KNOX
GRAMMAR
SCHOOL

2023 **TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Advanced

General Instructions

- Reading time – 10 minutes.
- Working time – 3 hours.
- Write using black pen.
- NESA approved calculators may be used.
- A reference sheet is provided with this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.

Total marks: 100

Section I – 10 marks (pages 2 – 5)
- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.

Section II – 90 marks (pages 6 – 34)
- Attempt Questions 11 – 18.
- Allow about 2 hours and 45 minutes for this section.

Question	**MC**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**Total**
Mark	**/10**	**/12**	**/12**	**/12**	**/12**	**/12**	**/10**	**/10**	**/10**	

Teachers:

Charlier C.
Cheah S.
Ekanayake P.
Meli S.
Menzies D.
Naidoo V.
Willcocks A.
Zaidi S.

Examiner: MED

Number of Students in Course: 175

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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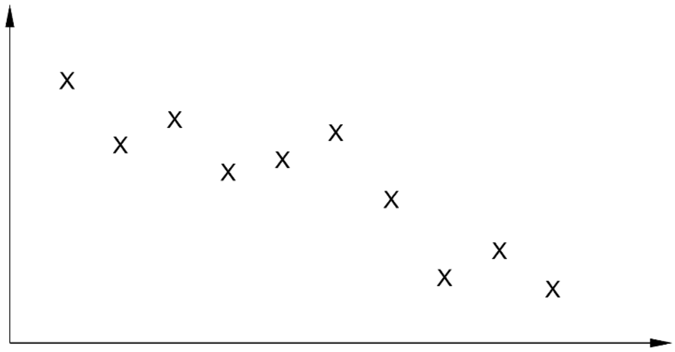
1. Consider the bivariate data shown on the scatterplot below.



Which one of the following values is the best estimate for Pearson’s correlation coefficient for this data?

(A) –0.9

(B) –0.2

(C) 0.2

(D) 0.9

2. The graph of the function $y = f(x)$ is moved 3 units to the left.
Which one of the following represents the new function?

(A) $y = f(x+3)$

(B) $y = f(x)+3$

(C) $y = f(x-3)$

(D) $y = f(x)-3$

3. The height of the tide in a harbour can be modelled using the sine function.
The time, t in hours, between high tide and low tide is 6 hours.
Which one of the following could be the function representing the height of the tide?

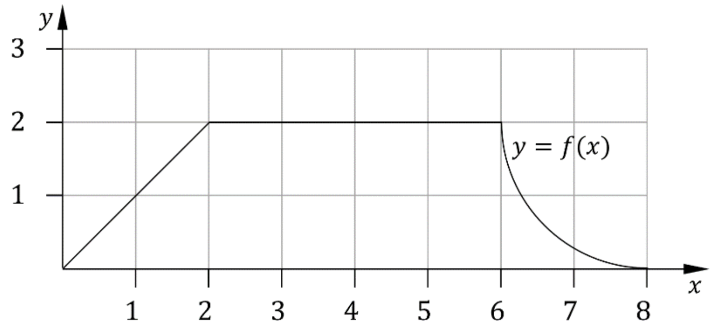
(A) $h = \sin\left(\dfrac{\pi t}{3}\right)$

(B) $h = \sin\left(\dfrac{\pi t}{6}\right)$

(C) $h = \sin\left(\dfrac{\pi t}{12}\right)$

(D) $h = \sin(2\pi t)$

4. The graph $y = f(x)$ is shown below:



DRAWN
TO
SCALE

What is the exact value of $\int_0^8 f(x)\,dx$?

(A) $10 + \pi$

(B) $10 - \pi$

(C) $14 + \pi$

(D) $14 - \pi$

5. The product of the 1st term and 2nd term of a geometric sequence is 48 while the product of the 3rd term and the 4th term of the sequence is 3888.
What is the product of the 4th term and the 5th term of the sequence?

(A) 11 664

(B) 15 552

(C) 34 992

(D) 186 624

6. The displacement of a particle moving along the x axis at time t is given by:

$$x = t^3 + \ln\left(2t^3 + 1\right) \text{ for } t \geq 0$$

How many times is the particle stationary?

(A) 0

(B) 1

(C) 2

(D) 3

7. ΔABC is an isosceles triangle such that $AB = AC = 8$ cm.
Given that the area of ΔABC is 16 cm^2, which one of the following gives the possible values of $\angle ABC$?

(A) 15°, 70°

(B) 15°, 75°

(C) 30°, 120°

(D) 30°, 150°

8. It is given that $P(A) = \dfrac{1}{2}$ and $P(A \cup B) = \dfrac{3}{5}$

What is the value of $P(\bar{B} \mid \bar{A})$?

(A) $\dfrac{1}{4}$

(B) $\dfrac{3}{10}$

(C) $\dfrac{3}{4}$

(D) $\dfrac{4}{5}$

9. What is the derivative of $\left(\tan^2 x+1\right)^2$?

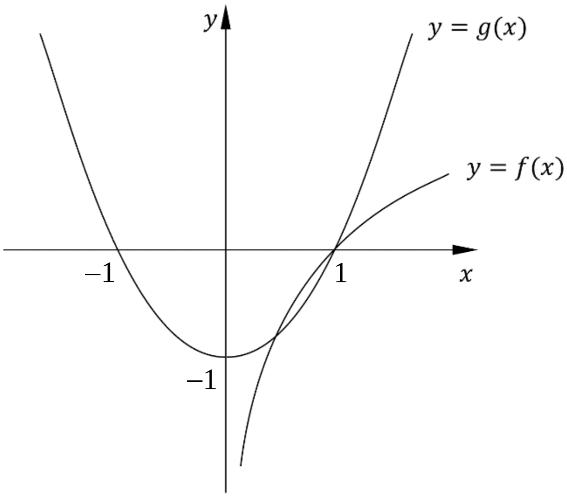
(A) $4\sec^3 x$

(B) $4\sec^4 x$

(C) $4\sec^3 x\tan x$

(D) $4\sec^4 x\tan x$

10. The graph shows $y=f(x)$ and $y=g(x)$, where $f(x)=\ln x$ and $g(x)=x^2-1$.



How many solutions does the equation $\left[f(x)\right]^2-\left[g(x)\right]^2=0$ have?

(A) 0

(B) 1

(C) 2

(D) 3

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SCHOOL

Student Number

2023 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced

Section II Answer Booklet

Section II

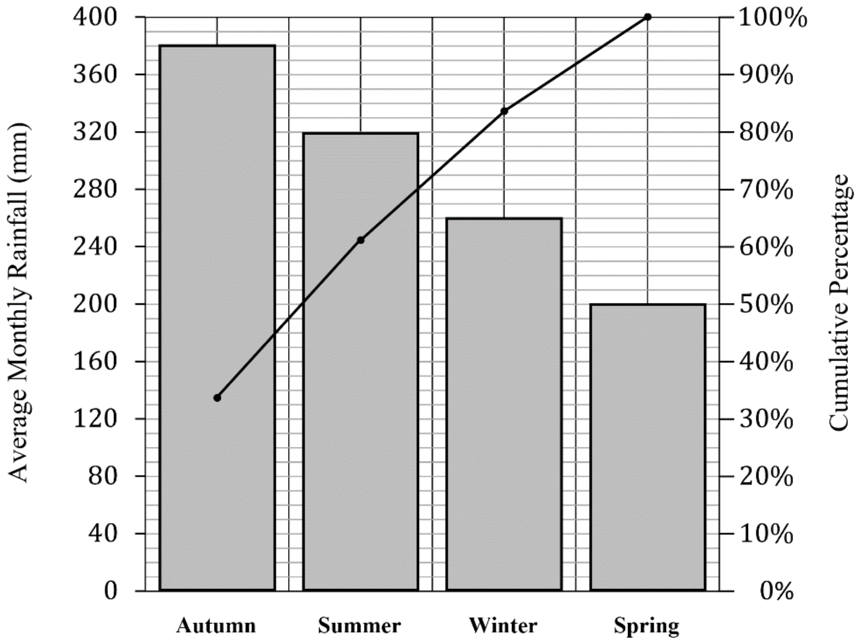
90 marks
Attempt Questions 11-18
Allow about 2 hours 45 minutes for this section

Instructions

- Answer the questions in the spaces provided. These spaces generally provide guidance for the expected length of the response.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of this booklet.
  If you use this space, clearly indicate which question you are answering.

Question 11 (12 marks)

(a) Below is a Pareto chart showing the average seasonal rainfall in Sydney.



What percentage of the annual rainfall occurs in the two driest seasons? Give your answer as a whole percentage. **1**

(b) Find the primitive of $f(x) = \dfrac{1}{2x-3}$ **2**

(c) Calculate $\int_{\ln 2}^{2\ln 2} e^{2x}dx$ **3**

(d) It is given that $f''(x) = 6x$ and that $f(x)$ has a stationary point at $(-1, 2)$. **3**
Find $f(x)$.

(e) It is given that $1+2x-3x^2 \geq 0$ in the domain [0, 1].

Prove that $f(x)=1+2x-3x^2$ is a probability density function for [0, 1]. **3**

Question 12 (12 marks)

(a) A normally distributed variable, X, has a mean of 2 and a standard deviation of 0.5. Use the empirical rule to find $P(0.5 \le X \le 3)$. **2**

(b) Find the second derivative of $y = x\left(x+1\right)^3$, leaving your answer in fully factorised form. **3**

(c) The circle $x^2 + y^2 - 6x + 8y - 11 = 0$ is transformed by a horizontal translation to the left by 4 units and a vertical translation up 3 units. **3**

What is the centre and radius of the new circle?

(d) The table below shows the probability distribution of a discrete random variable X.

x	0	2	4	5	8	9
$P(X = x)$	k^2	0.16	0.18	0.3	k	0.12

(i) Show that $k = 0.2$ **1**

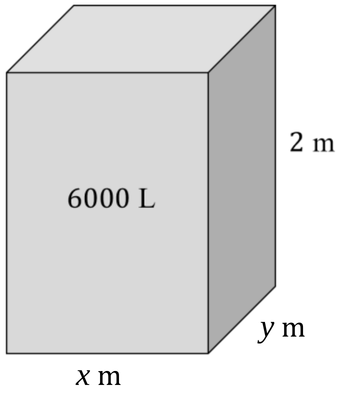
(ii) Calculate E(X). **1**

(e) (i) Show that $\frac{d}{dx}\left(x\log_e x - x\right) = \log_e x$ **1**

(ii) Hence find $\int_1^e \log_e x \, dx$ as an exact value. **1**

Question 13 (12 marks)

(a) A rainwater tank manufacturer wants to make a new storage tank that has a standard height, to reduce the obstruction of views. The manufacturer decides that it will be a 2 metre high rectangular prism with all faces made from sheet metal, including the top and bottom, and needs to hold 6000 litres of water,
Let the other dimensions be x metres and y metres, as shown in the diagram.



NOT TO SCALE

For cost reasons, the manufacturer wants to use the least amount of area of sheet metal possible.

(i) Show that for the given capacity of the tank, $y = \dfrac{3}{x}$ **1**

(ii) Hence, calculate the exact dimensions of the tank that uses the least amount of area of sheet metal as possible. Show all reasoning. **3**

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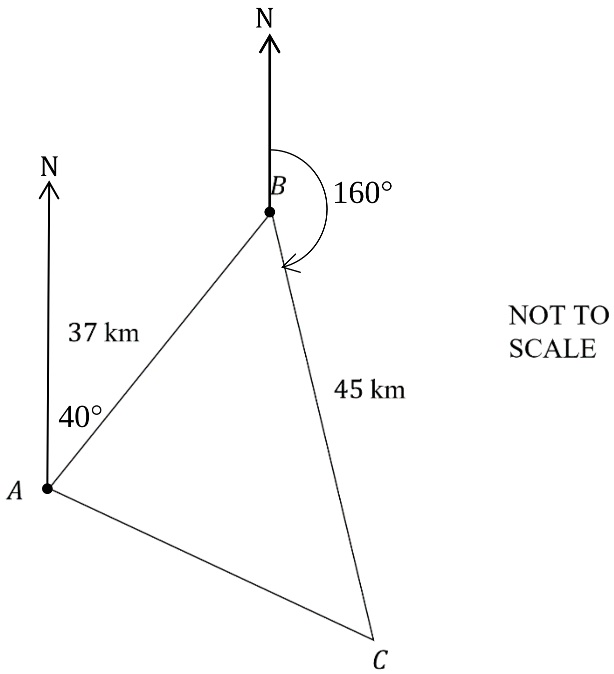
(b) There are three towns in a rural area. Town B is 37 km from Town A at a bearing of 040°T. Town C is 45 km from Town B at a bearing of 160°T.



(i) Find the size of $\angle ABC$. **1**

(ii) Find the distance from Town A to Town C, correct to 1 decimal place. **2**

Question 13 (b) *continues on the next page*

Question 13 (b) (continued)

(iii) Find the bearing of Town *A* from Town *C*. **2**

(c) (i) Show that $f(x) = x^3 - 5x + \dfrac{xe^{-x^2}}{1+x^4}$ is an odd function. **2**

(ii) Hence evaluate $\displaystyle\int_{-5}^{5} \left(x^3 - 5x + \frac{xe^{-x^2}}{1+x^4} \right) dx$. **1**

Question 14 (12 marks)

(a) A quantity Q of the radioactive isotope Carbon-14 decays according the equation of natural decay $Q = Q_0 e^{-kt}$ where Q_0 is the initial mass of Carbon-14, k is a constant and t is the time in years.
The half-life of a radioactive substance is the time it takes to reduce to half its original mass. Carbon-14 has a half-life of 5730 years.

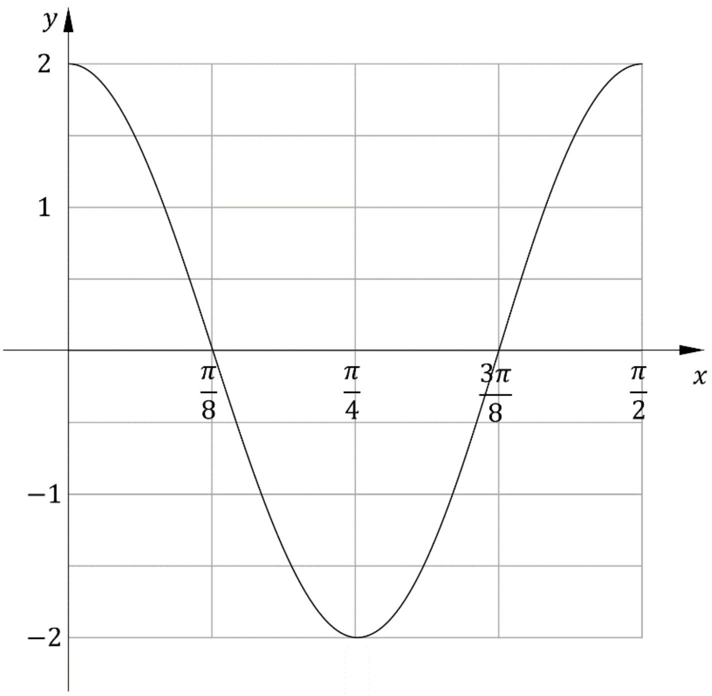
(i) Find the exact value of k. **2**

(ii) A fossilised leaf contains 9% of its original amount of Carbon-14. **2**
How old is the fossilised leaf? Give your answer correct to the nearest year.

(b) (i) Simplify $\dfrac{\cos^2\theta}{1-\sin\theta}-\dfrac{\cos^2\theta}{1+\sin\theta}$ **2**

(ii) Hence, solve $\dfrac{\cos^2\theta}{1-\sin\theta}-\dfrac{\cos^2\theta}{1+\sin\theta}=1$ for $0\le\theta\le\dfrac{\pi}{2}$ **1**

(c) The graph shown below is $y = A\cos bx$



(i) Write down the values of A and b for this graph. **2**

(ii) On the axes above, draw the graph of $y = \sin 2x + 1$ for $0 \le x \le \frac{\pi}{2}$. **1**

(d) Simplify $\sqrt{x^2 + 2x + 1} - \sqrt{x^2 - 2x + 1}$. **2**

Question 15 (12 marks)

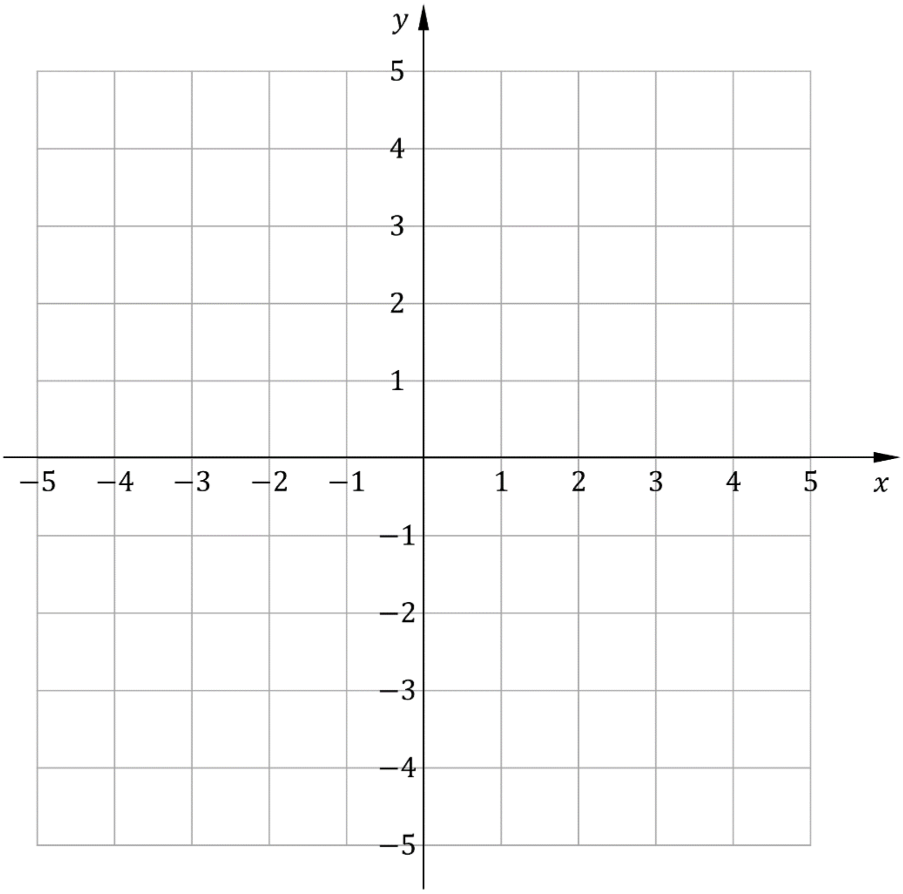
(a) Solve $\ln\left(2\sin^2\theta - \cos\theta\right) = 0$ for $0 \le \theta \le 2\pi$. **4**

(b) (i) Find the stationary points and determine their nature, and any points of inflection for $y = x^4 - 2x^3 + 1$, showing all reasoning. **4**

Question 15 (b) *continues on next page*

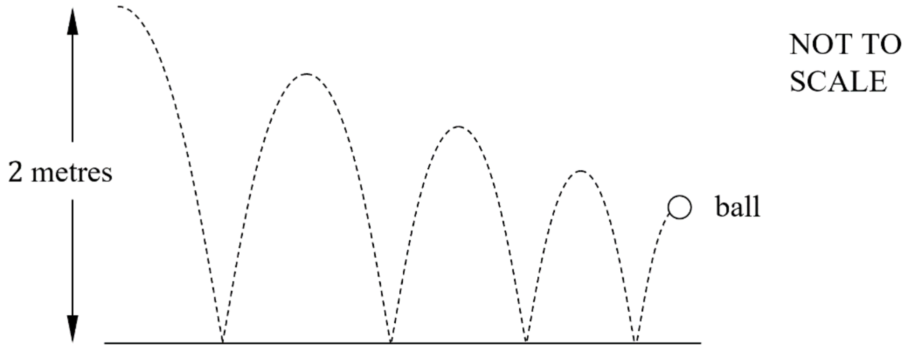
Question 15 (b) (continued)

(ii) Sketch the graph of $y = x^4 - 2x^3 + 1$ on the axes below, clearly showing all important points including any turning points and points of inflection. **1**
It is not necessary to find all x-intercepts.



(c) The Moon has a lower gravity than Earth, and there is no atmosphere to cause air resistance, so a ball would bounce higher and for much longer on the Moon than on Earth. **3**

When a ball is dropped on the Moon, each bounce is 95% as high as the previous bounce. When an identical ball is dropped on Earth, each bounce is 50% as high as the previous bounce.

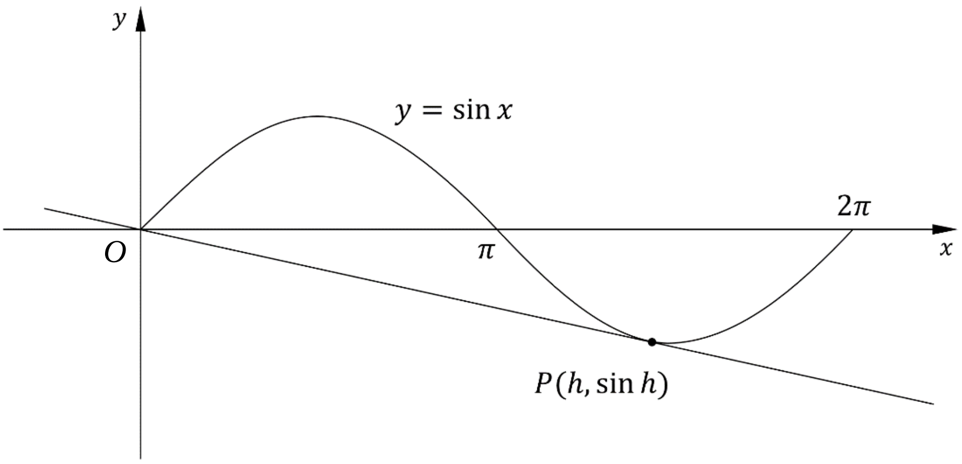


Two identical balls are dropped on the Moon and on Earth, each from a height of two metres, and they both eventually stop bouncing.
Calculate the difference in the total vertical distance travelled by these balls.

Question 16 (10 marks)

(a) The graph $y = \sin x$ only has one tangent in the domain $[\pi, 2\pi]$ that passes through the origin as shown in the diagram below. **4**

Let the point of contact of this tangent be $P(h, \sin h)$.



Prove that $h = \tan h$.

(b) The table below gives the present value interest factors for an annuity of $1 for various interest rates r (as a decimal), and the number of periods N.

Present value interest factors per $1					
	Interest rate r (as a decimal)				
N	**0.0025**	**0.005**	**0.0075**	**0.01**	**0.0125**
71	64.98140	59.64121	54.89293	50.66190	46.88363
72	65.81686	60.33951	55.47685	51.15039	47.29247
73	66.65023	61.03434	56.05643	51.63405	47.69627
74	67.48153	61.72571	56.63169	52.11292	48.09508
75	68.31075	62.41365	57.20267	52.58705	48.48897
76	69.13791	63.09815	57.76940	53.05649	48.87800

(i) Anson plans to invest $200 each month for 6 years at a rate of 3% p.a. compounded monthly. Use the table of values to calculate the present value of this annuity to the nearest dollar. **2**

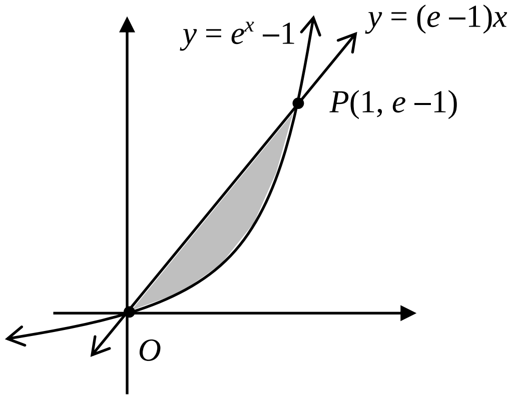
(ii) Monique uses the same table of values to calculate the loan repayments for a business loan for her dance school. She has borrowed $2 000 000 and she is repaying it in **quarterly** repayments over 19 years, at a rate of 5% p.a. Using the table of values above or otherwise, calculate the amount of each quarterly repayment, correct to the nearest dollar. **2**

(c) Give the simplest exact value of $\int_0^1 3^x \, dx$. **2**

Question 17 (10 marks)

(a) Use the trapezoidal rule, with 3 function values, to find the area between the curve $y = x\sin^2 x$, the x-axis, and the lines $x = 0$ and $x = \dfrac{\pi}{2}$ leaving your answer in exact terms. **3**

(b) The curve $y = e^x - 1$ meets the line $y = (e-1)x$ at the origin and at the point $P(1, e-1)$, as shown in the diagram below. **3**



.

Find the exact value of the shaded area.

(c) The equation $f(x)=\cos\left(\frac{x}{2}\right)$ is a probability density function for $\frac{\pi}{3}\le x\le\pi$.

(DO NOT PROVE THIS)

(i) Using the definition of a cumulative distribution function **2**

$$F(x)=\int_a^x f(t)\,dt \text{ for } a\le x\le b$$

prove that the cumulative distribution function for the given domain is:

$$F(x)= \quad 2\sin\left(\frac{x}{2}\right)-1 \qquad\qquad \frac{\pi}{3}\le x\le\pi$$

Question 17 (c) *continues on the next page*

(ii) Hence, or otherwise, find the median of this data, correct to one decimal place. **2**

Question 18 (10 marks)

(a) Sergio decides to buy a top-of-the-line gaming laptop for \$12 500 from the retailer Jay-Bee. Jay-Bee offers the following terms:

- The customer pays for the laptop with equal monthly repayments of $\$R$ at the end of each month for 5 years.
- The first six months are interest free. After six months, interest is charged on the balance owing at 9% p.a. compounded monthly.
- The equal monthly repayment is still made at the end of each month during the first six months.

(i) Let A_n be the amount owing after n months. **3**

Show that after seven months, the total amount Sergio owes is:

$$A_7 = 12\,500 \times 1.0075 - R(6 \times 1.0075 + 1)$$

Question 18 (a) *continues on the next page*

Question 18 (a) (continued)

(ii) Calculate the monthly repayment $\$R$ required to pay back the loan, correct to the nearest dollar. **4**

(b) The graph of $y = \dfrac{1}{2x}$ is shown below. **3**

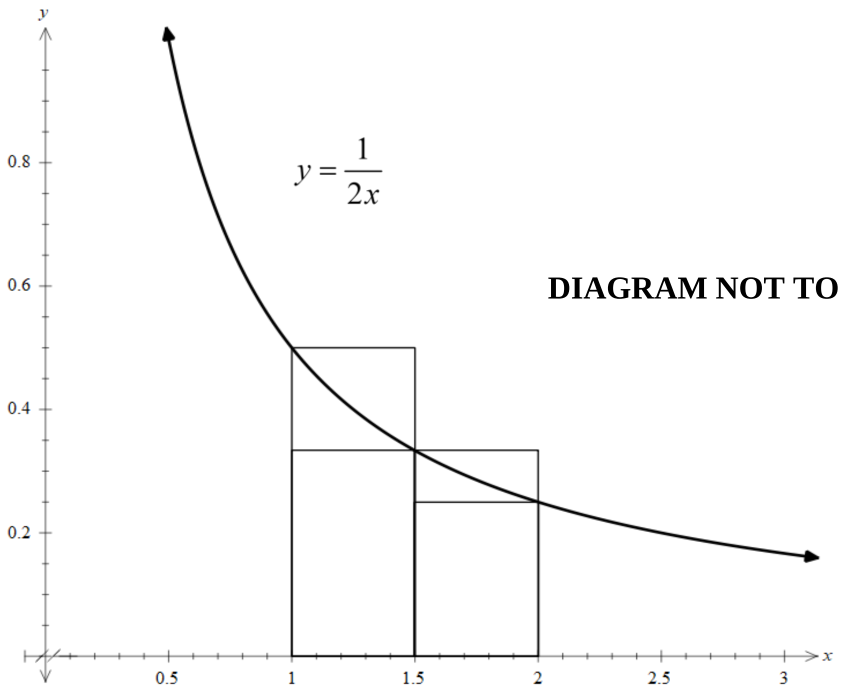


DIAGRAM NOT TO SCALE

Upper and lower rectangles of width 0.5 units are drawn from the x-axis to the curve between $x = 1$ and $x = 2$ as shown in the diagram.

By considering the areas of the rectangles, show that $\dfrac{7}{24} < \ln\sqrt{2} < \dfrac{5}{12}$.

Working space continues on the next page

End of paper

Section II extra writing space

If you use this space, clearly indicate which question you are answering.

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Mathematics Advanced

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- Attempt Questions 1 – 10.
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- Allow about 2 hours and 45 minutes for this section.

Question	MC	11	12	13	14	15	16	17	18	Total
Mark	/10	/12	/12	/12	/12	/12	/10	/10	/10	

SOLUTIONS

Teachers:

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Examiner: MED

Number of Students in Course: 175

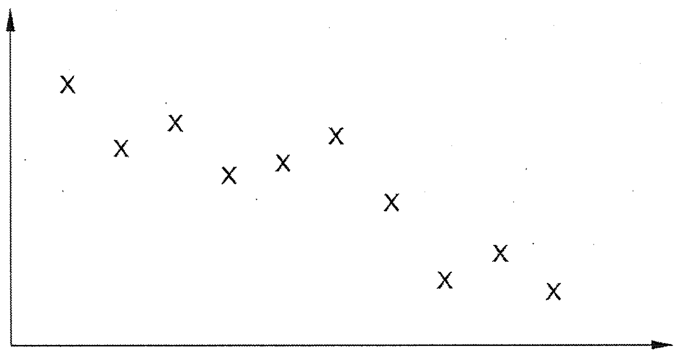
Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

---

1. Consider the bivariate data shown on the scatterplot below.



Which one of the following values is the best estimate for Pearson's correlation coefficient for this data?

(A) -0.9 strong negative correlation.

(B) -0.2

(C) 0.2

(D) 0.9

2. The graph of the function $y = f(x)$ is moved 3 units to the left.
Which one of the following represents the new function?

(A) $y = f(x+3)$ horizontal left by 3 units

(B) $y = f(x)+3$

(C) $y = f(x-3)$

(D) $y = f(x)-3$

3. The height of the tide in a harbour can be modelled using the sine function.
The time, t in hours, between high tide and low tide is 6 hours.
Which one of the following could be the function representing the height of the tide?

(A) $h = \sin\left(\frac{\pi t}{3}\right)$

(B) $h = \sin\left(\frac{\pi t}{6}\right)$

(C) $h = \sin\left(\frac{\pi t}{12}\right)$

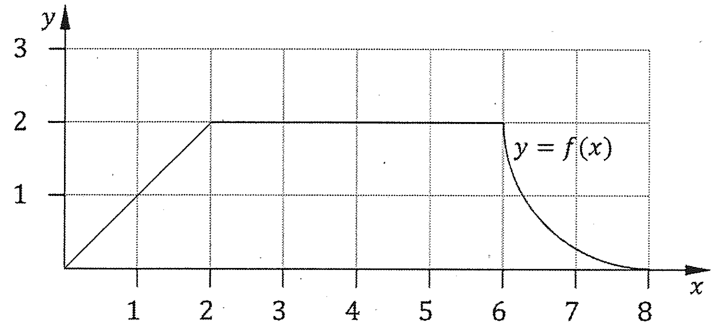
(D) $h = \sin(2\pi t)$

$T = \frac{2\pi}{n}$ $\therefore$ $n = \frac{2\pi}{T}$

$= \frac{2\pi}{12}$

$= \frac{\pi}{6}$

4. The graph $y = f(x)$ is shown below:



DRAWN TO SCALE

What is the exact value of $\int_0^8 f(x)\,dx$?

(A) $10 + \pi$

(B) $10 - \pi$

(C) $14 + \pi$

(D) $14 - \pi$

$\frac{1}{2} \times 2 \times 2 + (2\times4) + 2\times2 - \frac{1}{4}\pi(2)^2$

$= 2 + 8 + 4 - \pi$

$= 14 - \pi.$

5. The product of the 1st term and 2nd term of a geometric sequence is 48 while the product of the 3rd term and the 4th term of the sequence is 3888.
What is the product of the 4th term and the 5th term of the sequence?

(A) 11 664

(B) 15 552

(C) 34 992

(D) 186 624

$T_1 \times T_2 = a^2 r = 48$ #1

$T_3 \times T_4 = a^2 r^5 = 3888$ #2

#2 ÷ #1 $r^4 = \frac{3888}{48} = 81$

$r = 3$ ($r > 0$ $\because$ products are positive)

$a^2 = \frac{48}{3} = 16$

$T_4 \times T_5 = a^2 r^7 = 16 \times 3^7 = 34992$

6. The displacement of a particle moving along the x axis at time t is given by:

$$x = t^3 + \ln\left(2t^3 + 1\right) \text{ for } t \geq 0$$

How many times is the particle stationary?

$$\frac{dx}{dt} = 3t^2 + \frac{6t^2}{2t^3+1}$$

(A) 0

(B) 1

stationary: $\frac{dx}{dt} = 0 \quad 3t^2 + \frac{6t^2}{2t^3+1} = 0$

(C) 2

$3t^2(2t^3+1) + 6t^2 = 0$

(D) 3

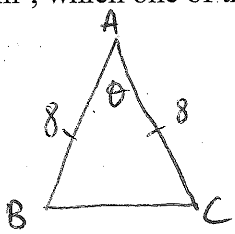
$3t^2(2t^3+1+2) = 0$

$3t^2(2t^3+3) = 0$

$t = 0, \ t = -\sqrt[3]{3/2} \ \therefore t = 0.$

1 solution.

7. ΔABC is an isosceles triangle such that $AB = AC = 8$ cm.
Given that the area of ΔABC is 16 cm^2, which one of the following gives the possible values of $\angle ABC$?



$16 = \frac{1}{2} ab \sin\theta$

$16 = \frac{1}{2} \times 8 \times 8 \times \sin\theta$

$\therefore \sin\theta = \frac{1}{2}$

$\theta = 30°, 150°$

$\therefore \angle ABC = \frac{30°}{2} = 15°$

OR $= \frac{150°}{2} = 75°$

(A) 15°, 70°

(B) 15°, 75°

(C) 30°, 120°

(D) 30°, 150°

8. It is given that $P(A) = \frac{1}{2}$ and $P(A \cup B) = \frac{3}{5}$

What is the value of $P(\bar{B} \mid \bar{A})$?

$$P(\bar{B} \mid \bar{A}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{A})}$$

$$= \frac{1 - P(A \cup B)}{1 - P(A)}$$

$$= \frac{1 - 3/5}{1 - \frac{1}{2}} = \frac{2/5}{1/2} = 4/5$$

(A) $\frac{1}{4}$

(B) $\frac{3}{10}$

(C) $\frac{3}{4}$

(D) $\frac{4}{5}$

9. What is the derivative of $\left(\tan^2 x+1\right)^2$?

(A) $4\sec^3 x$

(B) $4\sec^4 x$

(C) $4\sec^3 x \tan x$

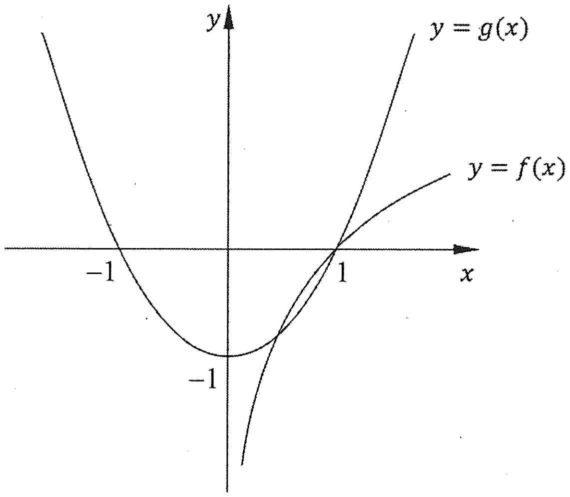
(D) $4\sec^4 x \tan x$

$\frac{d}{dx}\left(\tan^2 x+1\right)^2 = 2\left(\tan^2 x+1\right)\times 2\tan x\sec^2 x$

$= 2\sec^2 x \times 2\tan x \sec^2 x$

$= 4\sec^4 x \tan x$

10. The graph shows $y=f(x)$ and $y=g(x)$, where $f(x)=\ln x$ and $g(x)=x^2-1$.



How many solutions does the equation $[f(x)]^2-[g(x)]^2=0$ have?

(A) 0

(B) 1

(C) 2

(D) 3

$[f(x)]^2-[g(x)]^2 = [f(x)+g(x)][f(x)-g(x)]$

$f(x)+g(x)=0$ only when $f(x)=g(x)=0$, @ $x=1$

$f(x)-g(x)=0 \therefore f(x)=g(x) \therefore$

i.e. at pts of intersection. Already counted $x=1$.

one other point of intersection

$\therefore$ 2 points of intersection

KNOX
GRAMMAR
SCHOOL

Student Number

2023 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Section II Answer Booklet

Section II

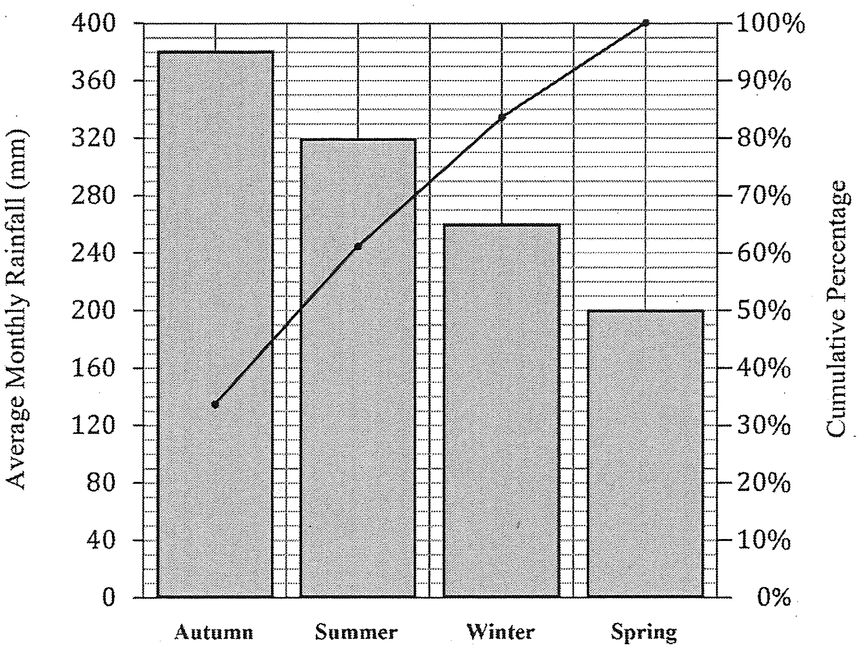
90 marks
Attempt Questions 11-18
Allow about 2 hours 45 minutes for this section

Instructions

- Answer the questions in the spaces provided. These spaces generally provide guidance for the expected length of the response.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate which question you are answering.

Question 11 (12 marks)

(a) Below is a Pareto chart showing the average seasonal rainfall in Sydney.



What percentage of the annual rainfall occurs in the two driest seasons? **1**
Give your answer as a whole percentage.

FROM GRAPH: $100\% - 61\% = 39\%$ (or 40%) ✓

Alternatively: total $= 380 + 320 + 260 + 200 = 1160$ OR

2 driest seasons $= 260 + 200 = 460$.

$\frac{460}{1160} \times 100\% \doteqdot 39.7\% = 40\%$ ✓

(b) Find the primitive of $f(x) = \dfrac{1}{2x-3}$ **2**

$\int f(x)\,dx = \frac{1}{2} \log_e |2x-3| + C$ ✓ ✓

OR $\int \frac{1}{2x-3}\,dx = \frac{1}{2} \int \frac{2}{2x-3}\,dx$

$= \frac{1}{2} \log_e |2x-3| + C.$

(c) Calculate $\int_{\ln 2}^{2\ln 2} e^{2x}\,dx$ **3**

$$= \frac{1}{2}\left[e^{2x}\right]_{\ln 2}^{2\ln 2} \checkmark$$

$$= \frac{1}{2}\left[e^{4\ln 2} - e^{2\ln 2}\right] \checkmark$$

$$= \frac{1}{2}\left[e^{\ln 16} - e^{\ln 4}\right]$$

$$= \frac{1}{2}\left[16 - 4\right] = 6 \checkmark$$ (or by calculator)

(d) It is given that $f''(x) = 6x$ and that $f(x)$ has a stationary point at $(-1, 2)$. **3**
Find $f(x)$.

$$f'(x) = \int 6x \cdot dx = 6\left[\frac{x^2}{2}\right] + C$$

$$= 3x^2 + C \checkmark$$

$f'(x) = 0$ when $x = -1$.

$$\therefore 3(-1)^2 + C = 0 \quad \therefore C = -3$$

$$f'(x) = 3x^2 - 3 \checkmark$$

$$f(x) = \int (3x^2 - 3) \cdot dx$$

$$= \frac{3x^3}{3} - 3x + D$$

$$= x^3 - 3x + D$$

when $x = -1$, $f(-1) = 2$

$$2 = (-1)^3 - 3(-1) + D$$

$$\therefore D = 0$$

$$f(x) = x^3 - 3x \checkmark$$

(e) It is given that $1+2x-3x^2 \geq 0$ in the domain [0, 1].

Prove that $f(x)=1+2x-3x^2$ is a probability density function for [0, 1]. **3**

$\int_0^1 f(x).dx = 1$ for PDF.

$\int_0^1 (1+2x-3x^2)\,dx = \left[x+x^2-x^3\right]_0^1$

$= \left[(1+1-1)-(0)\right]$

$= 1$ ✓

since $\int f(x)\,dx = 1$ and $f(x) \geq 0$ ✓ for [0, 1]

$\therefore$ $f(x)$ is a PDF.

Question 12 (12 marks)

(a) A normally distributed variable, X, has a mean of 2 and a standard deviation of 0.5. Use the empirical rule to find $P(0.5 \le X \le 3)$. **2**

For $X = 0.5$, $z = \frac{0.5-2}{0.5} = -3$

For $X = 3$, $z = \frac{3-2}{0.5} = 2$

✓ for z scores

$\therefore P(0.5 \le X \le 3) = P(-3 \le Z \le 2)$

$= \frac{1}{2}(0.95 + 0.997)$

$= 0.9735$ or 97.35% ✓

(b) Find the second derivative of $y = \overset{u}{x}\overset{v}{(x+1)^3}$, leaving your answer in fully factorised form. **3**

$y' = (x+1)^3 \times 1 + 3(x+1)^2 \times x$

$= (x+1)^3 + 3x(x+1)^2$

$= (x+1)^2((x+1) + 3x)$

$= (x+1)^2(4x+1)$ ✓

$y'' = (4x+1)(x+1) \times 2 + (x+1)^2 \times 4$ ✓

$= 2(x+1)(4x+1) + 4(x+1)^2$

$= 2(x+1)[4x+1 + 2(x+1)]$

$= 2(x+1)(6x+3) = 6(x+1)(2x+1)$ ✓

(c) The circle $x^2+y^2-6x+8y-11=0$ is transformed by a horizontal translation to the left by 4 units and a vertical translation up 3 units. **3**

What is the centre and radius of the new circle?

$x^2-6x+9+y^2+8y+16=11+9+16$ ✓

$(x-3)^2+(y+4)^2=36$

centre $(3,-4)$ radius $=6$ units

✓ for both.

$\therefore$ new circle: ✓

centre $(-1,-1)$ radius $=6$ units

(d) The table below shows the probability distribution of a discrete random variable X.

x	0	2	4	5	8	9
$P(X=x)$	k^2	0.16	0.18	0.3	k	0.12

(i) Show that $k=0.2$ **1**

By SUBSTITUTION: $(0.2)^2+0.16+0.18+0.3+0.2+0.12$

$=1$ ✓

BY EQUATION: $K^2+K+0.76=1 \therefore K^2+K-0.24=0$

$(K-0.2)(K+1.2)=0 \quad \therefore K=0.2 \quad (K\geq 0)$

(ii) Calculate E(X). **1**

$E(X)=0\times 0.2^2+2\times 0.16+4\times 0.18+5\times 0.3\times 8\times 0.2$

$+9\times 0.12$

$=5.22$ ✓ (award answer only)

(e) (i) Show that $\frac{d}{dx}\left(x\log_e x - x\right) = \log_e x$ **1**

$\frac{d}{dx}(x\log_e x - x) = \log_e x \times 1 + x \times \frac{1}{x} - 1$ ✓

$= \log_e x + 1 - 1$

$= \log_e x$

(ii) Hence find $\int_1^e \log_e x \, dx$ as an exact value. **1**

$\therefore \int_1^e \log_e x = \left[x\log_e x - x\right]_1^e$

$= \left[(e\log_e e - e) - (1\log_e 1 - 1)\right]$

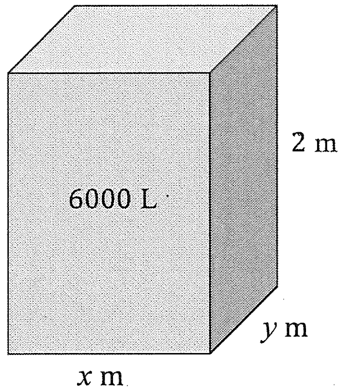
$= (e - e) - (0 - 1)$

$= 1$ ✓

(award answer only)

Question 13 (12 marks)

(a) A rainwater tank manufacturer wants to make a new storage tank that has a standard height, to reduce the obstruction of views. The manufacturer decides that it will be a 2 metre high rectangular prism with all faces made from sheet metal, including the top and bottom, and needs to hold 6000 litres of water,
Let the other dimensions be x metres and y metres, as shown in the diagram.



NOT TO SCALE

For cost reasons, the manufacturer wants to use the least amount of area of sheet metal possible.

(i) Show that for the given capacity of the tank, $y = \dfrac{3}{x}$ **1**

$6000\,L = 6\,m^3$

$\therefore\ x \times y \times 2 = 6$ ✓

$2xy = 6$

$y = \dfrac{6}{2x} = \dfrac{3}{x}$.

(ii) Hence, calculate the exact dimensions of the tank that uses the least amount of area of sheet metal as possible. Show all reasoning. **3**

$S.A. = 2(xy + 2x + 2y)$

Let $A = 2xy + 4x + 4y$

substitute from part (i) $A = 4x + \dfrac{12}{x} + 6$

$\dfrac{dA}{dx} = 4 - \dfrac{12}{x^2}$ ✓ or $\dfrac{dA}{dx} = 4 - \dfrac{12}{x^2} = 4 - 12x^{-2}$

working space continues over the page

Question 13 (a)(ii) (continued)

Areas will be a minimum when $\frac{dA}{dx} = 0$.

$4 - \frac{12}{x^2} = 0$

$4x^2 = 12$

$x^2 = 3$

$x = \sqrt{3}$ (since $x \geq 0$). ✓

Test for minimum.

$\frac{d^2A}{dx^2} = 24x^{-3} = \frac{24}{x^3}$

$\frac{24}{x^3} > 0$ for all $x > 0$ ✓

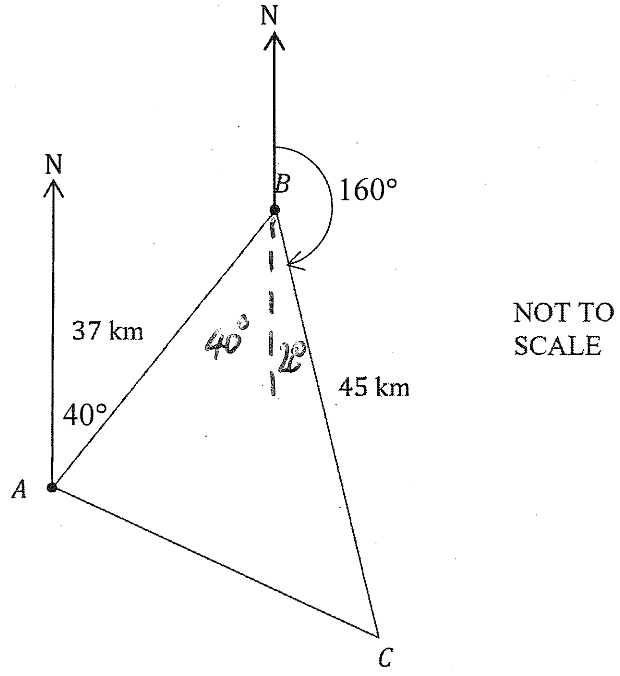
$\therefore$ concave up $\therefore$ Area minimum @ $x = \sqrt{3}$.

$y = \frac{3}{\sqrt{3}} = \sqrt{3}$

Dimensions of tank : $\sqrt{3} \times \sqrt{3} \times 2$

(no penalty for not writing dimension statement)

(b) There are three towns in a rural area. Town B is 37 km from Town A at a bearing of 040°T. Town C is 45 km from Town B at a bearing of 160°T.



(i) Find the size of $\angle ABC$. **1**

$\angle ABC = 40^\circ + 20^\circ = 60^\circ$.

(ii) Find the distance from Town A to Town C, correct to 1 decimal place. **2**

$AC^2 = 37^2 + 45^2 - 2 \times 37 \times 45 \times \cos 60^\circ$ ✓

$= 1729$

$AC = \sqrt{1729}$

$= 41.5812\ldots$

$\doteqdot 41.6$ Km ✓ (no penalty for incorrect rounding)

Question 13 (b) (continued)

(iii) Find the bearing of Town A from Town C. **2**

$$\cos \angle BAC = \frac{37^2 + 1729 - 45^2}{2 \times 37 \times \sqrt{1729}}$$

$\therefore \angle BAC = 70°$ (nearest degree) ✓

(no penalty for using 41.6 km).

Bearing of A to C = $110°T$ ✓

$\therefore$ Bearing of C to A = $110° + 180° = 290°T$

(c) (i) Show that $f(x) = x^3 - 5x + \dfrac{xe^{-x^2}}{1+x^4}$ is an odd function. **2**

$$f(-a) = (-a)^3 - 5(-a) + \frac{-ae^{-(-a)^2}}{1+(-a)^4} \checkmark$$

$$= -a^3 + 5a - \frac{ae^{-a^2}}{1+a^4} \checkmark$$

$$= -\left(a^3 - 5a + \frac{ae^{-a^2}}{1+a^4}\right) = -f(a)$$

Hence odd function.

(ii) Hence evaluate $\displaystyle\int_{-5}^{5} \left(x^3 - 5x + \frac{xe^{-x^2}}{1+x^4} \right) dx$. **1**

$\int_{-5}^{5} f(x).dx = 0$ ✓

Question 14 (12 marks)

(a) A quantity Q of the radioactive isotope Carbon-14 decays according the equation of natural decay $Q = Q_0 e^{-kt}$ where Q_0 is the initial mass of Carbon-14, k is a constant and t is the time in years.
The half-life of a radioactive substance is the time it takes to reduce to half its original mass. Carbon-14 has a half-life of 5730 years.

(i) Find the exact value of k. **2**

when $Q = \frac{Q_0}{2}$, $t = 5730$

$\therefore \; Q_0/2 = Q_0 e^{-K \times 5730}$

$0.5 = e^{-K \times 5730}$ ✓

$\ln 0.5 = -5730 K$

$K = -\frac{\ln 0.5}{5730}$ OR $\frac{\ln 2}{5730}$ ✓

No penalty for writing $K = 0.0001$ (4 d.p.)

(ii) A fossilised leaf contains 9% of its original amount of Carbon-14. How old is the fossilised leaf? Give your answer correct to the nearest year. **2**

$0.09 \cancel{Q_0} = \cancel{Q_0} e^{-Kt}$

$0.09 = e^{\ln 0.5/5730 \times t}$ ✓ OR similar

$t = \frac{5730 \ln 0.09}{\ln 0.5}$

$\doteqdot$ 19906 years old ✓

(no penalty for incorrect rounding)

(b) (i) Simplify $\dfrac{\cos^2\theta}{1-\sin\theta}-\dfrac{\cos^2\theta}{1+\sin\theta}$ **2**

$$= \frac{\cos^2\theta(1+\sin\theta) - \cos^2\theta(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{\cos^2\theta + \cos^2\theta\sin\theta - \cos^2\theta + \cos^2\theta\sin\theta}{1-\sin^2\theta} \checkmark$$

$$= \frac{2\cancel{\cos^2\theta}\sin\theta}{\cancel{\cos^2\theta}} = 2\sin\theta \checkmark$$

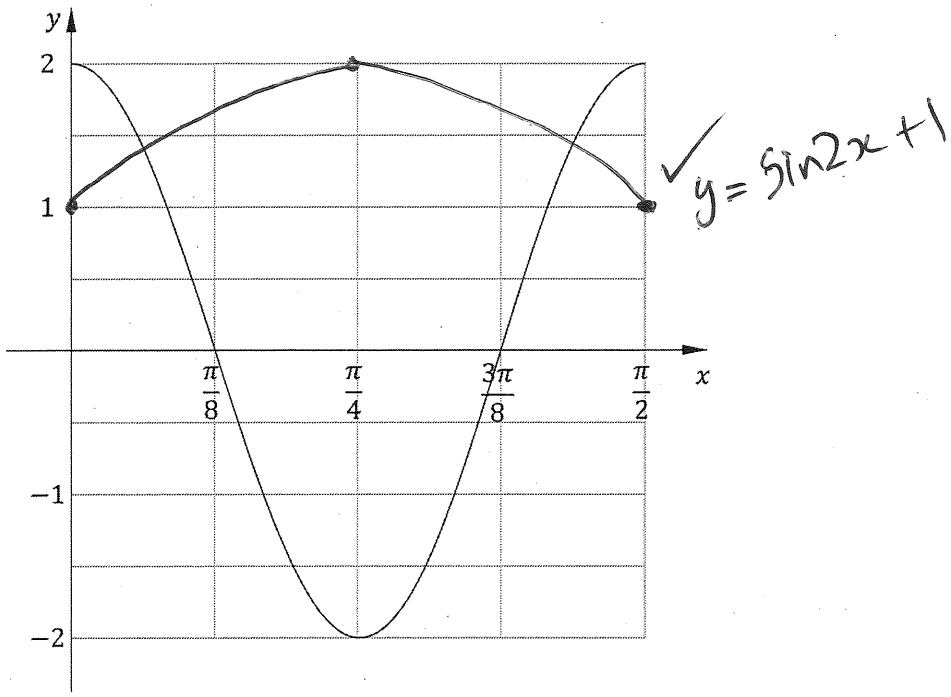
(ii) Hence, solve $\dfrac{\cos^2\theta}{1-\sin\theta}-\dfrac{\cos^2\theta}{1+\sin\theta}=1$ for $0\le\theta\le\dfrac{\pi}{2}$ **1**

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6} \checkmark$

(c) The graph shown below is $y = A\cos bx$



(i) Write down the values of A and b for this graph. **2**

$T = \frac{2\pi}{b}$ $\quad b = \frac{2\pi}{\pi/2} = 4.$ ✓

$A = 2$ ✓

(ii) On the axes above, draw the graph of. $y = \sin 2x + 1$ for $0 \le x \le \frac{\pi}{2}$ **1**

see above (✓)

(d) Simplify $\sqrt{x^2+2x+1} - \sqrt{x^2-2x+1}$. **2**

$= \sqrt{(x+1)^2} - \sqrt{(x-1)^2}$ ✓

$= (x+1) - (x-1)$

$= 2$ ✓

Question 15 (12 marks)

(a) Solve $\ln\left(2\sin^2\theta - \cos\theta\right) = 0$ for $0 \le \theta \le 2\pi$. **4**

Since $\ln X = 0$ when $X = 1$,

$2\sin^2\theta - \cos\theta = 1$ ✓

$2\sin^2\theta - \cos\theta - 1 = 0$

$2(1 - \cos^2\theta) - \cos\theta - 1 = 0$

$-2\cos^2\theta + 2 - \cos\theta - 1 = 0$

$\therefore\ 2\cos^2\theta + \cos\theta - 1 = 0$ ✓

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\therefore\ \cos\theta = \frac{1}{2}$ OR $\cos\theta = -1$ ✓

for $0 \leq \theta \leq 2\pi$

$\theta = \pi/3,\ 5\pi/3,\ \pi$ ✓

(b) (i) Find the stationary points and determine their nature, and any points of inflection for $y = x^4 - 2x^3 + 1$, showing all reasoning. **4**

$y' = 4x^3 - 6x^2$

$y'' = 12x^2 - 12x$

Stationary points occur when $y' = 0$

$4x^3 - 6x^2 = 0$

$2x^2(2x - 3) = 0$

$x = 0$ OR $x = \frac{3}{2}$ ✓

Test nature of stationary points:

$x = 0$ $y'' = 12 \times 0 - 12 \times 0$

$= 0$ $\therefore$ possible point of horizontal inflexion

$x = 3/2$ $y'' = 12 \times (3/2)^2 - 12 \times (3/2)$

$= 9$ $\therefore$ concave up ✓ correct test

minimum turning point @ $(3/2, -0.6875)$

OR $(3/2, -11/16)$

Possible pt of inflexion when $y'' = 0$

$12x^2 - 12x = 0$ $12x(x - 1)$

$\therefore x = 0$, $x = 1$. ✓

Test concavity

x	-1	0	0.5	1	2
y''	24	0	-3	0	24

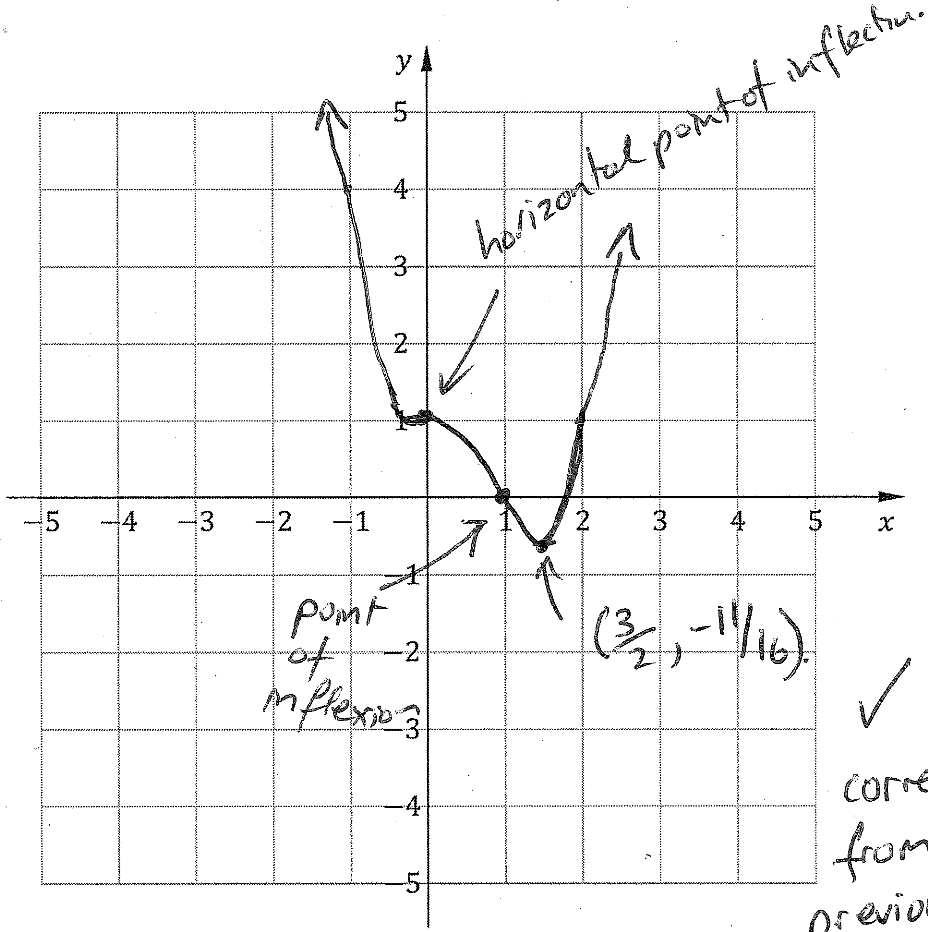
$\therefore$ Horizontal point of inflexion @ $(0, 1)$ ✓ correct identifying

point of inflexion @ $(1, 0)$

Question 15 (b) continues on next page

Question 15 (b) (continued)

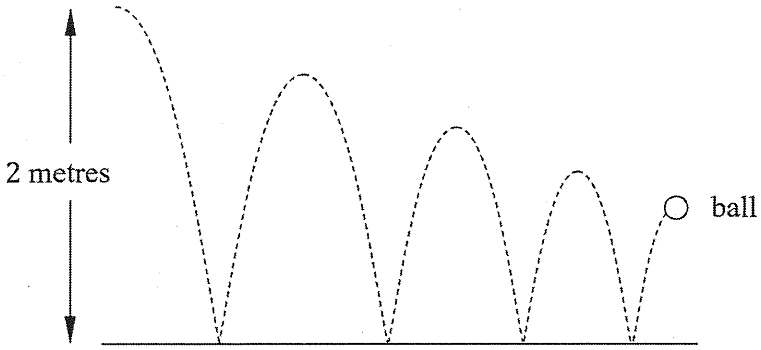
(ii) Sketch the graph of $y = x^4 - 2x^3 + 1$ on the axes below, clearly showing all important points including any turning points and points of inflection. It is not necessary to find all x-intercepts. **1**



✓ correct from previous

(c) The Moon has a lower gravity than Earth, and there is no atmosphere to cause air resistance, so a ball would bounce higher and for much longer on the Moon than on Earth. **3**

When a ball is dropped on the Moon, each bounce is 95% as high as the previous bounce. When an identical ball is dropped on Earth, each bounce is 50% as high as the previous bounce.



NOT TO SCALE

Two identical balls are dropped on the Moon and on Earth, each from a height of two metres, and they both eventually stop bouncing.
Calculate the difference in the total vertical distance travelled by these balls.

Total vertical distance for each bounce = twice its height + the 1st 2m.

$\therefore$ Limiting sum of a G.P.

MOON: $a = 2$ $r = 0.95$ subtracting initial 2m bounce up

$$D_{MOON} = \frac{2}{1-0.95} - 2 = 78 \text{ metres} \checkmark$$

EARTH: $a = 2$ $r = 0.5$

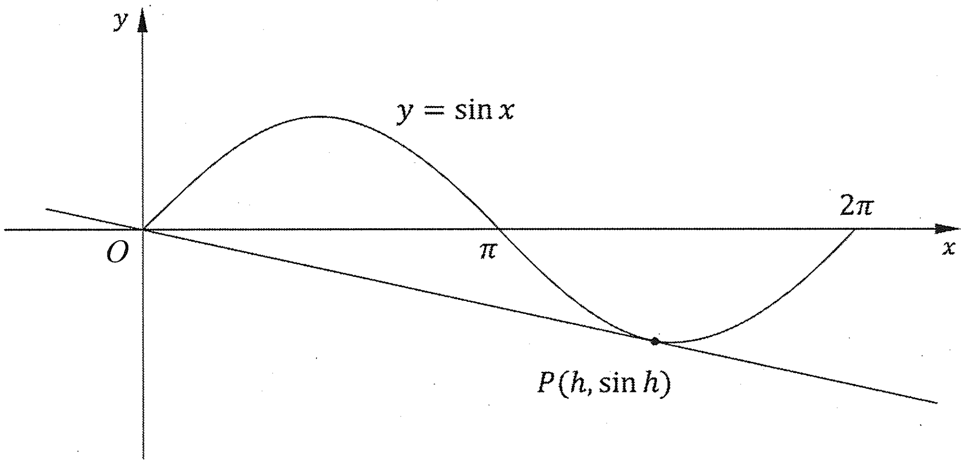
$$D_{EARTH} = \frac{2}{1-0.5} - 2 = 6 \text{ metres.} \checkmark$$

$\therefore$ Difference $78 - 6 = 72$ metres $\checkmark$

ALTERNATIVELY: if students neglect initial bounce up and find $80 - 8 = 72$m award full marks (with correct working)

Question 16 (10 marks)

(a) The graph $y = \sin x$ only has one tangent in the domain $[\pi, 2\pi]$ that passes through the origin as shown in the diagram below. **4**
Let the point of contact of this tangent be $P(h, \sin h)$.



Prove that $h = \tan h$.

$y' = \cos x$ ✓

at $x = h$ $y' = \cos h$. gradient $= \cos h$.

$\therefore$ $y - y_1 = m(x - x_1)$ $\therefore$ $y - \sin h = \cos h(x - h)$

$y = (\cos h)x - h\cos h + \sin h$ ✓

tangent passes through origin $(0,0)$

$\therefore$ $0 = 0 - h\cos h + \sin h$ ✓

$h\cos h = \sin h$

$h = \dfrac{\sin h}{\cos h} = \tan h.$ ✓

(b) The table below gives the present value interest factors for an annuity of $1 for various interest rates r (as a decimal), and the number of periods N.

Present value interest factors per $1					
	Interest rate r (as a decimal)				
N	**0.0025**	**0.005**	**0.0075**	**0.01**	**0.0125**
71	64.98140	59.64121	54.89293	50.66190	46.88363
72	65.81686	60.33951	55.47685	51.15039	47.29247
73	66.65023	61.03434	56.05643	51.63405	47.69627
74	67.48153	61.72571	56.63169	52.11292	48.09508
75	68.31075	62.41365	57.20267	52.58705	48.48897
76	69.13791	63.09815	57.76940	53.05649	48.87800

(i) Anson plans to invest $200 each month for 6 years at a rate of 3% p.a. compounded monthly. Use the table of values to calculate the present value of this annuity to the nearest dollar. **2**

$r = 3\% \div 12 = 0.0025$ ✓ , $n = 6 \times 12 = 72$

$PV = 65.81686 \times 200$

$= \$13\,163.37$ or $\$13\,163$ ✓

(ii) Monique uses the same table of values to calculate the loan repayments for a business loan for her dance school. She has borrowed $2 000 000 and she is repaying it in **quarterly** repayments over 19 years, at a rate of 5% p.a. Using the table of values above or otherwise, calculate the amount of each quarterly repayment, correct to the nearest dollar. **2**

$r = 5\% \div 12 = 0.0125$ $n = 19 \times 4 = 76$

$PV = PMT \times 48.87800$ ✓

$2\,000\,000 = PMT \times 48.878$

$PMT = \dfrac{2000000}{48.878} = \$40\,918$ ✓

(c) Give the simplest exact value of $\int_0^1 3^x\,dx$. **2**

$$\int_0^1 3^x\,dx = \left[\frac{3^x}{\ln 3}\right]_0^1 \checkmark$$

$$= \frac{3}{\ln 3} - \frac{1}{\ln 3}$$

$$= \frac{2}{\ln 3} \checkmark$$

Question 17 (10 marks)

(a) Use the trapezoidal rule, with 3 function values, to find the area between the curve $y = x\sin^2 x$, the x-axis, and the lines $x = 0$ and $x = \frac{\pi}{2}$ leaving your answer in exact terms. **3**

$y = x \sin^2 x$

$x = \pi/4,\ y = \pi/4 \times \sin^2 \pi/4$

$= \pi/4 \times \left(\frac{1}{\sqrt{2}}\right)^2$

$= \pi/8$

x	0	$\pi/4$	$\pi/2$
y	0	$\pi/8$	$\pi/2$

✓ for y-values

$$\int x\sin^2 x \, dx \doteqdot \frac{h}{2}\left\{f(0) + 2f(\pi/4) + f(\pi/2)\right\}$$

$h = \pi/4$

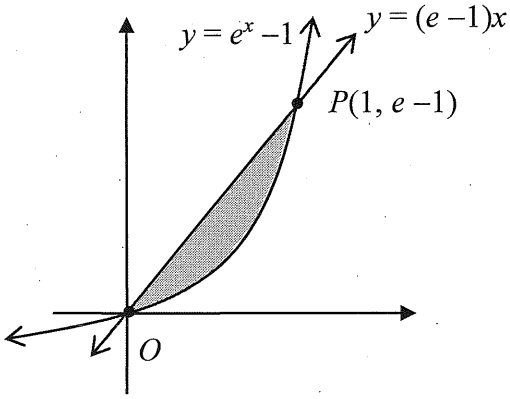
$$= \frac{\pi/4}{2}\left\{0 + 2 \times \pi/8 + \pi/2\right\} ✓$$

$$= \frac{\pi}{8} \times \left(3\pi/4\right)$$

✓ either.

$$= \frac{3\pi^2}{32}$$

(b) The curve $y = e^x - 1$ meets the line $y = (e-1)x$ at the origin and at the point $P(1, e-1)$, as shown in the diagram below. **3**



Find the exact value of the shaded area.

$$A = \int_0^1 \left((e-1)x - (e^x - 1)\right) . dx. \quad ✓$$

$$= \int_0^1 \left((e-1)x - e^x + 1\right) dx$$

$$= \left[\frac{(e-1)x^2}{2} - e^x + x\right]_0^1 \quad ✓$$

$$= \left[\frac{(e-1)1^2}{2} - e + 1 - (0 - 1 + 0)\right]$$

$$= \frac{e-1}{2} - e + 2$$

$$= \frac{3-e}{2}$$

✓ either

(c) The equation $f(x) = \cos\left(\frac{x}{2}\right)$ is a probability density function for $\frac{\pi}{3} \leq x \leq \pi$.

(DO NOT PROVE THIS)

(i) Using the definition of a cumulative distribution function **2**

$$F(x) = \int_a^x f(t)\,dt \text{ for } a \leq x \leq b$$

prove that the cumulative distribution function for the given domain is:

$$F(x) = 2\sin\left(\frac{x}{2}\right) - 1 \qquad \frac{\pi}{3} \leq x \leq \pi$$

$$\int_{\pi/3}^{x} \cos\frac{x}{2} \cdot dx$$

$$= \left[2\sin\frac{x}{2}\right]_{\pi/3}^{x} \checkmark$$

$$= \left[2\sin\frac{x}{2} - 2\sin\frac{\pi/3}{2}\right] \checkmark$$

$$= 2\sin\frac{x}{2} - 1 \quad \text{as required}$$

Question 17 (c) continues on the next page

(ii) Hence, or otherwise, find the median of this data, correct to one decimal place. **2**

For median $F(x) = 0.5$

$2 \sin \frac{x}{2} - 1 = \frac{1}{2}$ ✓

$2 \sin \frac{x}{2} = 3/2$

$\sin \frac{x}{2} = 3/4$

$x/2 = \sin^{-1} 3/4$

$x = 2 \sin^{-1} 3/4$

$\doteq 1.7$ ✓

Question 18 (10 marks)

(a) Sergio decides to buy a top-of-the-line gaming laptop for \$12 500 from the retailer Jay-Bee. Jay-Bee offers the following terms:

- The customer pays for the laptop with equal monthly repayments of \$$R$ at the end of each month for 5 years.
- The first six months are interest free. After six months, interest is charged on the balance owing at 9% p.a. compounded monthly.
- The equal monthly repayment is still made at the end of each month during the first six months.

(i) Let A_n be the amount owing after n months. **3**
Show that after seven months, the total amount Sergio owes is:

$$A_7 = 12500 \times 1.0075 - R(6 \times 1.0075 + 1)$$

$A_1 = 12\,500 - R$

$A_2 = (12\,500 - R) - R = 12\,500 - 2R$

$A_3 = A_2 - R = 12500 - 3R$ ✓

$\therefore A_6 = 12\,500 - 6R$

$r = \frac{0.09}{12} = 0.0075$

$A_7 = A_6 \times 1.0075 - R$ ✓

$= (12500 - 6R) \times 1.0075 - R$

$= 12500 \times 1.0075 - 6R \times 1.0075 - R$ ✓

$= 12\,500 \times 1.0075 - R(6 \times 1.0075 + 1)$

as required

Question 18 (a) continues on the next page

Question 18 (a) (continued)

(ii) Calculate the monthly repayment $\$R$ required to pay back the loan, correct to the nearest dollar. **4**

$A_8 = A_7 \times 1{\cdot}0075 - R$

$= 12500 \times 1{\cdot}0075^2 - R(6 \times 1{\cdot}0075^2 + 1{\cdot}0075) - R$

$= 12500 \times 1{\cdot}0075^2 - R(6 \times 1{\cdot}0075^2 + 1{\cdot}0075 + 1)$

$A_9 = A_8 \times 1{\cdot}0075 - R$

$= 12500 \times 1{\cdot}0075^3 - R(6 \times 1{\cdot}0075^3 + 1{\cdot}0075^2 + 1{\cdot}0075) - R$

develops series ✓

$= 12500 \times 1{\cdot}0075^3 - R(6 \times 1{\cdot}0075^3 + 1 + 1{\cdot}0075 + 1{\cdot}0075^2)$

Hence $A_n = 12500 \times 1{\cdot}0075^{n-6} - R(6 \times 1{\cdot}0075^{n-6} + 1 + 1{\cdot}0075 + 1{\cdot}0075^2 + \ldots + 1{\cdot}0075^{n-7})$

$n = 5 \times 12 = 60$

✓ $A_{60} = 12500 \times 1{\cdot}0075^{54} - R(6 + 1{\cdot}0075^{54} + (1 + 1{\cdot}0075 + \ldots + 1{\cdot}0075^{53}))$

correct for A_{60}

GP with $n = 54$
$a = 1$, $r = 1{\cdot}0075$

Also, $A_{60} = 0$

$\therefore\ 12500 \times 1{\cdot}0075^{54} = R\left(6 \times 1{\cdot}0075^{54} + \dfrac{1(1{\cdot}0075^{54} - 1)}{1{\cdot}0075 - 1}\right)$ ✓

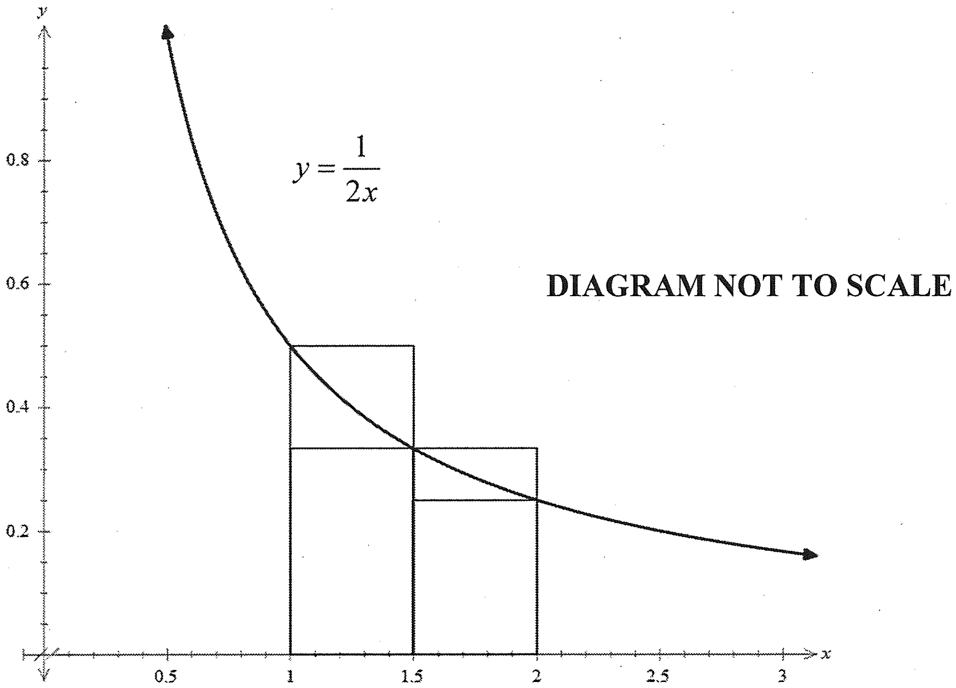
correct equation

$R = 12500 - 1{\cdot}0075^{54} \div \left(6 \times 1{\cdot}0075^{54} + \dfrac{1{\cdot}0075^{54} - 1}{0{\cdot}0075}\right)$

$= \$248{\cdot}66 \doteqdot \$249.$

✓ ANSWER.

(b) The graph of $y = \dfrac{1}{2x}$ is shown below. **3**



Upper and lower rectangles of width 0.5 units are drawn from the x-axis to the curve between $x = 1$ and $x = 2$ as shown in the diagram.

By considering the areas of the rectangles, show that $\dfrac{7}{24} < \ln\sqrt{2} < \dfrac{5}{12}$.

lower bound rectangle areas $< \int_1^2 \frac{1}{2x}\,dx <$ upper bound rectangle areas.

$x=1,\ y=\frac{1}{2}$; $x=1.5,\ y=\frac{1}{3}$; $x=2,\ y=\frac{1}{4}$

Hence: Area of lower rectangles

$= \frac{1}{3}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{2} = \frac{7}{24}$

Area of uppe rectangles

$= \frac{1}{2}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} = \frac{5}{12}$

✓ for both

$\int_1^2 \frac{dx}{2x} = \frac{1}{2}\left[\ln x\right]_1^2 = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$ ✓

$= \ln\sqrt{2}$ ✓

Henee $\frac{7}{24} < \ln\sqrt{2} < \frac{5}{12}$

Working space continues on the next page