Student Number:

Teacher: Mandile / Blair / Karagiannis (Circle)

Danebank
An Anglican School for Girls

Year 12 Advanced Mathematics
Task 4 – Trial HSC Examination
Wed 5th August 2020

General Instructions	
Write using black or blue pen Graphs can be drawn in pencil No liquid paper or correction tape to be used.	
Reading Time	10 minutes
Working time	3 hours

Structure of the Paper		
Section I	/10 Marks	Allow about 15 minutes for this section
Section II	/ 90 Marks	Allow about 2 hours 45 minutes for this section
Total	**/100 Marks**	**Weighting: 40%**

Special Instructions
NESA approved Calculator may be used. A separate reference sheet and Multiple Choice answer sheet is provided. Attempt all questions This examination must not be removed from the Examination Room. In Question 11-39 show relevant mathematical reasoning and/or calculations. Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response. Extra writing space is provided at the back of the examination paper on pages 26 - 28. If you use this space, clearly indicate which question you are answering.

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SECTION I	Multiple Choice	10 marks

Attempt Questions 1-10.

Allow about 15 minutes for this section.

Use the multiple choice answer sheet for questions 1 - 10.

1. What is the gradient of the line $\frac{x}{3}+\frac{y}{2}=1$?

   (A) $-\frac{3}{2}$

   (B) $-\frac{2}{3}$

   (C) $\frac{2}{3}$

   (D) $\frac{3}{2}$

2. What is the expression for $\frac{dy}{dx}$ if $y=\frac{1}{x^2}$?

   (A) $-\frac{2}{x^3}$

   (B) $-\frac{1}{x}$

   (C) $\frac{1}{x}$

   (D) $\frac{2}{x^3}$

3. What is the value of c for which the circle $(x-3)^2+(y-2)^2=c$ touches the x axis ?

   (A) 2

   (B) 3

   (C) 4

   (D) 9

4. What is the domain of the function $f(x)=\sqrt{x}+\dfrac{1}{\sqrt{2-x}}$?

(A) $(0,2)$

(B) $[0,2)$

(C) $(0,2]$

(D) $[0,2]$

5. The 7^{th} term of an arithmetic sequence is 45 and the 11^{th} term is 77.
Find the first term (a) and the common difference (d).

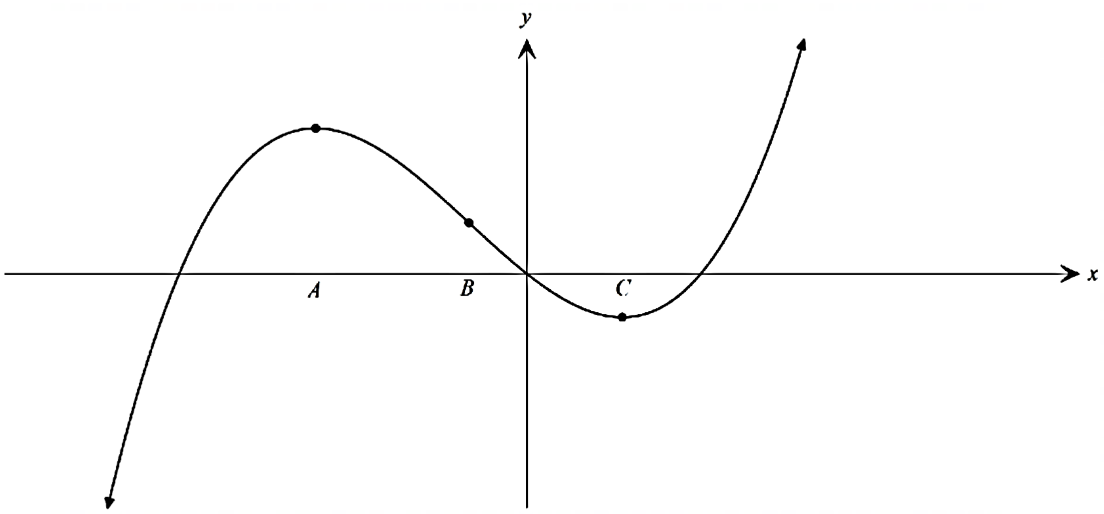
(A) $a=-3$ and $d=8$

(B) $a=3$ and $d=8$

(C) $a=8$ and $d=-3$

(D) $a=8$ and $d=3$

6. The graph of $y=f(x)$ is shown below.
$x=A$ and $x=C$ are stationary points, and $x=B$ is a point of inflection.



For $A<x<B$, which statement is true ?

(A) $f'(x)<0$ and $f''(x)>0$

(B) $f'(x)<0$ and $f''(x)<0$

(C) $f'(x)>0$ and $f''(x)>0$

(D) $f'(x)>0$ and $f''(x)<0$

7. Which one of the following is the set of all solutions to $2x^2 - 5x + 2 \geq 0$?

(A) $\left[\frac{1}{2}, 2\right]$

(B) $\left(\frac{1}{2}, 2\right)$

(C) $\left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$

(D) $\left(-\infty, \frac{1}{2}\right] \cup [2, \infty)$

8. What is the change in amplitude and period when the function $f(x) = \frac{1}{2}\sin 4x$ is transformed into $g(x) = \sin 2x$?

(A) The amplitude is halved and the period is halved.

(B) The amplitude is halved and the period is doubled.

(C) The amplitude is doubled and the period is halved.

(D) The amplitude is doubled and the period is doubled.

9. Which statement is true for an ungrouped data set with no outliers ?

(A) The largest possible range is 2 times the interquartile range.

(B) The largest possible range is 3 times the interquartile range.

(C) The largest possible range is 4 times the interquartile range.

(D) The largest possible range is 5 times the interquartile range.

10. What is the value of $\int_1^2 2^{-x}\,dx$?

(A) $\dfrac{1}{4\ln 2}$

(B) $\dfrac{1}{2\ln 2}$

(C) $\dfrac{\ln 2}{4}$

(D) $\dfrac{\ln 2}{2}$

END OF SECTION 1

SECTION II **90 marks**

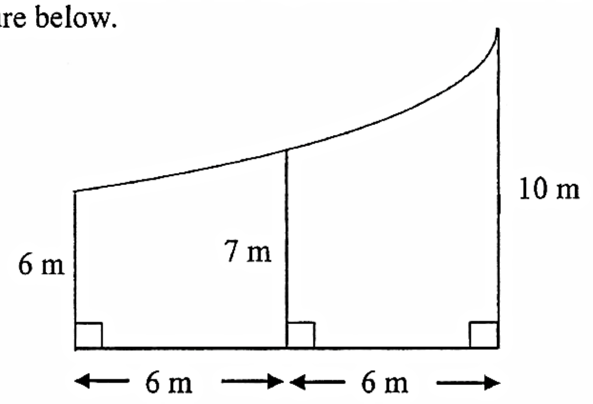
Attempt Questions 11-39

Allow about 2 hours and 45 minutes for this section

Show all necessary working in the space provided. If extra space is required, use the lined pages at the end of this section, carefully labelling the additional work with the appropriate question number and part.

Question 11 (2 marks)

Use two applications of the trapezoidal rule to find an approximation to the area of the figure below. **2**



NOT TO SCALE

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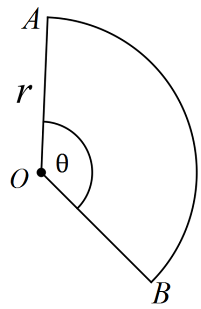
Question 12 (2 marks)

Tim was told that sector OAB has an area of $\frac{25\pi}{6}$ square units. The arc AB is $\frac{5\pi}{3}$ units long. He was asked to find the exact values of r and θ.



His working out is shown on the next page.

$l = r\theta$, $A = \frac{1}{2}r^2\theta$

$\therefore \quad \frac{1}{2}r^2\theta = \frac{25\pi}{6} \qquad (1)$

$r\theta = \frac{5\pi}{3} \qquad (2)$

$\frac{1}{2}r = \frac{5}{2} \qquad (3)$

$\therefore \quad r = 5$ units

(a) What operation did Tim perform on equations (1) and (2) to get to equation (3) **1**

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(b) What is the value of θ **1**

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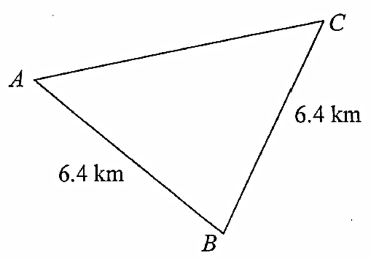
Question 13 (2 marks)



NOT TO SCALE

In the diagram, ABC is a triangular airfield with $AB = BC = 6 \cdot 4$ km . The bearing of B from A is 140° and the bearing of C from B is 032°.

(a) Show that $\angle ABC = 72°$. **1**

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(b) Find the area of the airfield correct to the nearest square kilometre. **1**

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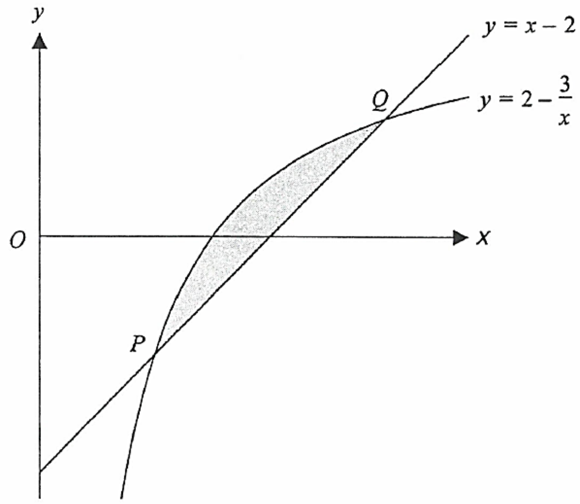
Question 14 (4 marks)



The diagram shows the curves $y = 2 - \frac{3}{x}$ and $y = x - 2$ for $x \geq 0$.

(a) Find the x coordinates of the two points P and Q where the two curves intersect. **2**

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(b) Hence find in simplest exact form the area of the shaded region contained between the two curves. **2**

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Question 15 (2 marks)

Solve the equation $|2\cos x - 1| = 1$ for $0 \le x \le \pi$. **2**

Question 16 (3 marks)

$A(-1, 16)$, $B(-3, 2)$ and $C(5, -2)$ are three points. D is the midpoint of AB and E is the midpoint of AC. Show that DE is parallel to BC. **3**

Question 17 (3 marks)

200 people took part in a new lie detector test with the following results.

	Test Positive	Test Negative
Person lying	96	24
Person not lying	8	72

(a) Find the probability that a person who was lying tested positive. **1**

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(b) Find the probability that the test was inaccurate. **2**

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Question 18 (3 marks)

Find in general form the equation of the normal to the curve $y = \sqrt{x}$ at the point where $x = 4$. **3**

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Question 19 (3 marks)

(a) Show that $\log_x 2 = \dfrac{1}{\log_2 x}$. **1**

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(b) Solve the equation $\log_2 x = 4\log_x 2$. **2**

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Question 20 (2 marks)

If $f'(x) = 6x^2 + 5x - 1$ and $f(-1) = 5$, find an expression for $f(x)$. **2**

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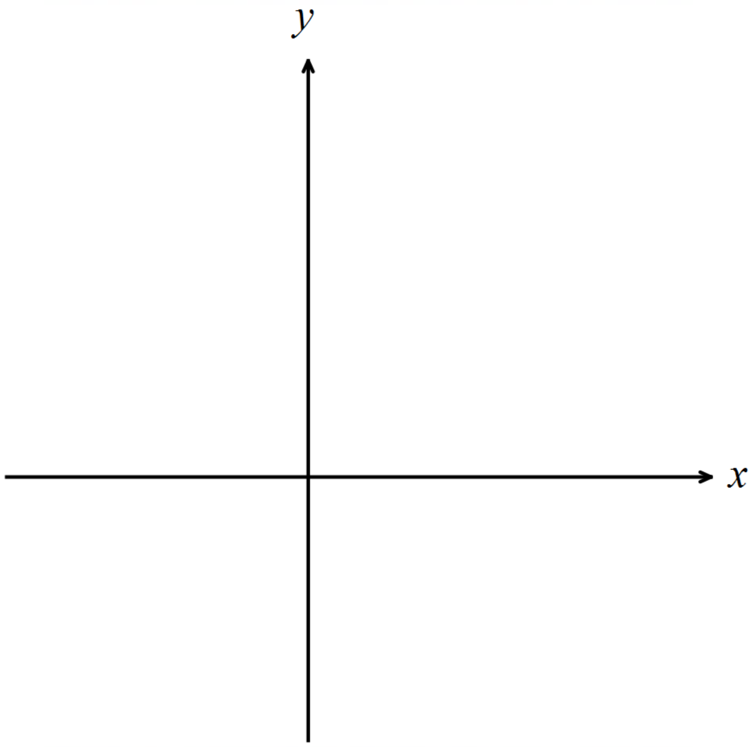
Question 21 (5 marks)

Consider the function $f(x) = 4x^3 - x^4$.

(a) Find the stationary points on the curve $y = f(x)$ and determine their nature. **3**

(b) Sketch the graph of the function showing the stationary points and the intercepts on the axes. **2**



Question 22 (4 marks)

The discrete random variable X has a probability distribution shown below and a mean of 2.

x	1	2	3	4
$p(x)$	0.3	0.45	a	b

(a) Find two equations in a and b. **2**

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(b) Hence find the values of a and b. **2**

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Question 23 (2 marks)

If $y = x\sin 2x$ find $\frac{dy}{dx}$. **2**

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Question 24 (2 marks)

Consider the functions $f(x) = e^x$ and $g(x) = \ln(x-2)$.

(a) Find the composite function $f(g(x))$. **1**

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(b) Find in interval notation the range of the composite function. **1**

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Question 25 (4 marks)

The table below shows the English marks (x) and the Mathematics marks (y) for a class of 12 students (A-L). Only the English mark is available for student L.

	A	B	C	D	E	F	G	H	I	J	K	L
x	67	61	65	67	75	75	69	85	85	89	87	80
y	58	64	66	68	70	72	72	76	80	82	84	

(a) Calculate the correlation coefficient between x and y for the students A to K. Describe the nature of the correlation between x and y. **2**

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(b) Find the equation of the least squares regression line of y on x for the students A to K. Estimate the Mathematics mark of student L. **2**

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Question 26 (2 marks)

If $y=\dfrac{e^x}{x+1}$ find $\dfrac{dy}{dx}$. **2**

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Question 27 (2 marks)

Find

$\int \cos 4x - \sin x \quad dx$ **2**

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Question 28 (2 marks)

Find in simplest form the limiting sum of the series $\sin^2 x+\sin^4 x+\sin^6 x+...$ for $0<x<\frac{\pi}{2}$. **2**

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Question 29 (5 marks)

A metal crate of fixed volume 9 m^3 is to be made in the shape of a rectangular prism with length $2x$ metres, width x metres and height h metres. **2**

(a) Show that the area A m^2 of metal required is given by $A = 4x^2 + \dfrac{27}{x}$.

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(b) Hence find the minimum area of metal required. **3**

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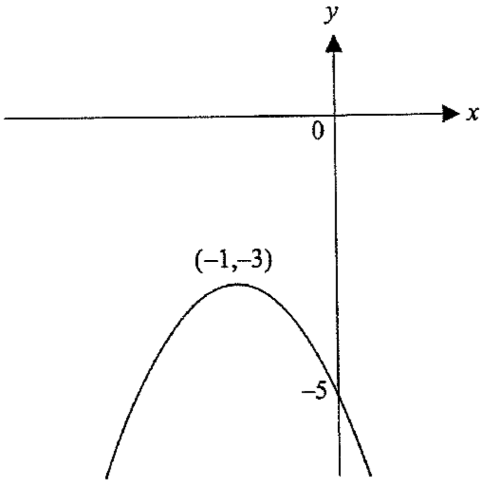
Question 30 (3 marks)

The function $f\left(x\right)=x^2$ is transformed into a new function whose graph is shown in the diagram below. **3**



Find the equation of the new function in the form $g\left(x\right)=k\ f\left(x+b\right)+c$ for some constants k, b and c.

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Question 31 (3 marks)

Solve $2\sin(3\theta)=1$ for $0\leq\theta\leq\pi$ **3**

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Question 32 (3 marks)

(a) On the same diagram draw the graphs of $y = \cos \pi x$ and $y = 1 - |x|$ for $-3 \leq x \leq 3$. **2**

(b) Hence find the number of solutions of the equation $\cos \pi x = 1 - |x|$ in the domain $(-\infty, \infty)$ **1**

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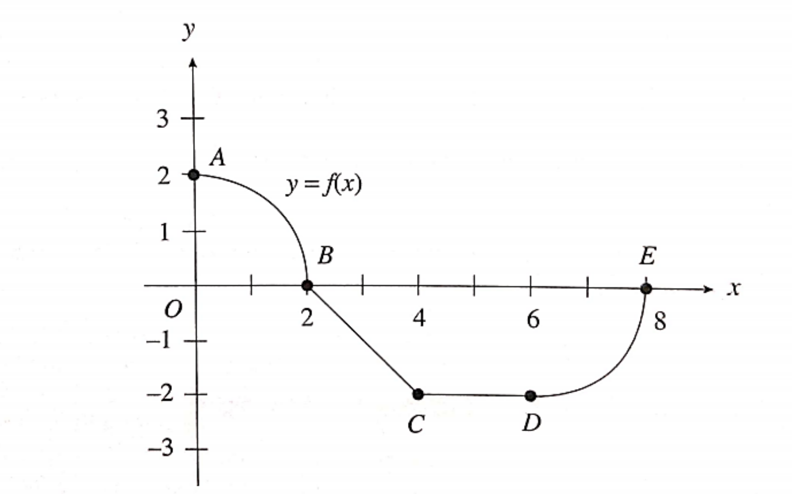
Question 33 (3 marks)



The graph of the function $y = f(x)$ drawn above, consists of a quarter circle AB, a straight line segment BC, a horizontal line segment CD and a quarter circle DE.

(a) Evaluate **2**

$$\int_0^8 f(x)\,dx$$

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(b) For what values of x in the domain $0 < x < 8$ is the function NOT differentiable? **1**

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Question 34 (4 marks)

The table below shows the future values of an annuity, for different rates of interest and for different numbers of compounding periods, where contributions of $1 are made at the end of each compounding period.

Table of future value interest factors

n	1%	2%	3%	4%	5%	6%
1	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
2	2·0100	2·0200	2·0300	2·0400	2·0500	2·0600
3	3·0301	3·0604	3·0909	3·1216	3·1525	3·1836
4	4·0604	4·1216	4·1836	4·2465	4·3101	4·3746
5	5·1010	5·2040	5·3091	5·4163	5·5256	5·6371
6	6·1520	6·3081	6·4684	6·6330	6·8019	6·9753

(a) An annuity account is opened and contributions of $1200 are made at the end of each half year for 3 years at an interest rate of 4% p.a. compounding half yearly. Calculate the final amount in the account immediately after the last contribution is made. **2**

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(b) Calculate the single lump sum amount that would need to be invested at the start to reach the same final amount at the end of the 3 years with the interest rate of 4% p.a. compounding half yearly. **2**

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Question 35 (3 marks)

If $y = \tan^2 x$ find the values of the constants a and b such that $\dfrac{d^2y}{dx^2} = ay^2 + by + 2$. **3**

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Question 36 (4 marks)

At time t years after it was purchased the value $\$V$ of a car is given by $V = 25\,000\, e^{-0.5t}$.

(a) Find the loss in value of the car during the third year. **2**

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(b) Find the year in which the car is losing value at a rate of \$100 per year. **2**

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Question 37 (3 marks)

Ozzie retires with a savings fund of $600 000. The fund earns interest at a rate of 3% p.a. compounded monthly. At the end of each month Ozzie withdraws $4000 from the fund. The amount A_n left in the fund after the n^{th} withdrawal is given by 3

$$A_n = 600\,000\left(1\cdot 0025\right)^n - 4000\left\{1 + 1\cdot 0025 + \left(1\cdot 0025\right)^2 + \ldots + \left(1\cdot 0025\right)^{n-1}\right\}.$$

(DO NOT PROVE THIS RESULT)

Find correct to the nearest month the time taken for the fund to reduce to half its initial value.

Question 38 (5 marks)

A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1}, and acceleration a ms^{-2} given by $a = 6t - 12$. Initially the particle is at rest at O.

(a) Find expressions for v and x in terms of t. **3**

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(b) Find when and where the particle is next at rest. **2**

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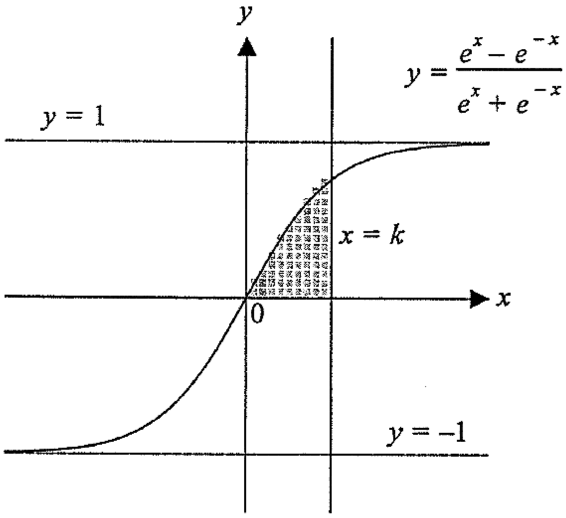
Question 39 (5 marks)



The diagram shows the graph of the curve $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

(a) Show that the shaded region bounded by the curve, the x axis and the line $x = k$, where $k > 0$, has area $\ln\left(\dfrac{e^k + e^{-k}}{2}\right)$. **2**

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(b) Find in simplest exact form the value of k such that the shaded region has area 1 unit squared **3**

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END OF EXAMINATION ☺

Section II Extra Writing Space

If you use this space, clearly indicate which questions you are answering.

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Student Number:

Student Name:______________________________

SECTION 1 — Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample

$2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ✗ C ○ D ○

correct

A ✗ B ✗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word correct and drawing an arrow as follows: correct

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1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

Teacher: Mandile / Blair / Karagiannis (Circle)

Student Number:

SOLUTIONS

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(A) $-\frac{3}{2}$

(B) $-\frac{2}{3}$

(C) $\frac{2}{3}$

(D) $\frac{3}{2}$

$\frac{y}{2} = 1 - \frac{x}{3}$

$y = 2 - \frac{2}{3}x$

2. What is the expression for $\frac{dy}{dx}$ if $y=\frac{1}{x^2}$?

(A) $-\frac{2}{x^3}$

(B) $-\frac{1}{x}$

(C) $\frac{1}{x}$

(D) $\frac{2}{x^3}$

$y = x^{-2}$

$\frac{dy}{dx} = -2x^{-3}$

3. What is the value of c for which the circle $(x-3)^2+(y-2)^2=c$ touches the x axis ?

r^2

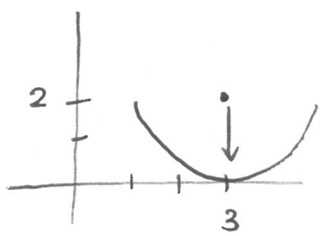
(A) 2

(B) 3

(C) 4

(D) 9

Centre $(3,2)$



$\therefore r = 2$

So

$c = r^2$

$= 2^2$

$= 4$

4. What is the domain of the function $f(x)=\sqrt{x}+\dfrac{1}{\sqrt{2-x}}$?

$x \geq 0$

$2-x>0$

$x<2$

(A) $(0,2)$

(B) $[0,2)$

(C) $(0,2]$

(D) $[0,2]$

5. The 7th term of an arithmetic sequence is 45 and the 11th term is 77.

Find the first term (a) and the common difference (d).

(A) $a=-3$ and $d=8$

(B) $a=3$ and $d=8$

(C) $a=8$ and $d=-3$

(D) $a=8$ and $d=3$

$a+6d=45$

$a+10d=77$

$4d=32$

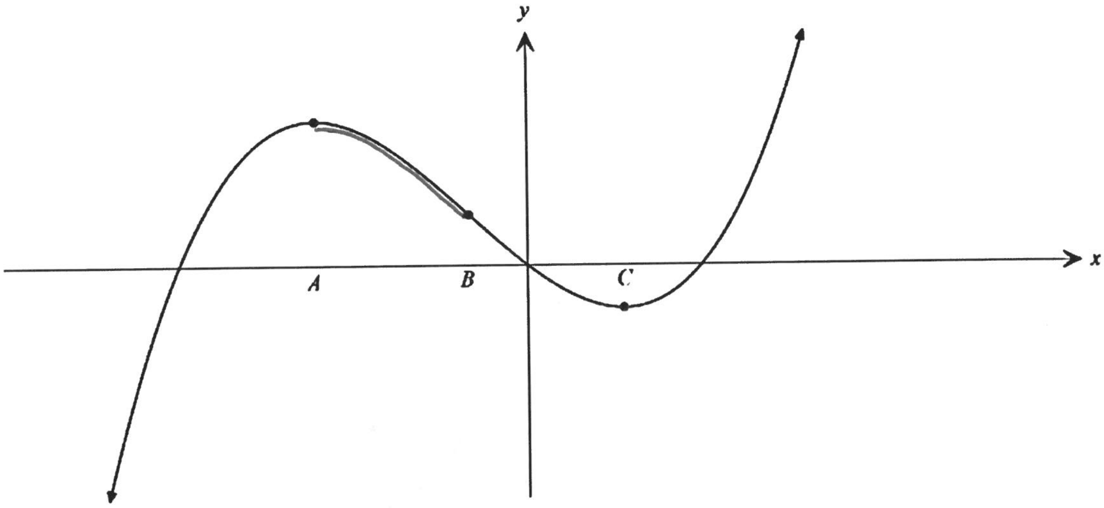
$d=8$

$a+6\times 8=45$

$a=-3$

6. The graph of $y=f(x)$ is shown below.

$x=A$ and $x=C$ are stationary points, and $x=B$ is a point of inflection.



For $A<x<B$, which statement is true ?

(A) $f'(x)<0$ and $f''(x)>0$

(B) $f'(x)<0$ and $f''(x)<0$

(C) $f'(x)>0$ and $f''(x)>0$

(D) $f'(x)>0$ and $f''(x)<0$

decreasing

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concave down

7. Which one of the following is the set of all solutions to $2x^2 - 5x + 2 \geq 0$?

(A) $\left[\frac{1}{2}, 2\right]$

(B) $\left(\frac{1}{2}, 2\right)$

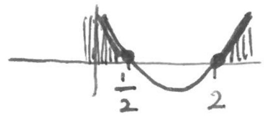
(C) $\left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$

(D) $\left(-\infty, \frac{1}{2}\right] \cup [2, \infty)$

$2x \quad -1$

$x \quad -2$

$(2x-1)(x-2) \geq 0$



$x \leq \frac{1}{2} \quad x \geq 2$

8. What is the change in amplitude and period when the function $f(x) = \frac{1}{2}\sin 4x$ is transformed into $g(x) = \sin 2x$?

(A) The amplitude is halved and the period is halved.

(B) The amplitude is halved and the period is doubled.

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(D) The amplitude is doubled and the period is doubled.

9. Which statement is true for an ungrouped data set with no outliers ?

(A) The largest possible range is 2 times the interquartile range.

(B) The largest possible range is 3 times the interquartile range.

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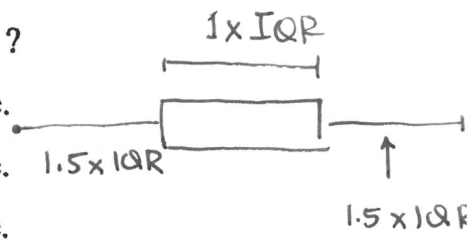
(D) The largest possible range is 5 times the interquartile range.



$(1.5 + 1.5 + 1) \times IQR$

$4 \times IQR$

10. What is the value of $\int_1^2 2^{-x}\, dx$?

(A) $\frac{1}{4\ln 2}$

(B) $\frac{1}{2\ln 2}$

(C) $\frac{\ln 2}{4}$

(D) $\frac{\ln 2}{2}$

$\left[\frac{2^{-x}}{-\ln 2}\right]_1^2$

$-\frac{2^{-2}}{\ln 2} - \left(\frac{2^{-1}}{-\ln 2}\right)$

$-\frac{1}{4\ln 2} + \frac{1}{2\ln 2} = \left(\frac{1}{2} - \frac{1}{4}\right) \cdot \frac{1}{\ln 2}.$

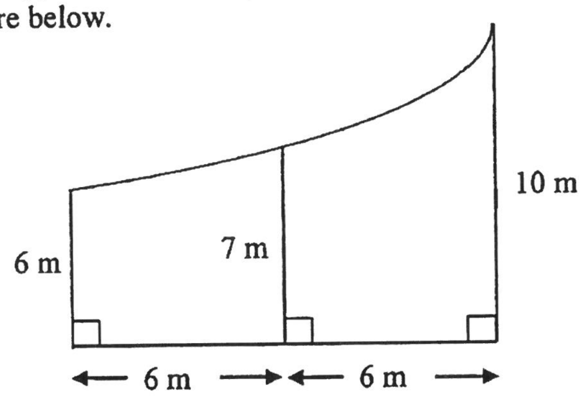
END OF SECTION 1

SECTION II **90 marks**

Attempt Questions 11-39
Allow about 2 hours and 45 minutes for this section
Show all necessary working in the space provided. If extra space is required, use the lined pages at the end of this section, carefully labelling the additional work with the appropriate question number and part.

Question 11 (2 marks)

Use two applications of the trapezoidal rule to find an approximation to the area of the figure below. **2**



NOT TO SCALE

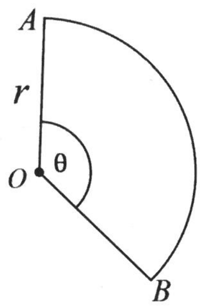
$A = \frac{6}{2}(6+7) + \frac{6}{2}(7+10)$

$= 39 + 51$

$= 90\ m^2$

Question 12 (2 marks)

Tim was told that sector OAB has an area of $\frac{25\pi}{6}$ square units. The arc AB is $\frac{5\pi}{3}$ units long. He was asked to find the exact values of r and θ.



His working out is shown on the next page.

$$l = r\theta, \quad A = \frac{1}{2}r^2\theta$$

$$\therefore \quad \frac{1}{2}r^2\theta = \frac{25\pi}{6} \quad (1)$$

$$r\theta = \frac{5\pi}{3} \quad (2)$$

$$\frac{1}{2}r = \frac{5}{2} \quad (3)$$

$$\therefore \quad r = 5 \text{ units}$$

(a) What operation did Tim perform on equations (1) and (2) to get to equation (3) **1**

(1) ÷ (2)

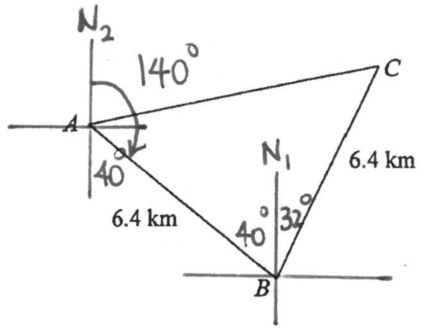
ie Divided equation 1 by equation 2

(b) What is the value of θ **1**

$$5\theta = \frac{5\pi}{3}$$

$$\theta = \frac{\pi}{3}$$

Question 13 (2 marks)



NOT TO SCALE

In the diagram, ABC is a triangular airfield with $AB = BC = 6 \cdot 4$ km. The bearing of B from A is 140° and the bearing of C from B is 032°.

(a) Show that $\angle ABC = 72°$. **1**

$\angle ABN_1 = 40°$ [alternate angles]

$\therefore \angle ABC = 40° + 32° = 72°$

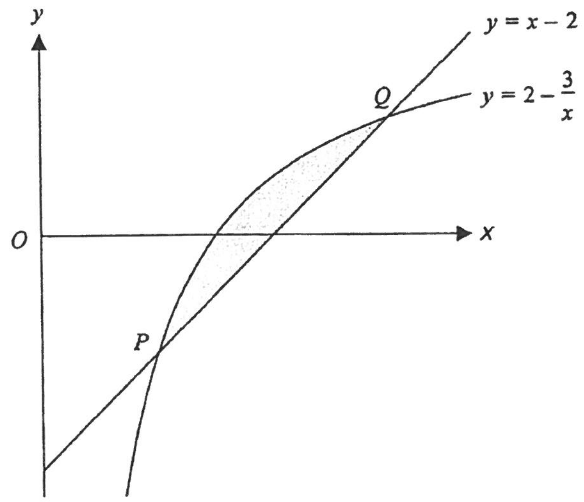
(b) Find the area of the airfield correct to the nearest square kilometre. **1**

$A = \frac{1}{2} ac \sin B$

$= \frac{1}{2} \times 6.4 \times 6.4 \times \sin 72°$

$= 19.4776\ldots \quad \therefore A \doteq 19 \text{ km}^2$

Question 14 (4 marks)



The diagram shows the curves $y = 2 - \frac{3}{x}$ and $y = x - 2$ for $x \geq 0$.

(a) Find the x coordinates of the two points P and Q where the two curves intersect. **2**

$\begin{cases} y = x - 2 & \text{— (1)} \\ y = 2 - \frac{3}{x} & \text{— (2)} \end{cases}$

(1) into (2) $\quad x - 2 = 2 - \frac{3}{x}$ $\qquad$ x coord of P is 1

$(\times x)$ $\quad x^2 - 2x = 2x - 3$ $\qquad$ x coord of Q is 3

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$\therefore x = 1$ or $x = 3$

(b) Hence find in simplest exact form the area of the shaded region contained between the two curves. **2**

$A = \int_1^3 \left(2 - \frac{3}{x}\right) - (x - 2) \, dx$

$= \int_1^3 \left(4 - \frac{3}{x} - x\right) dx$

$= \left[4x - 3\ln x - \frac{x^2}{2}\right]_1^3$

$= \left(4 \times 3 - 3\ln 3 - \frac{9}{2}\right) - \left(4 - 3\cancel{\ln 1} - \frac{1}{2}\right)$

$= 4 - 3\ln 3$

units^2.

Question 15 (2 marks)

Solve the equation $|2\cos x - 1| = 1$ for $0 \leq x \leq \pi$. **2**

$2\cos x - 1 = 1$ $\qquad$ $-(2\cos x - 1) = 1$

$2\cos x = 2$ $\qquad$ $-2\cos x + 1 = 1$

$\cos x = 1$ $\qquad$ $\cos x = 0$

$\therefore x = 0$ $\qquad$ $\therefore x = \frac{\pi}{2}$

Question 16 (3 marks)

$A(-1,16)$, $B(-3,2)$ and $C(5,-2)$ are three points. D is the midpoint of AB and E is the midpoint of AC. Show that DE is parallel to BC. **3**

$D\left(\frac{-1+-3}{2}, \frac{16+2}{2}\right)$ $\qquad$ $E\left(\frac{-1+5}{2}, \frac{16+-2}{2}\right)$

$D(-2, 9)$ $\qquad$ $E(2, 7)$

$m_{DE} = \frac{9-7}{-2-2}$ $\qquad$ $m_{BC} = \frac{2--2}{-3-5}$

$= \frac{2}{-4}$ $\qquad$ $= \frac{4}{-8}$

$= -\frac{1}{2}$ $\qquad$ $= -\frac{1}{2}$

Since $m_{DE} = m_{BC}$, $DE \parallel BC$

Question 17 (3 marks)

200 people took part in a new lie detector test with the following results.

	Test Positive	Test Negative	
Person lying	96	24	120
Person not lying	8	72	80
	104	96	

(a) Find the probability that a person who was lying tested positive. **1**

$P(\text{positive} \mid \text{were lying}) = \frac{96}{120} = \frac{4}{5}$

(b) Find the probability that the test was inaccurate. **2**

ie P (was lying & negative) or (was not lying & positive)

$= \frac{24+8}{200} = \frac{4}{25}.$

Question 18 (3 marks)

Find in general form the equation of the normal to the curve $y=\sqrt{x}$ at the point where $x=4$. **3**

$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ when $x=4$, $y=2$

when $x=4$ $\therefore (4, 2)$

$\frac{dy}{dx} = \frac{1}{4}$

normal $m = -4$

Eqn: $y-2 = -4(x-4)$

$y-2 = -4x+16$

$y = -4x+18$

$4x + y - 18 = 0$

Question 19 (3 marks)

(a) Show that $\log_x 2 = \dfrac{1}{\log_2 x}$. **1**

$\log_x 2 = \dfrac{\log_2 2}{\log_2 x}$ using change of base rule

$= \dfrac{1}{\log_2 x}$

(b) Solve the equation $\log_2 x = 4\log_x 2$. **2**

$\log_2 x = 4 \times \dfrac{1}{\log_2 x}$ using (a)

$(\log_2 x)^2 = 4$

$\log_2 x = 2$ or $\log_2 x = -2$

$\therefore 2^2 = x$ $\quad$ $2^{-2} = x$

$\therefore x = 4$ $\quad$ $x = \dfrac{1}{4}$

Question 20 (2 marks)

If $f'(x) = 6x^2 + 5x - 1$ and $f(-1) = 5$, find an expression for $f(x)$. **2**

$f(x) = \int (6x^2 + 5x - 1)\, dx$

$f(x) = 2x^3 + \dfrac{5x^2}{2} - x + c$

$5 = 2 \times (-1)^3 + \dfrac{5}{2}(-1)^2 - (-1) + c$

$5 = -2 + \dfrac{5}{2} + 1 = c$

$\therefore c = 3\frac{1}{2}$

$\therefore f(x) = 2x^3 + \dfrac{5}{2}x^2 - x + \dfrac{7}{2}$

Question 21 (5 marks)

Consider the function $f(x) = 4x^3 - x^4$.

(a) Find the stationary points on the curve $y = f(x)$ and determine their nature. **3**

$f'(x) = 12x^2 - 4x^3$

$= 4x^2(3 - x)$

$f''(x) = 24x - 12x^2$

$= 12x(2 - x)$

Stat points when $f'(x) = 0$

$\therefore x = 0$ or $x = 3$

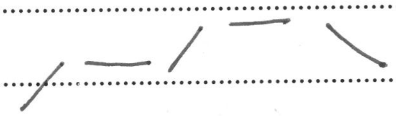
When $x = 0$, $y = 0$

$x = 3$, $y = 27$

$f''(0) = 0$

$f''(3) = -36$

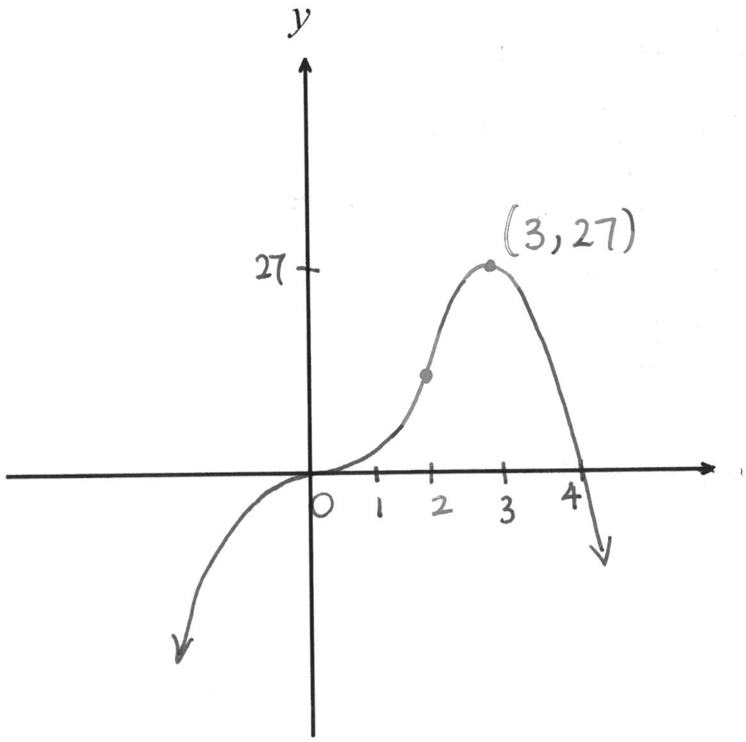
x	-1	0	1	3	4
$f'(x)$	16	0	8	0	-64



$\therefore$ $(0,0)$ is a horizontal point of inflection

$(3,27)$ is a maximum turning point

(b) Sketch the graph of the function showing the stationary points and the intercepts on the axes. **2**



Question 22 (4 marks)

The discrete random variable X has a probability distribution shown below and a mean of 2.

x	1	2	3	4
$p(x)$	0.3	0.45	a	b
$x\,p(x)$	0.3	0.9	$3a$	$4b$

(a) Find two equations in a and b. **2**

$0.3 + 0.45 + a + b = 1$ $\quad$ $0.3 + 0.9 + 3a + 4b = 2$

$a + b = 0.25$ —— ① $\quad$ $3a + 4b = 0.8$ —— ②

(b) Hence find the values of a and b. **2**

① × 3 $\quad 3a + 3b = 0.75$

② $\quad 3a + 4b = 0.8$

$b = 0.05$

$a + 0.05 = 0.25$

$\therefore a = 0.2$

$\begin{cases} a = 0.2 \\ b = 0.05 \end{cases}$

Question 23 (2 marks)

If $y = x\sin 2x$ find $\frac{dy}{dx}$. **2**

$y' = uv' + vu'$ $\quad$ $u = x$ $\quad$ $v = \sin 2x$

$\frac{dy}{dx} = x \cdot 2\cos 2x + \sin 2x \cdot 1$ $\quad$ $u' = 1$ $\quad$ $v' = 2\cos 2x$

$= 2x\cos 2x + \sin 2x$

Question 24 (2 marks)

$x > 2$

Consider the functions $f(x)=e^x$ and $g(x)=\ln(x-2)$.

(a) Find the composite function $f(g(x))$. **1**

$f(\ln(x-2)) = e^{\ln(x-2)}$

$= x-2$

(b) Find in interval notation the range of the composite function. **1**

Natural restriction is $x>2$ for $y=x-2$

$\therefore$ Range is $y>0$

$\therefore$ Range $(0, \infty)$

Question 25 (4 marks)

The table below shows the English marks (x) and the Mathematics marks (y) for a class of 12 students (A-L). Only the English mark is available for student L.

	A	B	C	D	E	F	G	H	I	J	K	L	
x	67	61	65	67	75	75	69	85	85	89	87	80	English
y	58	64	66	68	70	72	72	76	80	82	84		Maths.

(a) Calculate the correlation coefficient between x and y for the students A to K. Describe the nature of the correlation between x and y. **2**

$r = 0.9$

strong positive correlation

(b) Find the equation of the least squares regression line of y on x for the students A to K. Estimate the Mathematics mark of student L. **2**

$A = 18$ $\quad y = Bx + A$

$B = 0.72$ $\quad y = 0.72x + 18$

$\therefore$ Maths $= 0.72 \times 80 + 18 = 75.6 \doteqdot 76$

Question 26 (2 marks)

If $y = \dfrac{e^x}{x+1}$ find $\dfrac{dy}{dx}$. 2

$u = e^x \qquad v = x+1$

$u' = e^x \qquad v' = 1$

$y' = \dfrac{v.u' - u.v'}{v^2}$

$\dfrac{dy}{dx} = \dfrac{(x+1).e^x - e^x.1}{(x+1)^2}$

$= \dfrac{xe^x + e^x - e^x}{(x+1)^2}$

$= \dfrac{xe^x}{(1+x)^2}$

Question 27 (2 marks)

Find

$\displaystyle\int \cos 4x - \sin x \;\; dx$ 2

$= \frac{1}{4}\sin 4x \;- -\cos x \;+ c$

$= \frac{1}{4}\sin 4x + \cos x \;+ c.$

Question 28 (2 marks)

Find in simplest form the limiting sum of the series $\sin^2 x + \sin^4 x + \sin^6 x + \ldots$ for $0 < x < \frac{\pi}{2}$. 2

$S = \dfrac{a}{1-r} \qquad r = \sin^2 x.$

$= \dfrac{\sin^2 x}{1 - \sin^2 x}$

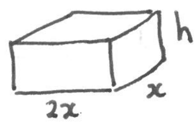
$= \dfrac{\sin^2 x}{\cos^2 x}$

$= \tan^2 x.$

Question 29 (5 marks)

A metal crate of fixed volume 9 m^3 is to be made in the shape of a rectangular prism with length $2x$ metres, width x metres and height h metres. 2

(a) Show that the area A m^2 of metal required is given by $A = 4x^2 + \dfrac{27}{x}$.



$SA = A = 2 \times 2x \times x + 2 \times x \times h + 2 \times 2x \times h$

$= 4x^2 + 2xh + 4xh$

$= 4x^2 + 6xh$

$V = 2x \times x \times h$

$2x^2h = 9$

$\therefore h = \dfrac{9}{2x^2}$

So $A = 4x^2 + 6x \cdot \dfrac{9}{2x^2}$

$A = 4x^2 + \dfrac{27}{x}$

(b) Hence find the minimum area of metal required. 3

$A = 4x^2 + 27x^{-1}$

$A' = 8x - 27x^{-2}$

Let $A' = 0$ $\quad 8x - \dfrac{27}{x^2} = 0$

$8x = \dfrac{27}{x^2}$

$8x^3 = 27$

$x^3 = \dfrac{27}{8}$

$x = \dfrac{3}{2}$

$A'' = 8 + 54x^{-3}$

when $x = 1.5$, $A'' = 24$

> 0

$\therefore$ min

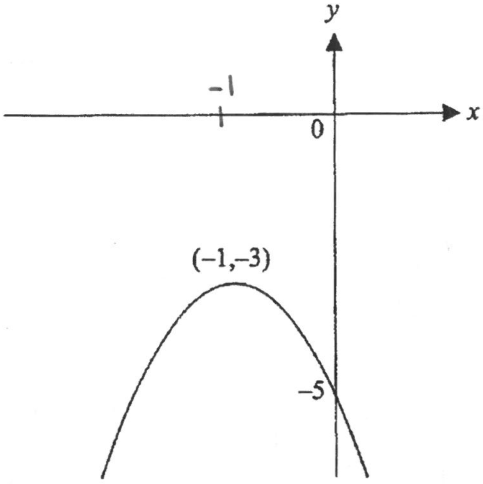
$\therefore$ Minimum Area is when $x = 1.5$ m

$A = 4 \times 1.5^2 + \dfrac{27}{1.5}$

$A = 27$ m^2

Question 30 (3 marks)

The function $f(x) = x^2$ is transformed into a new function whose graph is shown in the diagram below. **3**



Left 1 unit
Down 3 units.

Find the equation of the new function in the form $g(x) = k\,f(x+b)+c$ for some constants k, b and c.

x is replaced by $(x+1)$, $c = -3$

$\therefore g(x) = k\,f(x+1) - 3$

$(0,-5)$ lies on the graph

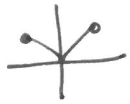
$-5 = k\,f(1) - 3$ but $f(1) = 1^2 = 1$

$-5 = k \times 1 - 3$

$\therefore k = -2$ $\therefore g(x) = -2f(x+1) - 3$

Question 31 (3 marks)

Solve $2\sin(3\theta) = 1$ for $0 \le \theta \le \pi$ **3**



$\sin 3\theta = \frac{1}{2}$

1st, 2nd Quads $3\theta = \frac{\pi}{6}, \frac{13\pi}{6}, \cancel{\frac{25\pi}{6}}$ out of domain

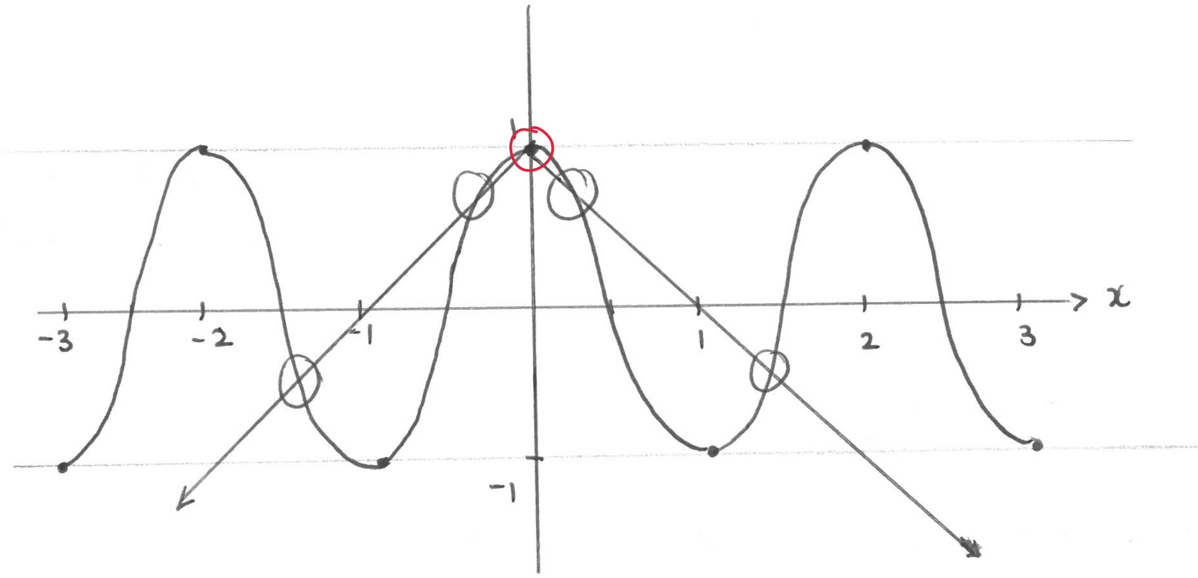
or $3\theta = \frac{5\pi}{6}, \frac{17\pi}{6}, \cancel{\frac{29\pi}{6}}$ out of domain

$\therefore \theta = \frac{\pi}{18}, \frac{13\pi}{18}, \frac{5\pi}{18}, \frac{17\pi}{18}$

$\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$

Question 32 (3 marks)

(a) On the same diagram draw the graphs of $y = \cos \pi x$ and $y = 1 - |x|$ for $-3 \leq x \leq 3$. **2**



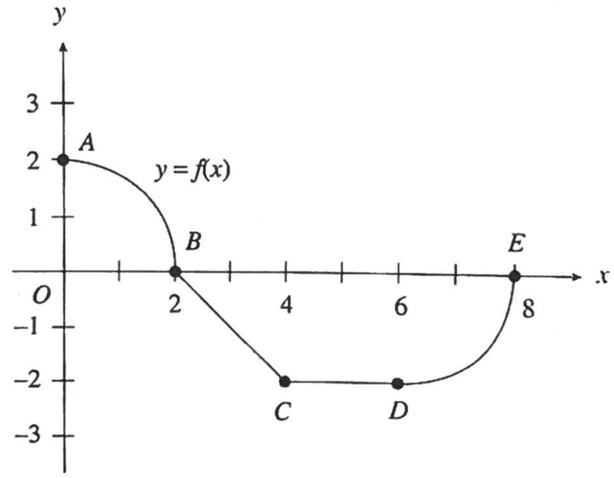
$y = \cos \pi x$

x	-3	-2	-1	0	1	2	3
y	-1	1	-1	1	-1	1	-1

(b) Hence find the number of solutions of the equation $\cos \pi x = 1 - |x|$ in the domain $(-\infty, \infty)$ **1**

5 points of intersection $\therefore$ 5 solutions

Question 33 (3 marks)



The graph of the function $y = f(x)$ drawn above, consists of a quarter circle AB, a straight line segment BC, a horizontal line segment CD and a quarter circle DE.

(a) Evaluate **2**

$$\int_0^8 f(x)\,dx$$

$\int_0^8 f(x)\,dx = \cancel{\frac{1}{4}\pi \times 2^2} + \left(-\frac{1}{2}\times 2\times 2\right) + (-2\times 2) + \cancel{\left(-\frac{1}{4}\times\pi\times 2^2\right)}$

$= -2 - 4$

$= -6$

(b) For what values of x in the domain $0 < x < 8$ is the function NOT differentiable? **1**

It must be smooth & continuous to be differentiable

$\therefore$ $x = 2$ and $x = 4$

Question 34 (4 marks)

The table below shows the future values of an annuity, for different rates of interest and for different numbers of compounding periods, where contributions of $1 are made at the end of each compounding period.

Table of future value interest factors

n	1%	2%	3%	4%	5%	6%
1	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
2	2·0100	2·0200	2·0300	2·0400	2·0500	2·0600
3	3·0301	3·0604	3·0909	3·1216	3·1525	3·1836
4	4·0604	4·1216	4·1836	4·2465	4·3101	4·3746
5	5·1010	5·2040	5·3091	5·4163	5·5256	5·6371
6	6·1520	6·3081	6·4684	6·6330	6·8019	6·9753

$n = 3 \times 2 = 6$

$r = 4\% \div 2 = 2\%$

(a) An annuity account is opened and contributions of $1200 are made at the end of each half year for 3 years at an interest rate of 4% p.a. compounding half yearly. Calculate the final amount in the account immediately after the last contribution is made. **2**

$FV = 1200 \times 6.3081$

$= \$7569.72$

(b) Calculate the single lump sum amount that would need to be invested at the start to reach the same final amount at the end of the 3 years with the interest rate of 4% p.a. compounding half yearly. **2**

$A = P(1+r)^n$

$7569.72 = P(1.02)^6$

$P = \dfrac{7569.72}{(1.02)^6}$

$P = \$6721.69$

Question 35 (3 marks)

If $y = \tan^2 x$ find the values of the constants a and b such that $\frac{d^2y}{dx^2} = ay^2 + by + 2$. 3

$y = (\tan x)^2$

$\frac{dy}{dx} = 2\tan x \sec^2 x$

$= 2\tan x(1+\tan^2 x)$

$= 2\tan x + 2\tan^3 x$

$y'' = 2.\sec^2 x + 2 \times 3(\tan x)^2 \times \sec^2 x$

$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$

$= 2 + 2\tan^2 x + 6\tan^2 x + 6\tan^4 x$

$= 8\tan^2 x + 6\tan^4 x + 2$

$= 6\tan^4 x + 8\tan^2 x + 2$ $\therefore a = 6$

$y'' = a y^2 + b y + 2$ $b = 8$

Question 36 (4 marks)

At time t years after it was purchased the value $\$V$ of a car is given by $V = 25\,000\,e^{-0.5t}$.

(a) Find the loss in value of the car during the third year. $(V_3 - V_2)$ 2

when $t = 3$ $V = 25000\,e^{-0.5 \times 3}$

$t = 2$ $V = 25000\,e^{-0.5 \times 2}$

$\therefore$ loss $= 25000\,e^{-1.5} - 25\,000\,e^{-1}$

$= -3618.73$ $\therefore$ Loss \$3618.73.

(b) Find the year in which the car is losing value at a rate of \$100 per year. 2

$\frac{dV}{dt} = -100$ $\frac{dV}{dt} = -0.5 \times 25\,000\,e^{-0.5t}$

$= -12500\,e^{-0.5t}$

$-100 = -12500\,e^{-0.5t}$

$0.08 = e^{-0.5t}$

$-0.5t = \log_e(0.08)$

$t = 9.656$ $\therefore$ during the 10^{th} year.

Question 37 (3 marks)

Ozzie retires with a savings fund of \$600 000. The fund earns interest at a rate of 3% p.a. compounded monthly. At the end of each month Ozzie withdraws \$4000 from the fund. The amount A_n left in the fund after the n^{th} withdrawal is given by 3

$A_n = 600\,000(1\cdot0025)^n - 4000\{1+1\cdot0025+(1\cdot0025)^2+...+(1\cdot0025)^{n-1}\}$.

(DO NOT PROVE THIS RESULT)

Find correct to the nearest month the time taken for the fund to reduce to half its initial value.

$300\,000 = 600\,000(1.0025)^n - 4000\left[\frac{1\times(1-1.0025^n)}{(1-1.0025)}\right]$

$300\,000 = 600\,000(1.0025)^n + 1\,600\,000(1-1.0025^n)$

$300\,000 = 1\,600\,000 - 1\,000\,000(1.0025^n)$

$1\,000\,000(1.0025)^n = 1\,300\,000$

$1.0025^n = 1.3$

$\ln 1.0025^n = \ln 1.3$

$n \times \ln 1.0025 = \ln 1.3$

$n = \frac{\ln 1.3}{\ln 1.0025}$

$n = 105.07$ months $\doteqdot$ 8 years 9 months

Question 38 (5 marks)

A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1}, and acceleration a ms^{-2} given by $a = 6t - 12$. Initially the particle is at rest at O.

(a) Find expressions for v and x in terms of t. **3**

$v = \int (6t - 12)\, dt$

$v = 3t^2 - 12t + c$

when $t = 0$, $v = 0$ $\therefore c = 0$

$v = 3t^2 - 12t$

$x = \int (3t^2 - 12t)\, dt$

$x = t^3 - 6t^2 + c_1$

$t = 0$, $x = 0$ $\therefore c_1 = 0$

$x = t^3 - 6t^2$

(b) Find when and where the particle is next at rest. **2**

at rest $v = 0$ $\quad 3t^2 - 12t = 0$

$3t(t - 4) = 0$

$t = 0$, $t = 4$ $\therefore$ next at rest when $t = 4$ seconds

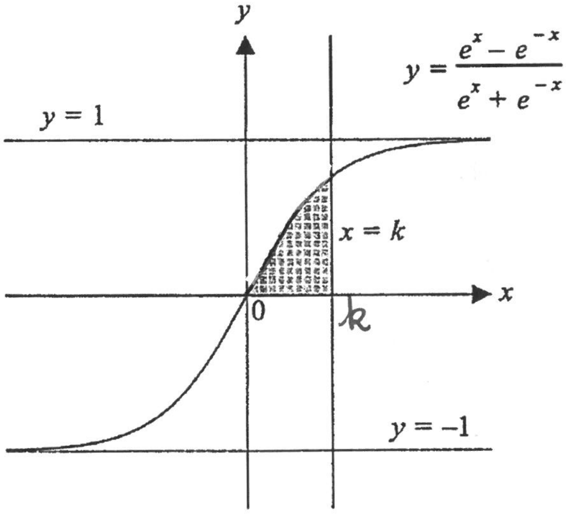
when $t = 4$

$x = 4^3 - 6 \times 4^2$

$x = -32$

$\therefore$ It is next at rest 32 m to left of origin.

Question 39 (5 marks)



The diagram shows the graph of the curve $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

(a) Show that the shaded region bounded by the curve, the x axis and the line $x = k$, where $k > 0$, has area $\ln\left(\frac{e^k + e^{-k}}{2}\right)$. **2**

$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C$

$A = \int_0^k \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

$= \ln\left[e^x + e^{-x}\right]_0^k$

$= \ln\left[(e^k + e^{-k})\right] - \ln\left[e^0 + e^0\right]$

$= \ln\left[e^k + e^{-k}\right] - \ln 2$

$= \ln\left[\frac{e^k + e^{-k}}{2}\right]$ as required

(b) Find in simplest exact form the value of k such that the shaded region has area 1 unit squared **3**

$1 = \ln\left[\frac{e^k + e^{-k}}{2}\right]$

$e = \frac{e^k + e^{-k}}{2}$

$2e = e^k + e^{-k}$

$$2e = e^k + \frac{1}{e^k} \qquad (\times e^k)$$

$$2e.e^k = (e^k)^2 + 1$$

$$(e^k)^2 - 2e.e^k + 1 = 0$$

Let $u = e^k$

$$u^2 - 2e.u + 1 = 0$$

$$u = \frac{-(-2e) \pm \sqrt{4e^2 - 4.1.1}}{2}$$

$$= \frac{\cancel{2}^{1}e \pm \cancel{2}^{1}\sqrt{e^2-1}}{\cancel{2}^{1}}$$

$$\therefore u = \frac{e \pm \sqrt{e^2-1}}{1}$$

$u > 0$ both $\pm$ are positive $\therefore$ viable solutions

So $e^k = e \pm \sqrt{e^2-1}$

$$k = \log_e\left(e \pm \sqrt{e^2-1}\right)$$

$k = \log_e\left(e + \sqrt{e^2-1}\right)$ or $\log_e\left(e - \sqrt{e^2-1}\right)$

$= \log_e 5.245\ldots \qquad = \log_e(0.1906\ldots)$

$= 1.657\ldots \qquad = \underbrace{-1.657\ldots}_{\text{but } k>0 \text{ from part (a)}} < 0$

$\therefore k > 0$

$\therefore k = \log_e\left(e + \sqrt{e^2-1}\right)$ only

END OF EXAMINATION ☺