2} \ln 4$$

$$= \ln 4^{1/2}$$

$$\therefore k = \ln 2$$

(iii) At $t=10$,

$$N = 100e^{10 \ln 2}$$

$$= 102400.$$

(iv) $N=1000$, $t=?$

$$1000 = 100e^{t \ln 2}$$

$$10 = e^{t \ln 2}$$

$$t \ln 2 = \ln 10$$

$$t = \frac{\ln 10}{\ln 2}$$

$$= 3.3219 \dots$$

$$= 3 \text{ mins } 19 \text{ sec (nrst sec).}$$

(c)(i) Similar triangles (equiangular)

$$\frac{r}{6} = \frac{20-h}{20} \quad (\text{corres. sides in sim. } \Delta \text{ s in same ratio})$$

$$20r = 120 - 6h$$

$$6h = 120 - 20r$$

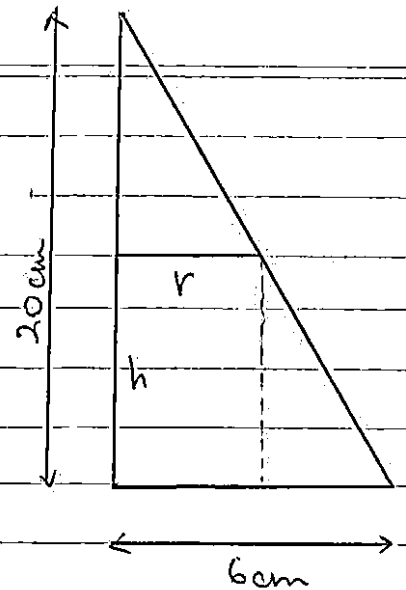
$$\therefore h = 20 - \frac{10}{3}r$$

$$\text{Then, } V = \pi r^2 h$$

$$= \pi r^2 \left(20 - \frac{10}{3}r \right)$$

$$= \pi r^2 \left(\frac{60 - 10r}{3} \right)$$

$$= \frac{10\pi r^2 (6-r)}{3}$$



$$(ii) \frac{dV}{dr} = \frac{10\pi}{3} [2r \cdot (6-r) + r^2 \cdot (-1)]$$

$$= \frac{10\pi}{3} [12r - 2r^2 - r^2]$$

$$= \frac{10\pi (12r - 3r^2)}{3}$$

$$= \frac{10\pi r (4-r)}{3}$$

$$= 0 \text{ for max/min } V.$$

$$r = 0, 4.$$

But $r > 0$, $\therefore r = 4$ only.

TEST for max V .

r	3	4	5
$\frac{dV}{dr}$	94	0	-157
	$\nearrow$	$\rightarrow$	$\searrow$

$\therefore \text{max } V$
at $r = 4$.

$$\text{When } r = 4, \quad h = 20 - \frac{10}{3}(4)$$

$$= \frac{20}{3}$$