SYDNEY
BOYS
HIGH
SCHOOL

NESA Number:

Name: ____________________

Tick next to your teacher's name

RCW	
JC	
WB	
MZ	

2025

YEAR 12
HSC TASK 4
TRIAL HSC

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- All answers, unless otherwise stated, should be left in simplified, exact form. Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations. Answers that rely totally on calculator technology may not necessarily receive full marks.

Total Marks: 100

Section I – 10 marks (pages 2 – 5)
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 38)
- Attempt all Questions in Section II
- Allow about 2 hours and 45 minutes for this section

Examiner: *External Examiner (AMG)*

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

1 Which of the following is the same as $\operatorname{cosec}(\pi + \theta)$?

A. $\dfrac{-1}{\sin\theta}$

B. $\dfrac{-1}{\cos\theta}$

C. $\dfrac{1}{\cos\theta}$

D. $\dfrac{1}{\sin\theta}$

2 Given that $f(x) = \dfrac{4x^5 - 8x}{x^3}$, what is the value of $f'(2)$?

A. 2

B. 8

C. 12

D. 18

3 The point $A(4, 2)$ lies on the graph of $y = f(x)$.

What are the coordinates of the point A', the image point of A on the graph of $y = g(x)$ if $g(x) = -f\left(\dfrac{x}{2}\right)$?

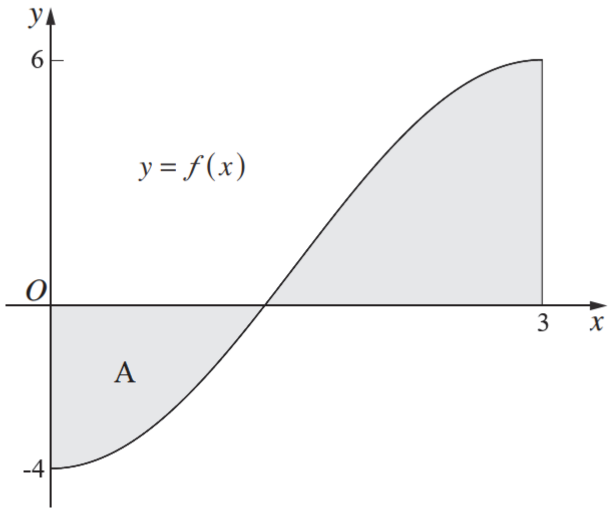
A. (4, –2)

B. (–4, 2)

C. (–8, 2)

D. (8, –2)

4 A graph of the function $y = f(x)$ is shown below.



It is given that:

- Area A is equal to 3 units^2
- $\displaystyle\int_0^3 f(x)\,dx = 4$

What is the area that the graph of $y = f(x)$ makes with the y-axis ?

A. 4

B. 10

C. 14

D. 20

5 The position, x metres, of a particle moving in a straight line from a fixed origin O at time, t seconds, is given by

$$x = e^{(k-1)t},$$

where $k > 1$.

What is the acceleration of the particle when $x = k + 1$?

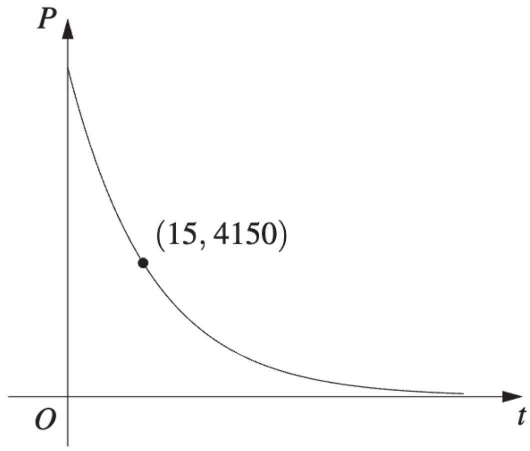
A. $k^2 - 1$

B. $(k^2 - 1)(k + 1)$

C. $(k^2 - 1)(k - 1)$

D. $(k - 1)^2$

6 The population, P, of an agricultural pest can be modelled by the equation $P = P_0 e^{-0.32t}$, where t is the time, in days, since the use of a new insecticide is introduced.



At what rate is the population of pests decreasing after 15 days?

A. 277 pests/day

B. 1328 pests/day

C. 1500 pests/day

D. 12 696 pests/day

7 Let $f(x)$ be a continuous odd function.

Given that $f(x - a)$ is an even function, which of the following must be true?

A. $f(x - a) = f(a - x)$

B. $f(x - a) + f(x + a) = 0$

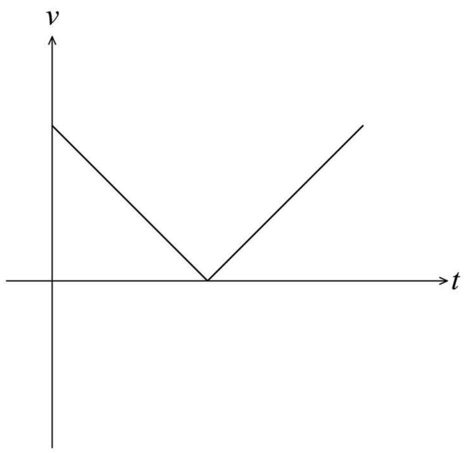
C. $f(0) > 0$

D. None of the above

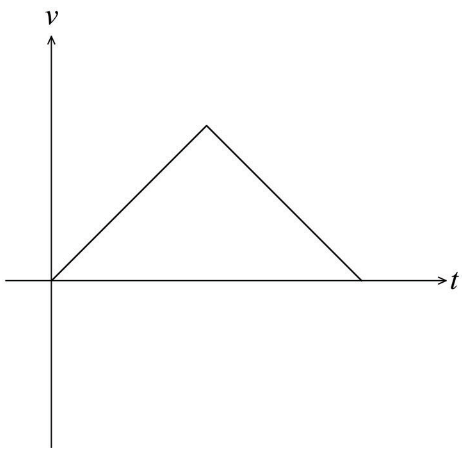
8 A body is thrown vertically upwards.

Which one of the following graphs correctly represent the velocity-time graph ?

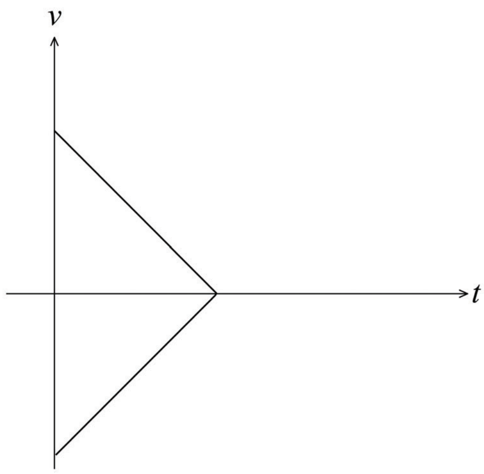
A.



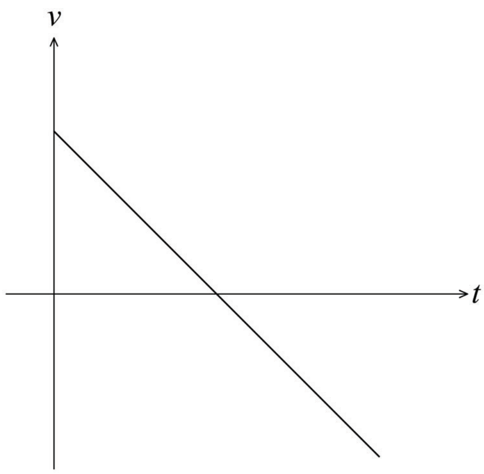
B.



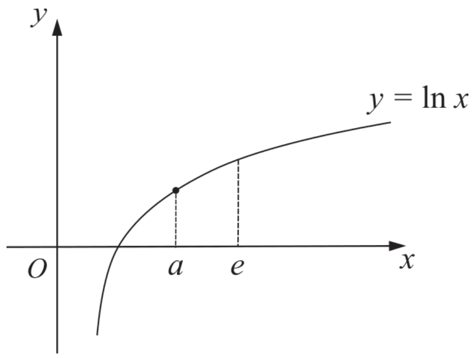
C.



D.



9 The line $y = mx + c$ is a tangent to the graph $y = \ln x$ at the point where $x = a$, as shown in the diagram below.



Which of the following statements is true ?

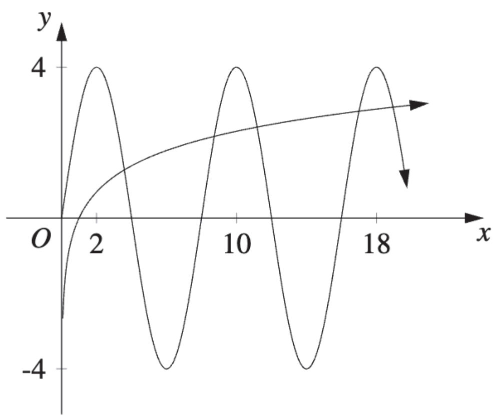
A. $\frac{1}{e} < m < 1$ and $-1 < c < 0$

B. $1 < m < e$ and $-1 < c < 0$

C. $\frac{1}{e} < m < 1$ and $0 < c < 1$

D. $1 < m < e$ and $0 < c < 1$

10 The graphs of $y = 4\sin\left(\frac{\pi}{4}x\right)$ and $y = \ln x$ are shown on the number plane below.



How many solutions exist to the equation $4\sin\left(\frac{\pi}{4}x\right) = \ln x$ in the domain $0 \le x \le 50$?

A. 10

B. 11

C. 12

D. 13

End of Section I

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Y12 Mathematics Advanced

HSC Task #4 (THSC)

Part A

Section II

Part A **14 marks**
Attempt Questions 11–17

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

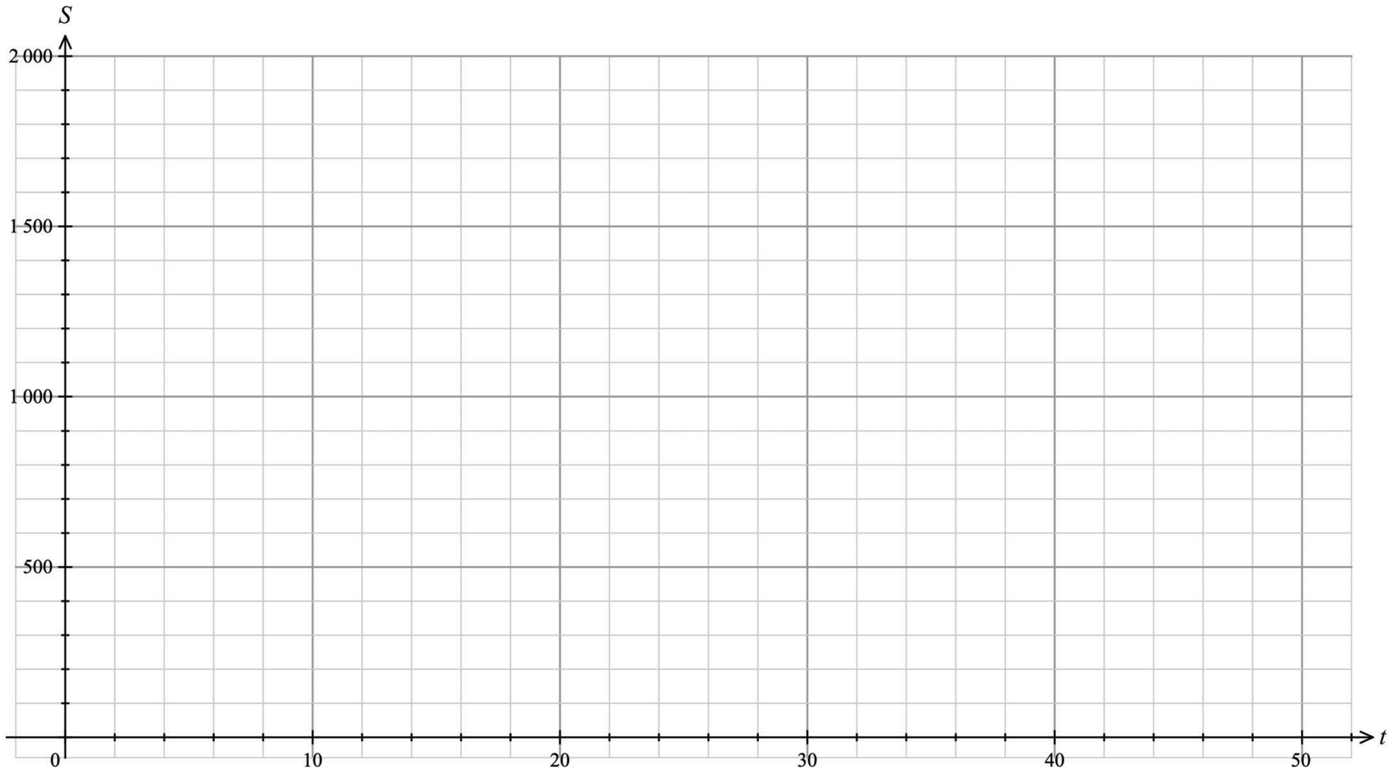
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (4 marks)

Jayme has \$100. He starts to save \$25 each week to buy a new racing bike.

His savings can be modelled by $S = 100 + 25t$, where S is the dollars saved and t is the time in weeks from when he started saving the money.

(a) On the grid below, draw the graph of this model and label it as Jayme's savings. **1**



(b) Jorge, Jayme's brother, wanted to start saving 8 weeks after Jayme began saving. He initially had no money but saves \$40 each week. **2**

By drawing a line on the grid above, or otherwise, find the value of t when the two brothers have the same money.

(c) Using the graph, or otherwise, find the value of t when Jorge's savings is \$300 more than Jayme's savings. **1**

Question 12 (2 marks)

Triangle ABC has $\angle A = 38°$, $BC = 16$ cm, and $AC = 18$ cm. **2**

Find $\angle B$, where $\angle B$ is obtuse, to the nearest minute.

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Question 13 (2 marks)

If $f(x) = \ln(\cos x)$, find the value of $f'\left(\frac{\pi}{4}\right)$. **2**

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Question 14 (2 marks)

Find $\frac{d}{dx}\left(\sqrt{x}e^{x}\right)$. **2**

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Question 15 (1 mark)

Find a primitive function y given that $\dfrac{dy}{dx} = \dfrac{2x}{3x^2 + 1}$. **1**

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Question 16 (1 mark)

Evaluate $\displaystyle\int \frac{6}{3^x}\, dx$. **1**

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Question 17 (2 marks)

Find the exact value of $\displaystyle\int_{\frac{1}{3}}^{\frac{1}{2}} \sec^2\left(\frac{\pi x}{2}\right) dx$. **2**

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End of Part A

NESA Number									

Y12 Mathematics Advanced

HSC Task #4 (THSC)

Part B

Section II

Part B **15 marks**
Attempt Questions 18 – 22

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 18 (4 marks)

Solve the simultaneous equations: **4**

$$x + 2y = 12$$
$$5x + y^2 = 39$$

Question 19 (2 marks)

The wavelength, λ metres, of a musical tone is inversely proportional to its frequency, f vibrations per second.

The frequency of middle C is approximately 260 vibrations per second and its wavelength is approximately 1.32 m.

(a) Write an equation to represent this, in the form $\lambda = g\,(f)$. **1**

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(b) Hence, find the frequency, in vibrations per second, of a sound wave with wavelength 0.8 m. **1**

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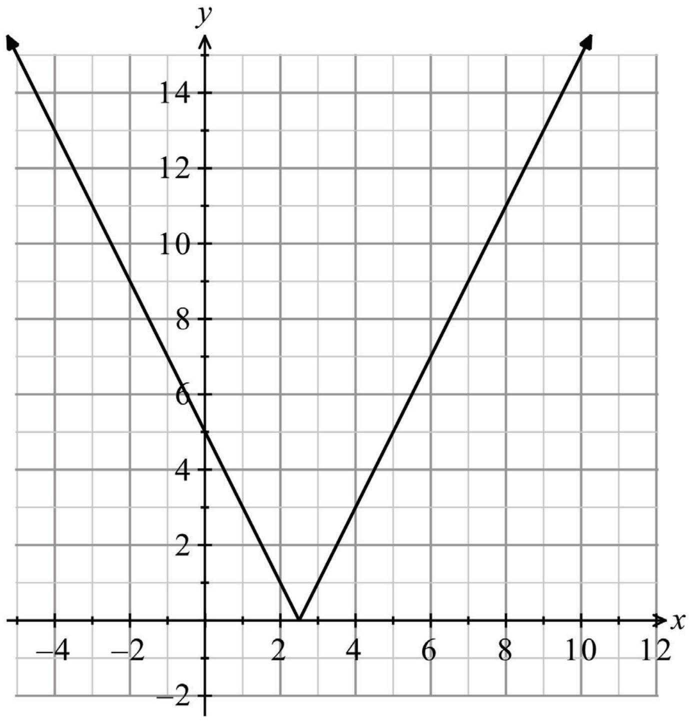
Question 20 (1 mark)

The graph of $y = |2x - 5|$ is shown below. By using it, solve $|2x-5| \leq 13$. **1**



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Question 21 (4 marks)

Consider the series $S_n = -18 + (-15) + (-12) + ... + u_n$.

(a) What is the value of u_7? **1**

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(b) What is the value of S_7? **1**

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(c) For what value of n is $S_n = 0$? **2**

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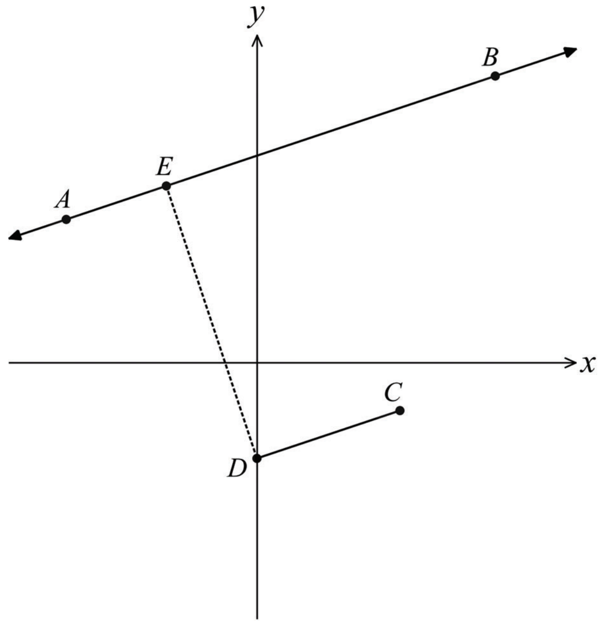
Question 22 (4 marks)

The points $A(-4, 3)$, $B(5, 6)$, $C(3, -1)$ and $D(0, -2)$ are shown on the diagram below.



NOT DRAWN TO SCALE

The point E lies on AB, such that $ED \perp AB$.

(a) Show that $AB \parallel DC$. **2**

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(b) Show that the equation of AB is $x - 3y + 13 = 0$. **1**

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(c) Find the equation of ED. **1**

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End of Part B

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Y12 Mathematics Advanced

HSC Task #4 (THSC)

Part C

Section II

Part C **16 marks**
Attempt Questions 23 – 28

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 23 (6 marks)

A curve has equation $y = x^3 - kx^2 + 1$, where k is a real constant.
When $x = 2$, the gradient of the curve is 6.

(a) Show that $k = 1.5$ **1**

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(b) Find the coordinates of the two stationary points, and determine their nature. **3**

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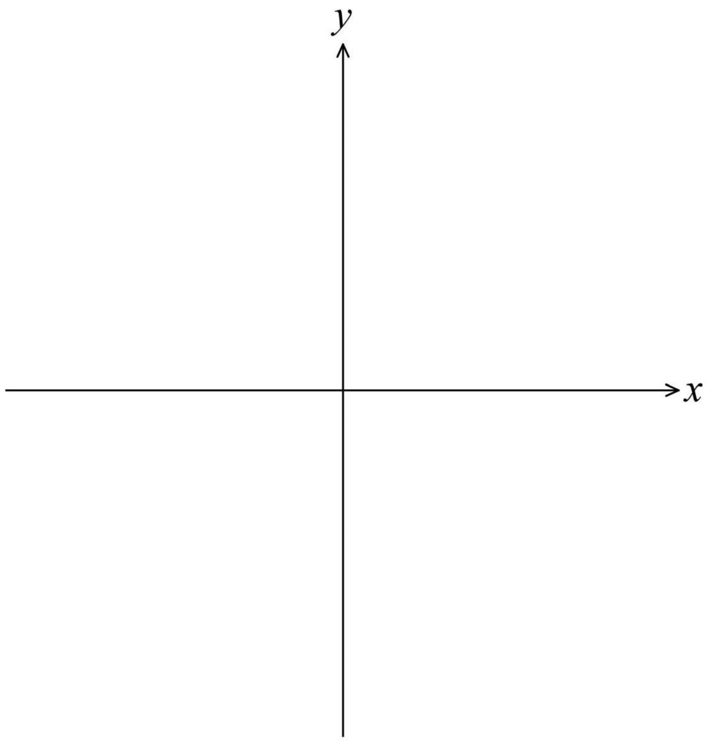
Question 23 continues on page 20

Question 23 (continued)

(c) Sketch the curve $y = x^3 - 1.5x^2 + 1$, indicating all of the above and the y-intercept. **2**



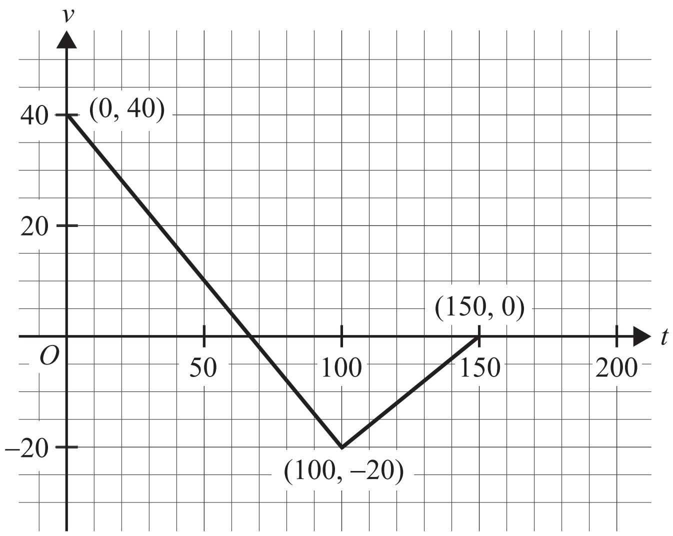
End of Question 23

Question 24 (2 marks)

Find $\displaystyle\int \frac{x}{\sqrt{3x^2+5}}\,dx$. **2**

Question 25 (2 marks)

The velocity-time graph of a particle moving along an east-west line with velocity v m/s at t seconds, starting 20 metres west of the origin, is shown below. **2**



How far, in metres, is the particle from the origin, 150 seconds later?

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Question 26 (2 marks)

Find $\dfrac{d}{dx}\left[\ln\left(e^x \sin x\right)\right]$. **2**

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Question 27 (2 marks)

Find $\int \cot 2x + \frac{1}{x\sqrt{x}}\,dx$ **2**

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Question 28 (2 marks)

An arithmetic sequence has the first term 5 and a common difference of d. **2**
The sum of the first 20 terms is 4 times the sum of the first 10 terms.

Find the value of d.

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End of Part C

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Y12 Mathematics Advanced

HSC Task #4 (THSC)

Part D

Section II

Part D **16 marks**
Attempt Questions 29 – 32

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 29 (2 marks)

Find $\displaystyle\int \frac{16-8x}{3x^2-12x+1}\,dx$. **2**

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Question 30 (3 marks)

The sum of an infinite geometric series, with positive terms, is 3. **3**

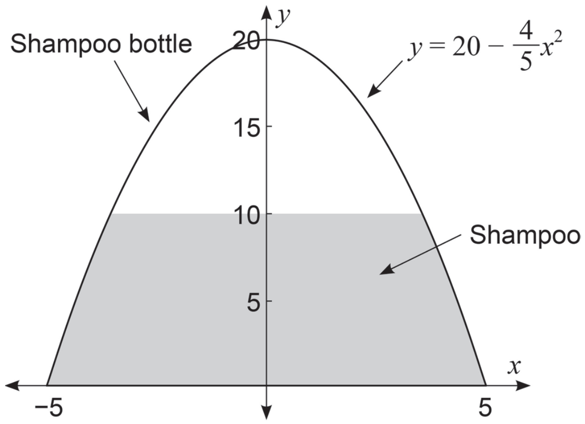
The sum of the cubes of its terms is $\frac{27}{19}$. What is the common ratio of the original series?

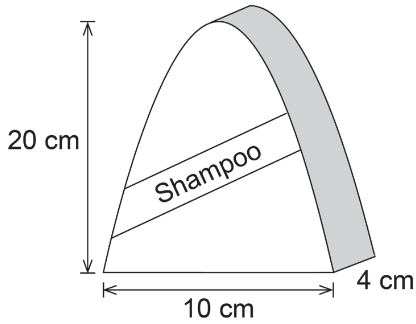
Question 31 (3 marks)

Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin x - \sin^3 x}\, dx$. **3**

Question 32 (8 marks)

A partially filled shampoo bottle has a uniform parabolic cross-section as shown in the diagram below.





Both x and y are measured in centimetres.
The width of the bottle perpendicular to the parabolic cross-section is 4 cm.

(a) Show that the volume V of shampoo in the bottle, if it is partially filled to a height of 10 cm, is **4**

$$\frac{1600 - 400\sqrt{2}}{3} \text{ cm}^3$$

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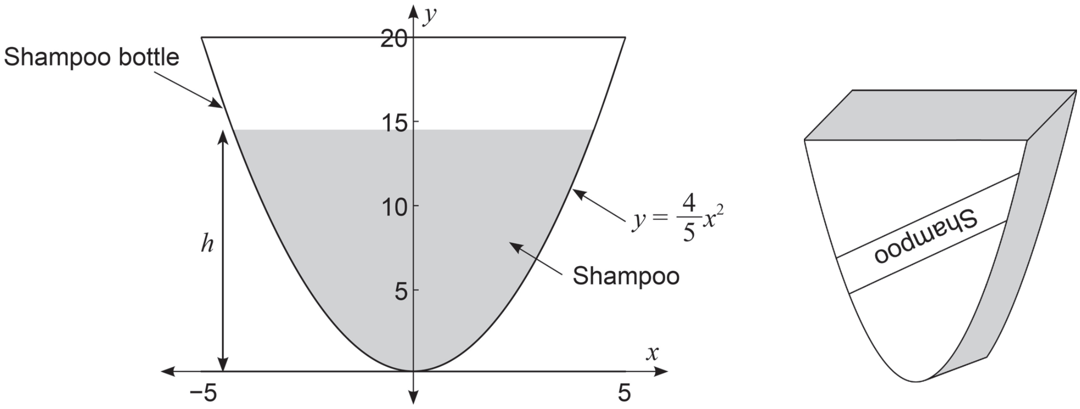
Question 32 continues on page 29

Question 32 (continued)

The bottle is turned upside down and the entire volume of the shampoo allowed to settle, as shown in the diagram below.



(b) Show that the shampoo level h is given by **4**

$$\left(40\sqrt{5} - 10\sqrt{10}\right)^{\frac{2}{3}} \text{ cm}$$

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End of Part D

NESA Number										

Y12 Mathematics Advanced

HSC Task #4 (THSC)

Part E

Section II

Part E 17 marks
Attempt Questions 33 – 35

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 33 (4 marks)

The table below shows the future value of annuity with a contribution of $1 at the end of each period. **4**

Period	0.75%	1.00%	1.50%	2.00%	2.50%	3.00%
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	2.00750	2.01000	2.01500	2.02000	2.02500	2.03000
3	3.02256	3.03010	3.04523	3.06040	3.07563	3.09090
4	4.04523	4.06040	4.09090	4.12161	4.15252	4.18363
5	5.07556	5.10101	5.15227	5.20404	5.25633	5.30914
6	6.11363	6.15202	6.22955	6.30812	6.38774	6.46841
7	7.15948	7.21354	7.32299	7.43428	7.54743	7.66246
8	8.21318	8.28567	8.43284	8.58297	8.73612	8.89234

Kevin opened an account and deposited $15 000 at the end of each year for 6 years.

At the end of the 7th year, he deposited $20 000 in the same account and continued to deposit $20 000 at end of each year until the end of the 10th year.

He made no more deposits after the end of the 10th year.
The money in his account was earning interest at 2.5% per annum compounded annually.

Question 34 continues on page 33

Question 33 (continued)

How much money will Kevin have in his account at the end of 15 years?

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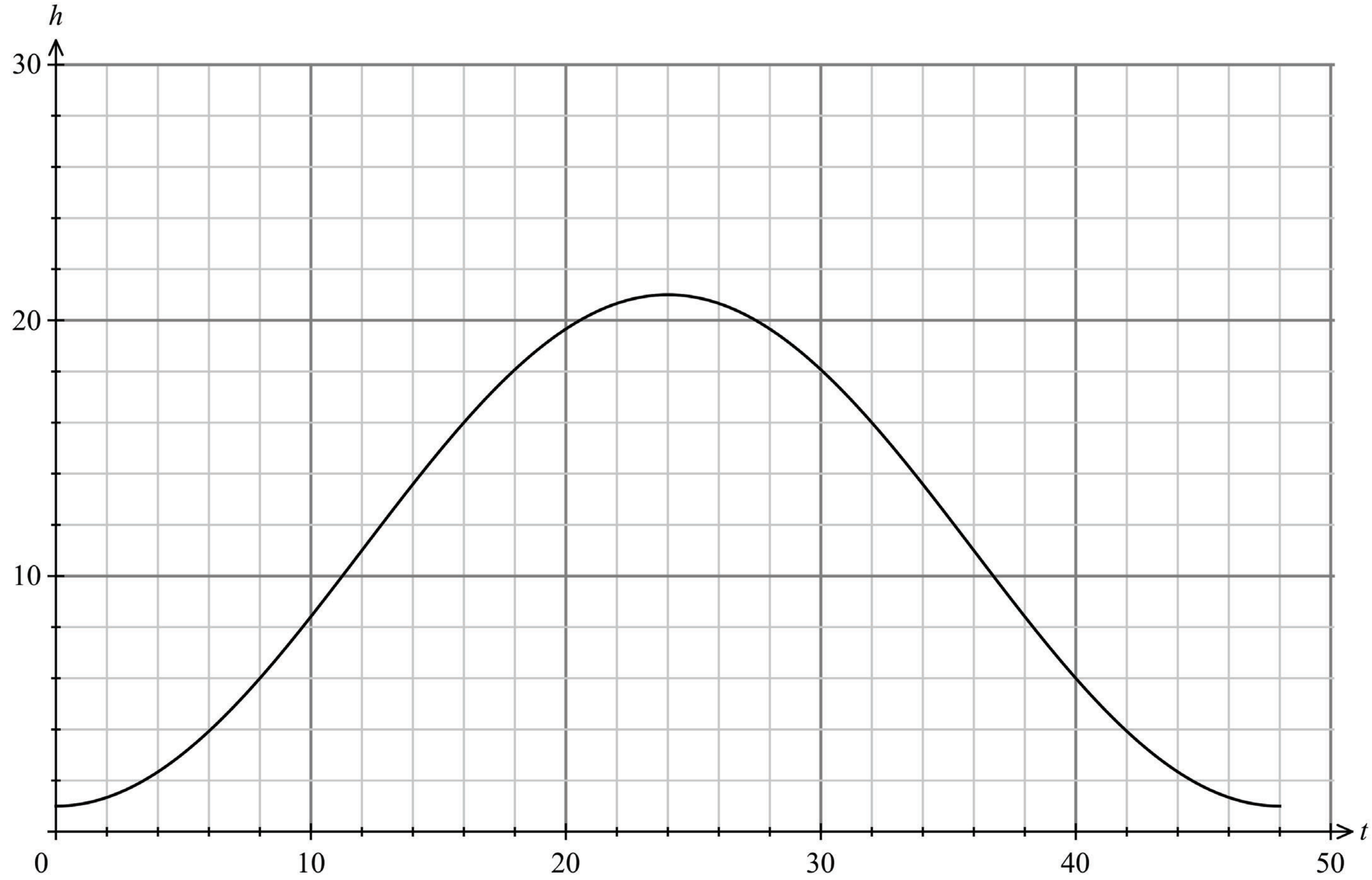
End of Question 34

Question 34 (7 marks)

The height of a car on a slow Ferris wheel in Funpark can be modelled by the formula
$h_1(t) = a\sin\left(\frac{\pi}{24}(t-c)\right)+11$, where t is the time in seconds and $0 \le t \le 48$.

The height of the car at $t = 0$ is 1 m and it reaches a maximum height of 21 m when $t = 24$, and then it decreases back to 1 m at $t = 48$.

The graph of $h_1(t)$ is as shown.



(a) What are the values of a and c? **2**

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Question 35 continues on page 35

Question 34 (continued)

(b) In Funpark there is a faster Ferris wheel than the one in part (a). **3**

The height of a car on this Ferris wheel can be modelled by the formula

$$h_2(t) = 12\sin\left(\frac{\pi}{8}(t-4)\right) + 13.$$

By drawing the graph of $h_2(t)$ on the graph of $h_1(t)$ find the values of the time t in $0 \le t \le 48$ at which the height of both cars is decreasing.

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(c) Find the rate of change of the height of the car on the slower Ferris wheel when the car on the faster Ferris wheel reaches its minimum height for the first time after its initial starting point. **2**

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Question 35 (6 marks)

(a) Find $\dfrac{d}{dx}\left(xe^{-x}\right)$. **1**

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(b) Hence, find $\displaystyle\int xe^{-x}\,dx$. **1**

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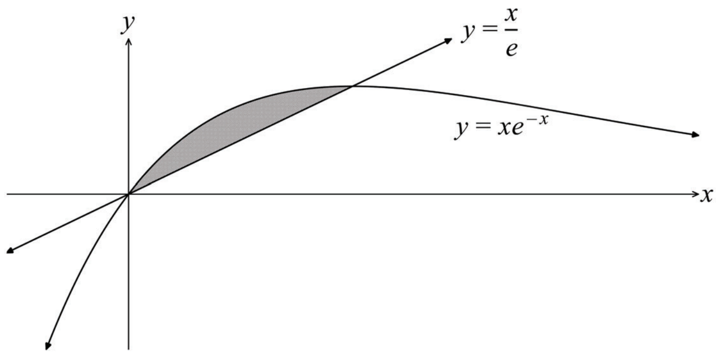
(c) The diagram below shows the shaded region bounded by the function $y = xe^{-x}$ and the line $y = \dfrac{x}{e}$. By using part (b), find the exact area of the shaded region. **4**



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End of Part E

NESA Number									

Y12 Mathematics Advanced

HSC Task #4 (THSC)

Part F

Section II

Part F **12 marks**
Attempt Questions 36 – 38

Answer each question in the space provided. A blank page is provided at the end of this question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 36 (6 marks)

A continuous random variable has the probability density function $f(x)$ given by

$$f(x) = \begin{cases} \dfrac{kx}{5-x^2}, & 0 \le x \le 2 \\ 0, & \text{for all other values of } x \end{cases}$$

where k is a positive constant.

(a) Show that the value of k is $\dfrac{2}{\ln 5}$. **2**

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Question 36 continues on page 40

Question 36 (continued)

(b) Find the cumulative distribution function, $F(x)$. **2**

(c) Hence, or otherwise, show that the median of X is $\sqrt{5-\sqrt{5}}$. **2**

End of Question 36

Question 37 (3 marks)

A continuous random variable, X, has a cumulative distribution function given by: **3**

$$F(x) = \frac{1}{16}x^2\left(5 - 2\sqrt{x}\right) \text{ for } 0 \le x \le 4.$$

Find the (exact) mode of X.

Question 38 (3 marks)

(a) Find the value of μ such that **1**

$$n(n+1)(n+2)(n+3) = (n^2 + \mu n)(n^2 + \mu n + 2).$$

(b) Let n be an integer. **2**

Deduce that $2022 \times 2023 \times 2024 \times 2025 + 1$ is a perfect square.

End of paper

SYDNEY
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2025

YEAR 12

HSC TASK 4

Mathematics Advanced

Sample Solutions

NOTE:

This process of checking your mark is about reading the solutions and the comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown some 'working' does not justify that your solution is worth any marks.

Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

MC Answers

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
A	D	D	C	C	B	B	D	A	C

Section I Multiple Choice Solutions

1 Which of the following is the same as $\operatorname{cosec}(\pi + \theta)$?

A. $\dfrac{-1}{\sin\theta}$

B. $\dfrac{-1}{\cos\theta}$

C. $\dfrac{1}{\cos\theta}$

D. $\dfrac{1}{\sin\theta}$

3rd quadrant results: $\sin\theta < 0$

$\operatorname{cosec}\theta$ is the reciprocal of $\sin\theta$.

A	63
B	11
C	6
D	6

2 Given that $f(x) = \dfrac{4x^5 - 8x}{x^3}$, what is the value of $f'(2)$?

A. 2

B. 8

C. 12

D. 18

$$f(x) = 4x^2 - 8x^{-2}$$
$$f'(x) = 8x + 16x^{-3}$$
$$f'(2) = 8 \times 2 + 16 \times 2^{-3} = 18$$

A	2
B	2
C	5
D	77

3 The point $A(4, 2)$ lies on the graph of $y = f(x)$.

What are the coordinates of the point A', the image point of A on the graph of $y = g(x)$ if $g(x) = -f\left(\dfrac{x}{2}\right)$?

A. $(4, -2)$

B. $(-4, 2)$

C. $(-8, 2)$

D. $(8, -2)$

The negative y-value, means it must be A or D.

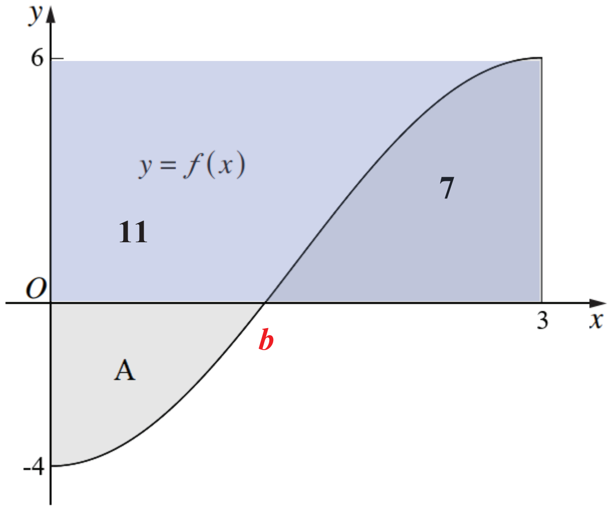
The positive x-value means it must be D.

OR

$$\frac{x}{2} = 4 \Rightarrow x = 8$$
$$g(8) = -f(4) = -2$$

A	6
B	1
C	15
D	64

4 A graph of the function $y = f(x)$ is shown below.



$$\int_0^3 f(x)\,dx = \int_0^b f(x)\,dx + \int_b^3 f(x)\,dx$$

$$\therefore\ 3 = -4 + \int_b^3 f(x)\,dx$$

$$\therefore\ \int_b^3 f(x)\,dx = 7$$

Area of rectangle $= 6 \times 3 = 18$

$\therefore$ remaining area of rectangle $= 18 - 7 = 11$

$\therefore$ Area next to y-axis $= 11 + 3 = 14$

It is given that:

- Area A is equal to 3 units^2
- $\displaystyle\int_0^3 f(x)\,dx = 4$

What is the area that the graph of $y = f(x)$ makes with the y-axis ?

A. 4

B. 10

C. 14

D. 20

Area of rectangle $= 6 \times 3 = 18$

Area next to y-axis $= 11 + 3 = 14$

A	1
B	40
C	40
D	5

5 The position, x metres, of a particle moving in a straight line from a fixed origin O at time, t seconds, is given by

$$x = e^{(k-1)t},$$

where $k > 1$.

What is the acceleration of the particle when $x = k + 1$?

A. $k^2 - 1$

B. $(k^2 - 1)(k + 1)$

C. $(k^2 - 1)(k - 1)$

D. $(k - 1)^2$

$v = (k-1)\,e^{(k-1)t} \Rightarrow a = (k-1)^2\,e^{(k-1)t}$

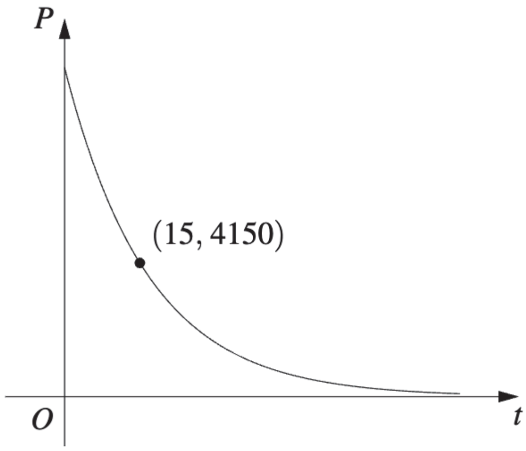
$x = k + 1 \Rightarrow k + 1 = e^{(k-1)t}$

$\therefore a = (k-1)^2 \times e^{(k-1)t} = (k-1)^2 \times (k+1)$

$\therefore a = (k^2-1)(k-1)$

A	8
B	23
C	50
D	5

6 The population, P, of an agricultural pest can be modelled by the equation $P = P_0 e^{-0.32t}$, where t is the time, in days, since the use of a new insecticide is introduced.



At what rate is the population of pests decreasing after 15 days?

A. 277 pests/day

B. 1328 pests/day

C. 1500 pests/day

D. 12 696 pests/day

$$P = P_0 e^{-0.32t} \Rightarrow \frac{dP}{dt} = -0.32\,P$$

$$t = 15 \Rightarrow \frac{dP}{dt} = -0.32 \times 4150 = -\ 1328$$

A	12
B	67
C	4
D	3

7 Let $f(x)$ be a continuous odd function.

Given that $f(x-a)$ is an even function, which of the following must be true?

A. $f(x-a) = f(a-x)$

B. $f(x-a) + f(x+a) = 0$

C. $f(0) > 0$

D. None of the above

Let $g(x) = f(x-a)$

$g(-x) = g(x)$:

$\therefore f(-x-a) = f(x-a)$

$\therefore f[-(x+a)] = f(x-a) \Rightarrow -f(x+a) = f(x-a)$ [$f(x)$ is odd]

$\therefore\ f(x-a) + f(x+a) = 0$

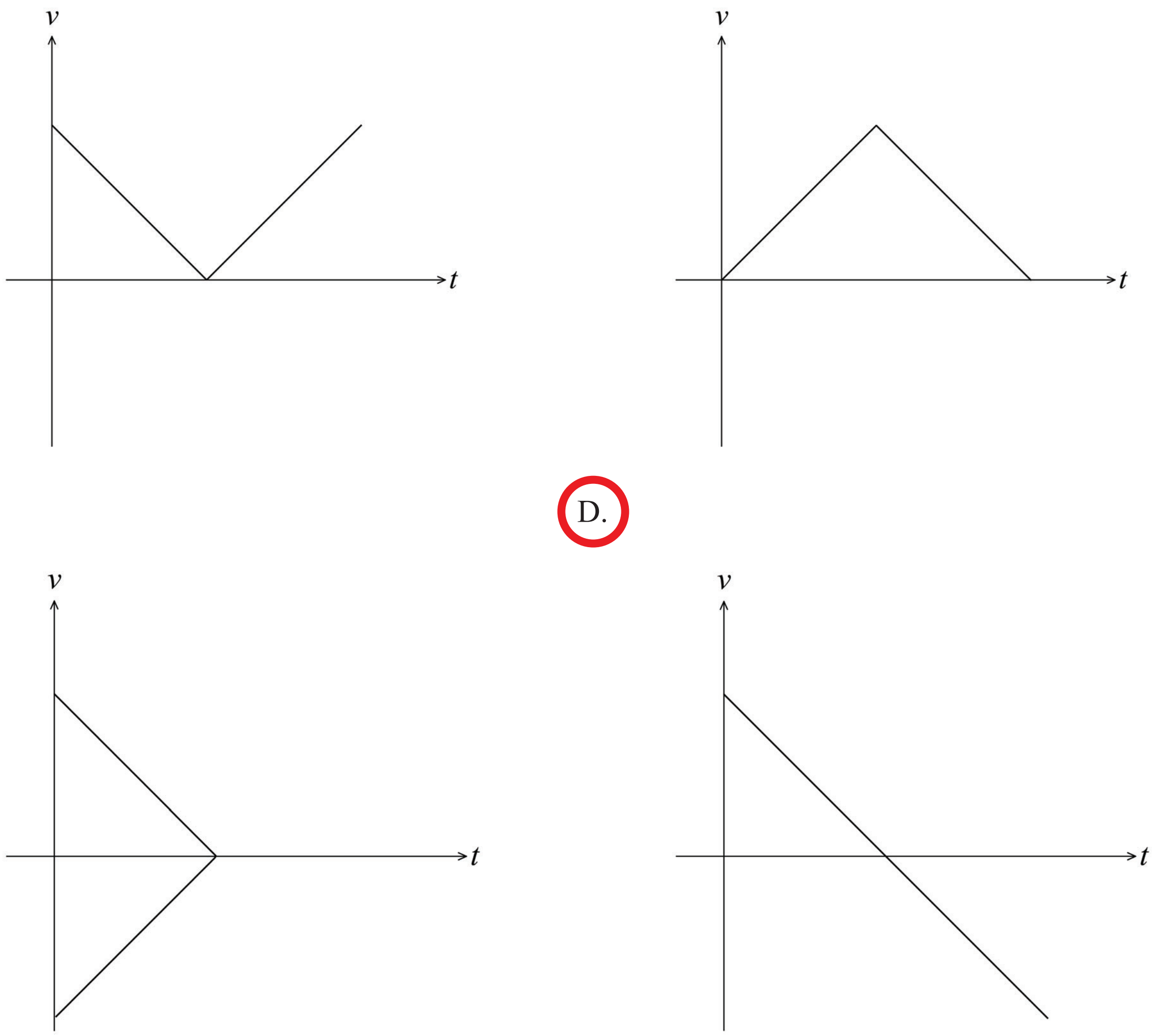
A	41
B	23
C	4
D	17

8 A body is thrown vertically upwards.

Which one of the following graphs correctly represent the velocity-time graph ?

A. B.

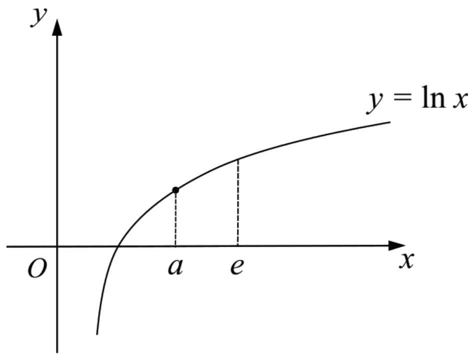
C. D.



The body changes direction after it stops, so there must be a change in sign.
Plus it increases in speed after it stops.

A	18
B	15
C	0
D	53

9 The line $y = mx + c$ is a tangent to the graph $y = \ln x$ at the point where $x = a$, as shown in the diagram below.



Which of the following statements is true ?

A. $\frac{1}{e} < m < 1$ and $-1 < c < 0$

B. $1 < m < e$ and $-1 < c < 0$

C. $\frac{1}{e} < m < 1$ and $0 < c < 1$

D. $1 < m < e$ and $0 < c < 1$

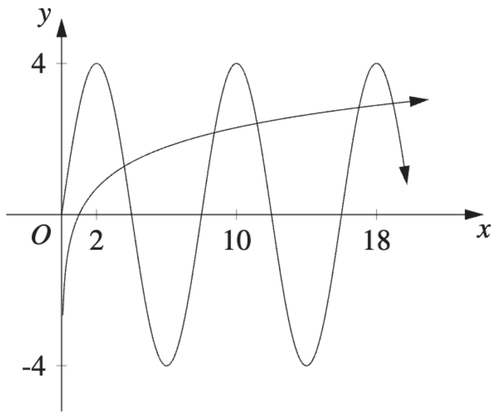
$y' = \frac{1}{x}$: $\quad 1 \le x \le e \Rightarrow \frac{1}{e} \le y' \le 1$

So options A and C only.

By drawing a tangent in at $x = 1$, the y-intercept must be negative.
(If needed one could prove this, but unnecessary for a MC question)

A	36
B	11
C	33
D	6

10 The graphs of $y = 4\sin\left(\frac{\pi}{4}x\right)$ and $y = \ln x$ are shown on the number plane below.



How many solutions exist to the equation $4\sin\left(\frac{\pi}{4}x\right) = \ln x$ in the domain $0 \le x \le 50$?

A. 10

B. 11

C. 12

D. 13

At $x = 50$, $4\sin\left(\frac{\pi}{4}x\right) = 4$.

The period of the sine function is $\frac{2\pi}{n} = 8$, where $n = \frac{\pi}{4}$.

From $x = 8$ to $x = 48$, there are 5 complete cycles where the sine function is intersected twice.
From $x = 0$ to $x = 8$, and from $x = 48$ to $x = 50$, there is only one intersection.

So there are $5 \times 2 + 1 + 1 = 12$ intersections.

A	5
B	14
C	42
D	25

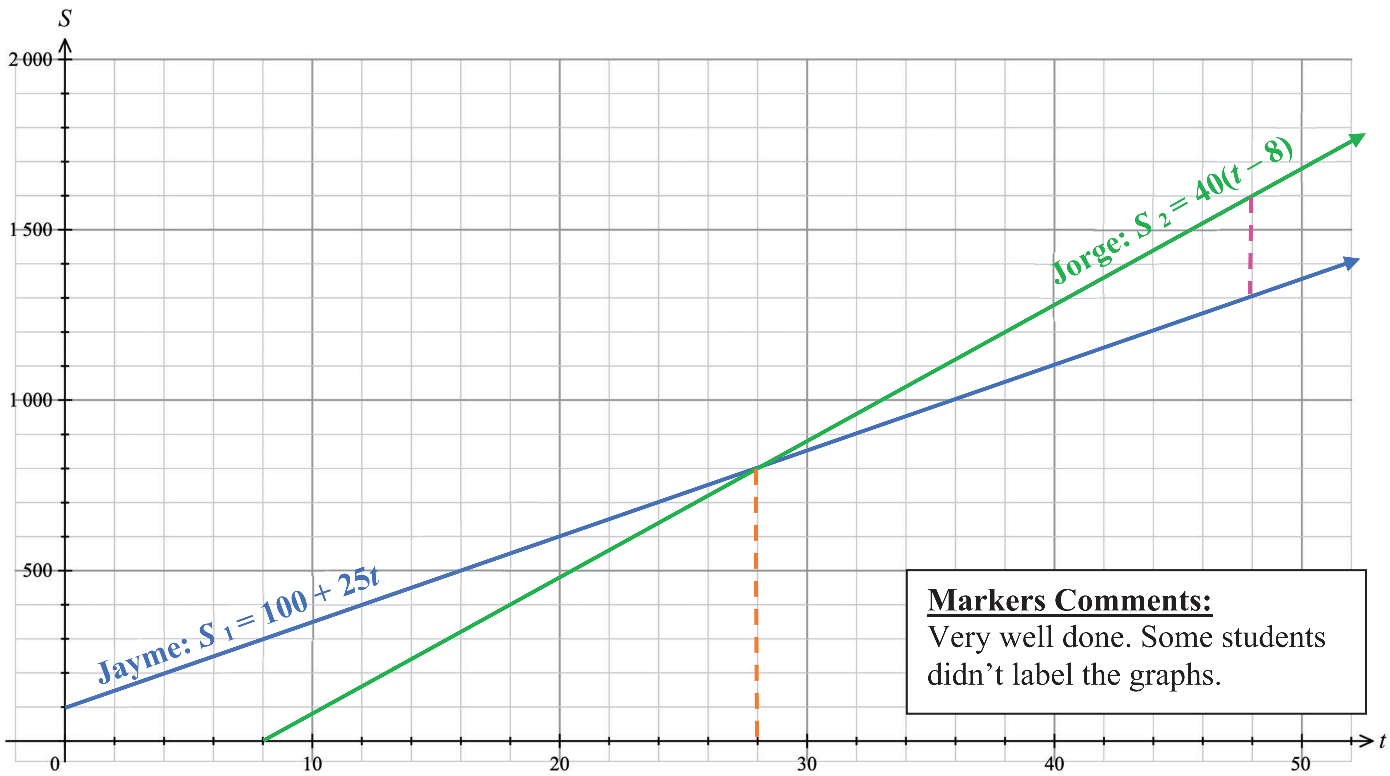
End of Section I

Question 11 (4 marks)

Jayme has \$100. He starts to save \$25 each week to buy a new racing bike.

His savings can be modelled by $S = 100 + 25t$, where S is the dollars saved and t is the time in weeks from when he started saving the money.

(a) On the grid below, draw the graph of this model and label it as Jayme's savings. **1**



Markers Comments:
Very well done. Some students didn't label the graphs.

(b) Jorge, Jayme's brother, wanted to start saving 8 weeks after Jayme began saving. He initially had no money but saves \$40 each week. **2**

By drawing a line on the grid above, or otherwise, find the value of t when the two brothers have the same money.

Reading from the graph = 28 weeks **OR**

$$40(t-8) = 100 + 25t \text{.........................[1]}$$
$$40t - 320 = 100 + 25t$$
$$15t = 420$$
$$t = 28 \text{..................................[1]}$$
$$\therefore 28 \textit{ weeks}$$

Markers Comments:
Very well done. Some students didn't factor in that Jorge 'started saving 8 weeks after Jayme and initially had no money.'

Question 11 Solutions continue on page A 2

Question 11 (continued)

(c) Using the graph, or otherwise, find the value of t when Jorge's savings is \$300 more than Jayme's savings. **1**

Markers Comments:
Very well done.

Reading from the graph = 48 weeks **OR**

$$40(t-8)-(100+25t)=300$$
$$40t-320-100-25t=300$$
$$15t-420=300$$
$$15t=720$$
$$t=48$$
$$\therefore 48\ weeks$$

Question 12 (2 marks)

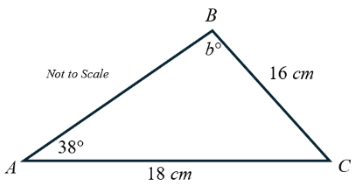
Triangle ABC has $\angle A = 38°$, $BC = 16$ cm, and $AC = 18$ cm. **2**

Find $\angle B$, where $\angle B$ is obtuse, to the nearest minute.

$$obtuse \rightarrow b^\circ = 180^\circ - \theta$$
$$\frac{\sin(180^\circ - \theta)}{18} = \frac{\sin 38^\circ}{16}$$
$$\sin(180^\circ - \theta) = \frac{18\sin 38^\circ}{16}$$
$$180^\circ - \theta^\circ = \sin^{-1}\left(\frac{18\sin 38^\circ}{16}\right)$$
$$b^\circ = 180^\circ - \sin^{-1}\left(\frac{18\sin 38^\circ}{16}\right) \ldots\ldots\ldots\ldots[1]$$
$$= 136^\circ 9' 43.93''$$
$$= 136^\circ 10' \quad (\text{nearest minute}) \ldots\ldots\ldots\ldots[1]$$

OR

$$\frac{\sin\theta}{18} = \frac{\sin 38^\circ}{16}$$
$$\sin\theta = \frac{18\sin 38^\circ}{16}$$
$$\theta^\circ = \sin^{-1}\left(\frac{18\sin 38^\circ}{16}\right)$$
$$\theta = 43^\circ 50' \ldots\ldots\ldots\ldots[1]$$
$$b^\circ = 180^\circ - 43^\circ 50'$$
$$= 136^\circ 10' \quad (\text{nearest minute}) \ldots\ldots\ldots\ldots[1]$$



Markers Comments:
Done reasonably well. However, many students didn't take into consideration that b was obtuse. Also, many students didn't round correctly to the nearest minute.

Question 13 (2 marks)

If $f(x) = \ln(\cos x)$, find the value of $f'\left(\dfrac{\pi}{4}\right)$. **2**

Using the reference sheet $y = \ln f(x) \rightarrow \dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

$$f'(x) = \frac{-\sin x}{\cos x}$$
$$= -\tan x \text{........................[1]}$$

when $x = \dfrac{\pi}{4}$:

$$f'(x) = -\tan\left(\frac{\pi}{4}\right)$$
$$= -1 \text{.............................[1]}$$

Markers Comments:
Poorly done. Many students either differentiated incorrectly or substituted incorrectly.

Question 14 (2 marks)

Find $\dfrac{d}{dx}\left(\sqrt{x}e^x\right)$. **2**

$u = \sqrt{x} \quad \left(x^{\frac{1}{2}}\right), \quad v = e^x$

$\dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} \qquad \dfrac{dv}{dx} = e^x$

$= \dfrac{1}{2\sqrt{x}}$ ....[1]

$$\frac{d}{dx}\left(\sqrt{x}\,e^x\right) = \sqrt{x}\,e^x + \frac{e^x}{2\sqrt{x}} \text{....................[1]}$$
$$= \frac{e^x(2x+1)}{2\sqrt{x}}$$

Markers Comments:
Reasonably well done.
Common mistake was that x to the power of a half wasn't differentiated correctly.

Question 15 (1 mark)

Find a primitive function y given that $\dfrac{dy}{dx} = \dfrac{2x}{3x^2+1}$. **1**

Using the reference sheet $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$

$f(x) = 3x^2 + 1$

$f'(x) = 6x$

$$\frac{2}{6}\int \frac{6x}{3x^2+1}\,dx = \frac{1}{3}\ln\left|3x^2+1\right| + c$$

Markers Comments:
Reasonably well done.
Common mistake was students multiplied by 3 instead of dividing.

Question 16 (1 mark)

Evaluate $\int \frac{6}{3^x}\,dx$. **1**

Using the reference sheet $\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a}+c$

$$\int \frac{6}{3^x}\,dx = 6\int 3^{-x}\,dx$$

$$= \frac{-6\times 3^{-x}}{\ln 3}+c$$

$$= \frac{-6}{3^x \ln 3}+c$$

Markers Comments:
Poorly done. Students who forgot C lost half a mark.

Question 17 (2 marks)

Find the exact value of $\int_{\frac{1}{3}}^{\frac{1}{2}} \sec^2\left(\frac{\pi x}{2}\right)dx$. **2**

Using the reference sheet $\int f'(x)\sec^2 f(x)\,dx = \tan f(x)+c$

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \sec^2\left(\frac{\pi x}{2}\right)dx = \frac{2}{\pi}\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\pi}{2}\sec^2\,dx$$

$$= \frac{2}{\pi}\left[\tan\left(\frac{\pi x}{2}\right)\right]_{\frac{1}{3}}^{\frac{1}{2}} \text{ ....[1]}$$

$$= \frac{2}{\pi}\left[\tan\left(\frac{\pi}{4}\right)-\tan\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{2}{\pi}\left[1-\frac{1}{\sqrt{3}}\right] \text{ ....[1]}$$

$$= \frac{6-2\sqrt{3}}{3\pi}$$

$$= \frac{2\left(2-\sqrt{3}\right)}{3\pi}$$

Markers Comments:
Reasonably well done. Common mistakes were students integrated if without having the fraction out the front or substituted the values from the integrand incorrectly.

End of Part A Solutions

Question 18 (4 marks)

Solve the simultaneous equations: **4**

$$x + 2y = 12 \quad (1)$$
$$5x + y^2 = 39 \quad (2)$$

from (1): $x = 12 - 2y \quad (3)$

sub (3) into (2): $5(12-2y) + y^2 = 39$

$60 - 10y + y^2 = 39$

$y^2 - 10y + 21 = 0$

$(y-7)(y-3) = 0$

when $y = 7$, $x = 12 - 2 \times 7$

$= -2 \longrightarrow (-2, 7)$

when $y = 3$, $x = 12 - 2 \times 3$

$= 6 \longrightarrow (6, 3)$

Comments

The answers of (6, 3) and (–2,7) as modelled in the solutions is the perfect response.

The pairings are explicitly stated and crystal clear.

***Common mistakes*:**

- unclear pairings even though the answers are correct.
  Responses looking like :

  $y = 7, 3$

  $x = -2,6$

  were penalised accordingly due to the ambiguity of the answer pairs.

Question 19 (2 marks)

The wavelength, λ metres, of a musical tone is inversely proportional to its frequency, f vibrations per second.

The frequency of middle C is approximately 260 vibrations per second and its wavelength is approximately 1.32 m.

(a) Write an equation to represent this, in the form $\lambda = g(f)$. **1**

$\lambda \propto \frac{1}{f}$

$\lambda = \frac{k}{f}$

$1{\cdot}32 = \frac{k}{260}$

$k = 260 \times 1{\cdot}32$

$\lambda = \frac{343{\cdot}2}{f}$

Comments:

Some students were confused by the function notation of this question:

λ is equivalent to y, and the frequency (f) is equivalent to x.

Essentially, this is a $y = f(x)$ question, but using $\lambda = g(f)$.
We use the function g because the letter 'f' has already been used.

***Common mistakes*:**

- not recognising that it was an inversely proportional relationship, or
- writing the equation but not calculating k the constant, and re-writing the equation with the numerical value of the constant in the equation.

(b) Hence, find the frequency, in vibrations per second, of a sound wave with wavelength 0.8 m. **1**

$f = \frac{343{\cdot}2}{\lambda}$

$= \frac{343{\cdot}2}{0{\cdot}8}$

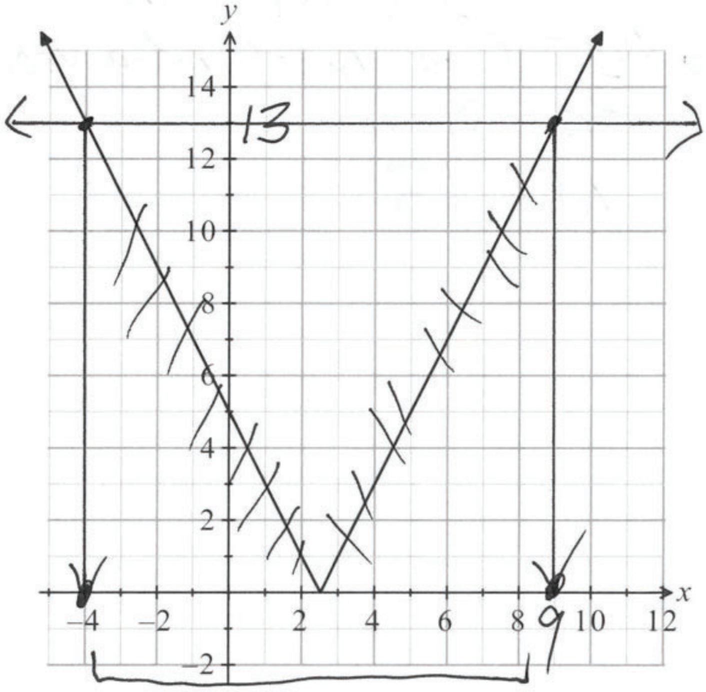
$= 429$ Hz or vib/s

Comments:

Overall very well done, and more correctly answered than part (a).

Question 20 (1 mark)

The graph of $y = |2x - 5|$ is shown below. By using it, solve $|2x-5| \le 13$. **1**



The absolute value graph is lower than or = to horizontal line at $y = 13$, $\therefore$ included region (intersection)

$-4 \le x \le 9$

Comments:

Most students used the graph appropriately to answer the question successfully.

***Common mistakes*:**

- A few students mistook the value of $y = 11$ for $y = 13$ on the graph.
- Other students went the wrong way, hence $x \ge 9$ and $x \le -4$,
- Finally, consider this response, where the 2 components are presented separately and not combined into an intersection: $x \le 9$ or $x \ge -4$.
  This response actually is equivalent to all real x, which is clearly not correct.

Question 21 (4 marks)

Consider the series $S_n = -18 + (-15) + (-12) + \ldots + u_n$.

a

(a) What is the value of u_7? **1**

$d = 3,\ a = -18$

$T_7 = a + (n-1)d$

$= -18 + (7-1) \times 3$

$= 0$

Comments:
Brilliantly done.

Common mistakes:

- The only problem occurred when some students mistook $d = 3$ for $d = -3$.

(b) What is the value of S_7? **1**

$S_7 = \frac{n}{2}[2a + (n-1)d] = \frac{7}{2}(2 \times -18 + 6 \times 3)$

$= -63$

Comments:
Most students got this right.

Common mistakes:

- problems occurred when some students mistook $d = 3$ for $d = -3$.
- Arithmetic mistakes.

(c) For what value of n is $S_n = 0$? **2**

$0 = \frac{n}{2}(-36 + 3(n-1))$

$= \frac{n}{2}(-39 + 3n)$

can discard $\frac{n}{2} = 0$ as $n = 0$ not a valid soln.

$3n - 39 = 0$

$n = 13$

Comments:

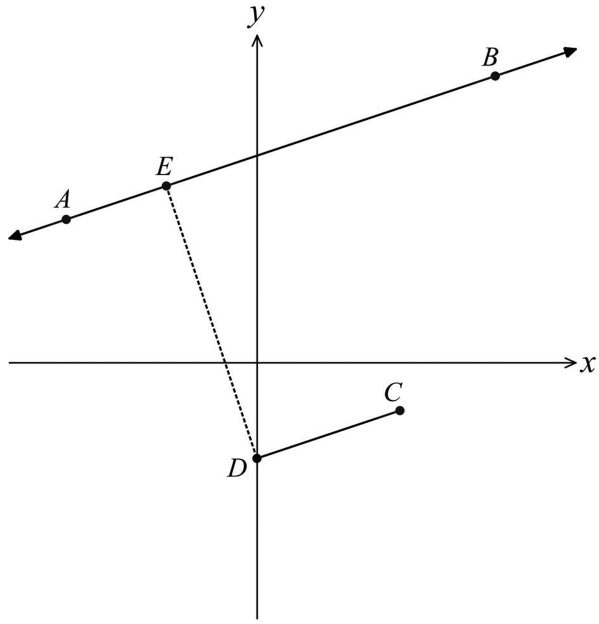
Very insightful by a lot of students that you do not need to process the $\frac{n}{2}$ component because it gives you $n = 0$ which is invalid as a series answer.

Common mistakes:

- Arithmetic mistakes, especially with regards to the quadratic formula.

Question 22 (4 marks)

The points $A(-4, 3)$, $B(5, 6)$, $C(3, -1)$ and $D(0, -2)$ are shown on the diagram below.



NOT DRAWN TO SCALE

The point E lies on AB, such that $ED \perp AB$.

(a) Show that $AB \parallel DC$. **2**

$$m_{AB} = \frac{6-3}{5+4} = \frac{3}{9} = \frac{1}{3}$$

$$m_{CD} = \frac{-2+1}{0-3} = \frac{-1}{-3} = \frac{1}{3}$$

$\therefore m_{AB} = m_{CD}$ which proves $AB // CD$

Comments

Once you proved that the 2 gradients were the same, you <u>must</u> have a concluding statement that lines with the same gradient are parallel.

(b) Show that the equation of AB is $x - 3y + 13 = 0$. **1**

using $A(-4, 3)$

$$y - 3 = \tfrac{1}{3}(x + 4)$$
$$3y - 9 = x + 4$$
$$x - 3y + 13 = 0$$

and

using $B(5, 6)$

$$y - 6 = \tfrac{1}{3}(x - 5)$$
$$3y - 18 = x - 5$$
$$0 = x - 3y + 13$$

Comments:

Generally very well done

Question 22 (continued)

(c) Find the equation of ED. **1**

$m_{ED} = -\frac{1}{\left(\frac{1}{3}\right)} = -3$

using $D(0, -2)$

$y + 2 = -3(x - 0)$

$3x + y + 2 = 0$

OR $y = -3x - 2$

Comments:

Generally very well done

***Common mistakes*:**

- Arithmetic mistakes, especially because you used $m = 3$ or $m = \frac{1}{3}$.

End of Part B Solutions

Part C Solutions **16 marks**

Question 23 (6 marks)

A curve has equation $y = x^3 - kx^2 + 1$, where k is a real constant.
When $x = 2$, the gradient of the curve is 6.

(a) Show that $k = 1.5$ **1**

$$f'(x) = 3x^2 - 2kx$$
$$f'(2) = 3(4) - 4k = 6$$
$$\therefore\ 4k = 6$$
$$k = 1.5$$

OR

$$f'(2) = 3(2^2) - 2\times 1.5\times 2 = 6$$

Criteria	Mark
Provides the correct solution.	1
Correctly differentiate $y = x^3 - kx^2 + 1$	0.5

Marker's Comment:
Generally, well done. Students must differentiate the expression.

(b) Find the coordinates of the two stationary points and determine their nature. **3**

$$y = x^3 - 1.5x^2 + 1$$
$$y' = 3x^2 - 3x$$
$$y' = 0,\quad 3x(x-1) = 0$$
$$\therefore x = 0, y = 1 \qquad x = 1, y = 0.5$$
$$y'' = 6x - 3$$

$f''(0) = -3 \quad < 0$ $\therefore$ Maximum at (0, 1)

$f''(1) = 3 \quad > 0$ $\therefore$ Minimum at (1, 0.5)

Criteria	Mark
Correctly identifies all points of the stationary points	3
Find the nature of the stationary points	2
Find the first derivative, and the x-values of both stationary points	1

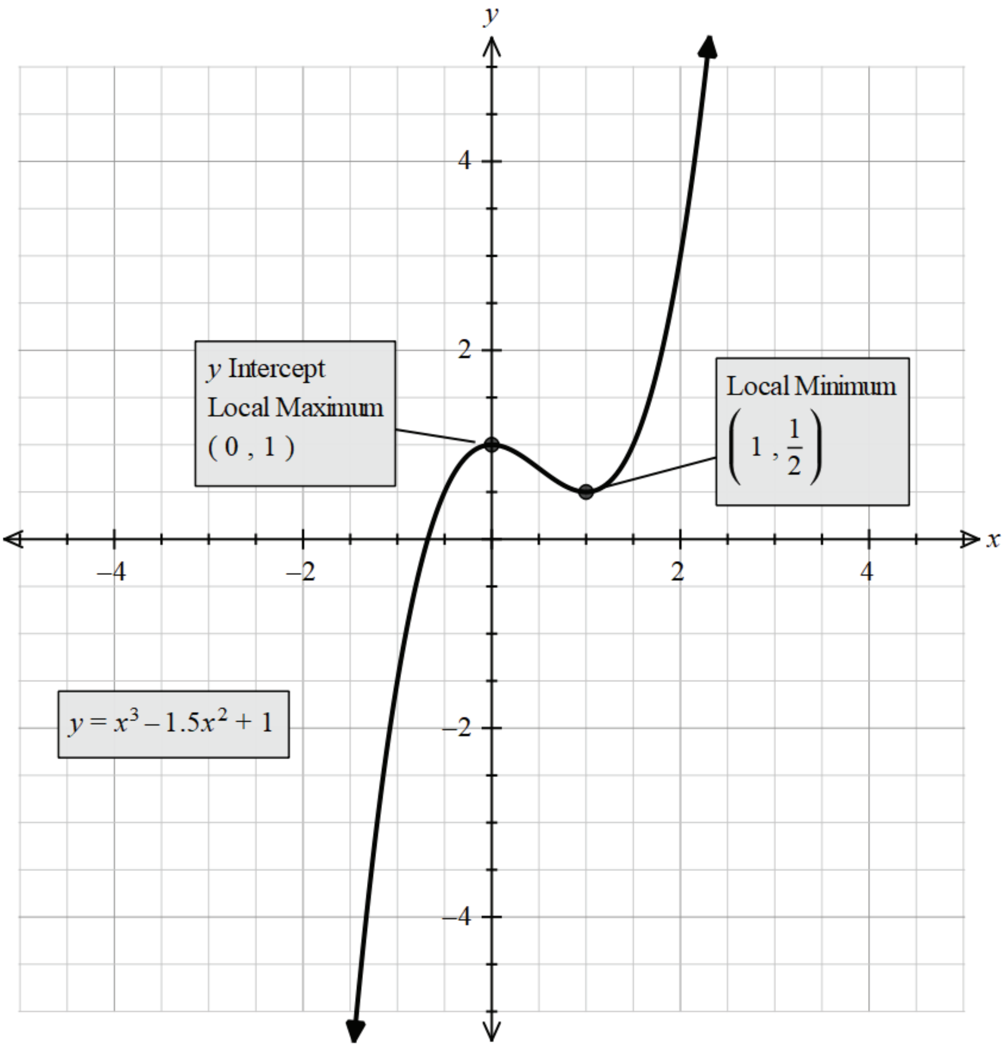
Marker's comments:

Students MUST show the correct values in the first or second derivatives for the maximum or minimum turning points.
Error Carries Forward (ECF) from an incorrect derivative.
There were a number of transcription errors into the equation: $x^3 - 1.5x^2 + 1$ NOT $1.5x$.
Students must substitute x-values into the equation for the corresponding y-values as a point consists of x and y values i.e. (x, y).

Question 23 continues on page C 2

Question 23 (continued)

(c) Sketch the curve $y = x^3 - 1.5x^2 + 1$, indicating all of the above and the y-intercept. **2**



Criteria	Mark
Provides the correct solution.	2
Correct shape and the maximum point, or the minimum point.	1.5
Correct shape and the y-intercept at $y = 1$	1

Marker's Comment:

The point of inflexion is at $\left(\frac{1}{2}, \frac{3}{4}\right)$ but it is not required for this question.

End of Question 23

Question 24 (2 marks)

Find $\displaystyle\int \frac{x}{\sqrt{3x^2+5}}\,dx$. **2**

Using the reference sheet $\displaystyle\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + C$

$$\begin{aligned}
\int \frac{x}{\sqrt{3x^2+5}}\,dx &= \int x\left(3x^2+5\right)^{-\frac{1}{2}} dx \\
&= \frac{1}{6}\int 6x\left(3x^2+5\right)^{-\frac{1}{2}} dx \\
&= \frac{1}{6}\left(\frac{1}{\frac{1}{2}}\right)\left(3x^2+5\right)^{\frac{1}{2}} + c \\
&= \frac{1}{3}\left(3x^2+5\right)^{\frac{1}{2}} + c \\
&= \frac{1}{3}\sqrt{3x^2+5} + c
\end{aligned}$$

Criteria	**Mark**
Provides the correct solution.	**2**
Correctly rewriting the integral as $\frac{1}{6}\int 6x\left(3x^2+5\right)^{-\frac{1}{2}} dx$	**1**

Marker's Comments: This question was completed reasonably well.

No penalty for not having '+ C' this time.

ECF for incorrect derivative for the product rule.

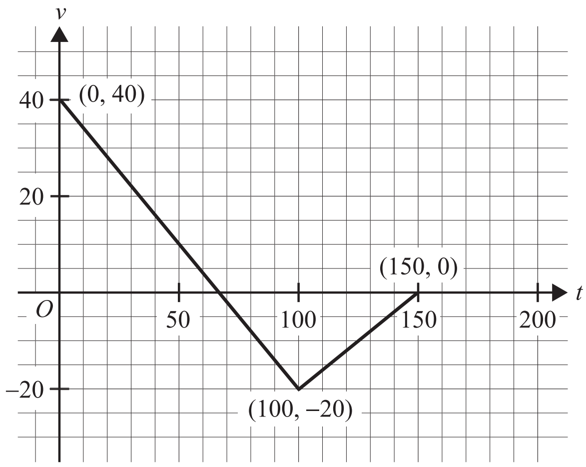
Students MUST be able to apply

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + C$$

from the reference sheet.

Question 25 (2 marks)

The velocity-time graph of a particle moving along an east-west line with velocity v m/s at t seconds, starting 20 metres west of the origin, is shown below. **2**



How far, in metres, is the particle from the origin, 150 seconds later?

From $t = 0$ to $t = 100$, $v = -0.6 + 40$
When $v = 0$, $t = 66\tfrac{2}{3}$.
Change in displacement:

$$\int_0^{150} v\,dt = \frac{1}{2} \times 66\tfrac{2}{3} \times 40 - \frac{1}{2} \times \left(150 - 66\tfrac{2}{3}\right) \times 20$$

$$= 1333\tfrac{1}{3} - 8333\tfrac{1}{3}$$

$$= 500$$

The particle's position from the origin is –20 + 500 = 480 metres.

Criteria	Mark
Provides the correct solution.	**2**
Correct calculation for the areas of the triangles from $t = 0$ to $t = 150$.	**1.5**
Correct calculation for the area of the triangle from $t = 0$ to $t = 66\tfrac{2}{3}$.	**1**
Correctly identified that $t = 66\tfrac{2}{3}$ when $v = 0$.	**0.5**

Marker's Comments: Not done well.

Students needed to identify the exact x-intercept. They could have done this by finding the equation of the line from $x = 0$ to $x = 100$.

It is more suitable to find the areas of the triangles from $x = 0$ to the x-intercept and the x-intercept to $x = 150$, than integration techniques.

Common error: leaving out that the particle starts at –20.
Students struggled to find the x-intercept and evaluate the areas correctly.

Question 26 (2 marks)

Find $\dfrac{d}{dx}\left[\ln\left(e^x \sin x\right)\right]$. **2**

$$\frac{d}{dx}\left[\ln\left(e^x \sin x\right)\right] = \frac{d}{dx}\left[\ln e^x + \ln\left(\sin x\right)\right]$$

$$= \frac{d}{dx}\left[x + \ln\left(\sin x\right)\right]$$

$$= 1 + \cot x$$

Alternative for those who forgot their 'log' rules:

$$\frac{d}{dx}\left(e^x \sin x\right) = e^x \cos x + e^x \sin x$$

$$\therefore \frac{d}{dx}\left[\ln\left(e^x \sin x\right)\right] = \frac{e^x\left(\cos x + \sin x\right)}{e^x \sin x}$$

$$= \frac{\cos x + \sin x}{\sin x}$$

$$= 1 + \cot x$$

Criteria	Mark
Provides the correct solution.	**2**
Attempted to use the product rule correctly.	**1**

Marker's Comments: This question was completed reasonably well.

Question 27 (2 marks)

Find $\int \cot 2x + \frac{1}{x\sqrt{x}}\,dx$ **2**

First: $\int \frac{\cos 2x}{\sin 2x}\,dx + \int x^{-\frac{3}{2}}\,dx$

Since $f(x) = \sin 2x \Rightarrow f'(x) = 2\cos 2x$

$$\therefore \int \frac{1}{2}\left(\frac{2\cos 2x}{\sin 2x}\right)dx + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c = \frac{1}{2}\ln\left|\sin 2x\right| - 2x^{-\frac{1}{2}} + c$$

$$= \frac{1}{2}\ln\left|\sin 2x\right| - \frac{2}{\sqrt{x}} + c$$

Criteria	Mark
Provides the correct solution with absolute value.	**2**
$\frac{1}{2}\ln(\sin 2x) - 2x^{-\frac{1}{2}} + c$ i.e. left out the absolute value.	**1.5**
Rewrite the integral to $\int \frac{\cos 2x}{\sin 2x}\,dx + \int x^{-\frac{3}{2}}\,dx$	**0.5**

Marker's comments: No penalty with leaving out '+ *C*'.
Some students struggled with trig and log integration.

Question 28 (2 marks)

An arithmetic sequence has the first term 5 and a common difference of d. **2**
The sum of the first 20 terms is 4 times the sum of the first 10 terms.

Find the value of d.

$$a = 5$$
$$S_{20} = 4 \times S_{10}$$
$$\therefore \frac{20}{2}(10 + 19d) = 4 \times \frac{10}{2}(10 + 9d)$$
$$\therefore 10(10 + 19d) = 20(10 + 9d)$$
$$10 + 19d = 2(10 + 9d)$$
$$10 + 19d = 20 + 18d$$
$$\therefore d = 10$$

Criteria	Mark
Provides the correct solution.	**2**
Correct equate the expressions of S_{20} and 4 × S_{10}	**1**

Marker's Comment: Generally, well done.

Some students got T_n and S_n confused, and some need to practise their algebra to avoid errors.

End of Part C Solutions

Part D Solutions 16 marks

Question 29 (2 marks)

Find $\displaystyle\int \frac{16-8x}{3x^2-12x+1}\,dx$. **2**

Let $f(x)=3x^2-12x+1$

$\therefore f'(x)=6x-12$

$\therefore 16-8x=-\dfrac{4}{3}(6x-12)$

$$\int \frac{16-8x}{3x^2-12x+1}\,dx=-\frac{4}{3}\int \frac{6x-12}{3x^2-12x+1}\,dx$$

$$=-\frac{4}{3}\ln\left|3x^2-12x+1\right|+C$$

Extremely well done.

Students need to simplify their final answers.

Some students forgot the absolute value sign and/or the constant '+C'.
There was no penalty for this question, but marks may be deducted next time.

Common incorrect answers $\dfrac{4}{3}\ln\left|3x^2-12x+1\right|+C,\ -\dfrac{3}{4}\ln\left|3x^2-12x+1\right|+C$

Question 30 (3 marks)

The sum of an infinite geometric series, with positive terms, is 3. **3**

The sum of the cubes of its terms is $\frac{27}{19}$. What is the common ratio of the original series?

Let the sum of the original geometric series be S_1 and the new series be S_2.

$$S_1 = \frac{a}{1-r} = 3 \quad \Rightarrow \quad a = 3(1-r) \qquad ①$$

$$S_2 = \frac{a_2}{1-r_2} = \frac{a^3}{1-r^3} = \frac{27}{19} \qquad ②$$

Substitute ① to ②

$$\therefore \frac{\left(3(1-r)\right)^3}{1-r^3} = \frac{27}{19} \quad \Rightarrow \quad \frac{(1-r)^3}{(1-r)\left(1+r+r^2\right)} = \frac{1}{19}$$

Dividing both sides by 27 can simplify the calculation

$$\therefore \frac{1-2r+r^2}{1+r+r^2} = \frac{1}{19}$$

$$\therefore 6r^2 - 13r + 6 = 0$$

$$\therefore r = \frac{3}{2}, \frac{2}{3}$$

$$\because |r| < 1$$

$$\therefore r = \frac{2}{3}$$

Poorly done.

No marks were awarded for formulae that students could directly find on the reference sheet.

Too many careless mistakes. ECF was accepted.

Some students did not use the correct geometric series sum formula because they did not read the question carefully. Many students also failed to recognise that the new geometric series is infinite.

Some students could not identify the first term and/or the common ratio of the new geometric series.

Many students struggled with solving the equation.
Improvement in algebraic techniques is recommended.

Question 31 (3 marks)

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin x - \sin^3 x}\, dx$. **3**

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sqrt{\sin x - \sin^3 x}\, dx &= \int_0^{\frac{\pi}{2}} \sqrt{\sin x\left(1-\sin^2 x\right)}\, dx \\
&= \int_0^{\frac{\pi}{2}} \sqrt{\sin x \cos^2 x}\, dx \\
&= \int_0^{\frac{\pi}{2}} \cos x\left(\sin x\right)^{\frac{1}{2}}\, dx \qquad \because x \in \left[0, \frac{\pi}{2}\right],\quad \cos x > 0 \\
&= \int_0^{\frac{\pi}{2}} \frac{2}{3}\times\frac{3}{2}\cos x\left(\sin x\right)^{\frac{1}{2}}\, dx \\
&= \frac{2}{3}\left[\left(\sin x\right)^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}} \\
&= \frac{2}{3}\left[\left(\sin\frac{\pi}{2}\right)^{\frac{3}{2}} - \left(\sin 0\right)^{\frac{3}{2}}\right] \\
&= \frac{2}{3}(1-0) \\
&= \frac{2}{3}
\end{aligned}$$

Poorly done

Some students did not attempt this question.

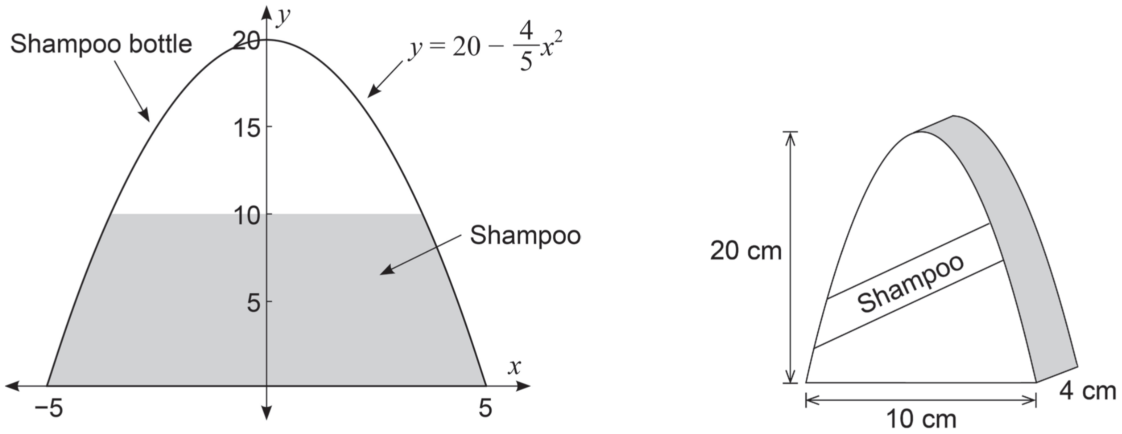
Many students struggled in simplifying $\sqrt{\sin x - \sin^3 x}$

Many students misused the reverse chain rule, which resulted in an incorrect expression with division by zero. Students need to know how to apply this.

Common incorrect answers: $0, -\dfrac{2}{3}$

Question 32 (8 marks)

A partially filled shampoo bottle has a uniform parabolic cross-section as shown in the diagram below.



Both x and y are measured in centimetres.
The width of the bottle perpendicular to the parabolic cross-section is 4 cm.

(a) Show that the volume V of shampoo in the bottle, if it is partially filled to a height of 10 cm, is **4**

$$\frac{1600 - 400\sqrt{2}}{3}\text{ cm}^3$$

Method 1

$$\because y = 20 - \frac{4}{5}x^2$$

$$\therefore x = \pm\frac{\sqrt{100-5y}}{2}$$

$$A = 2\int_0^{10} \frac{\sqrt{100-5y}}{2}\,dy$$

$$= \int_0^{10} \sqrt{100-5y}\,dy$$

$$= \left[-\frac{2}{15}(100-5y)^{\frac{3}{2}}\right]_0^{10}$$

$$= -\frac{2}{15}\left(50^{\frac{3}{2}} - 100^{\frac{3}{2}}\right)$$

$$= \frac{400}{3} - \frac{100\sqrt{2}}{3}$$

$$V = Ah$$

$$= \left(\frac{400}{3} - \frac{100\sqrt{2}}{3}\right)\times 4$$

$$= \frac{1600-400\sqrt{2}}{3}\quad cm^3$$

Method 2

$$\text{Let } y = 20 - \frac{4}{5}x^2 = 10$$

$$x = \pm\frac{5\sqrt{2}}{2}$$

$$A = 2\times\frac{5\sqrt{2}}{2}\times 10 + 2\int_{\frac{5\sqrt{2}}{2}}^{5}\left(20 - \frac{4}{5}x^2\right)dx$$

$$= 50\sqrt{2} + 2\left[20x - \frac{4}{15}x^3\right]_{\frac{5\sqrt{2}}{2}}^{5}$$

$$= 50\sqrt{2} + 2\left(100 - \frac{100}{3} - \left(50\sqrt{2} - \frac{25}{3}\sqrt{2}\right)\right)$$

$$= 50\sqrt{2} + 2\left(\frac{200}{3} - \frac{125\sqrt{2}}{3}\right)$$

$$= \frac{400}{3} - \frac{100\sqrt{2}}{3}\quad cm^2$$

$$V = Ah$$

$$= \left(\frac{400}{3} - \frac{100\sqrt{2}}{3}\right)\times 4$$

$$= \frac{1600-400\sqrt{2}}{3}\quad cm^3$$

Method 3

Let $y = 20 - \frac{4}{5}x^2 = 10$

$x = \pm\frac{5\sqrt{2}}{2}$

$$A = \int_{-5}^{5}\left(20 - \frac{4}{5}x^2\right)dx - \int_{-\frac{5\sqrt{2}}{2}}^{\frac{5\sqrt{2}}{2}}\left(20 - \frac{4}{5}x^2 - 10\right)dx$$

$$= \left[20x - \frac{4}{15}x^3\right]_{-5}^{5} - \left[10x - \frac{4}{15}x^3\right]_{-\frac{5\sqrt{2}}{2}}^{\frac{5\sqrt{2}}{2}}$$

$$= \left(200 - \frac{200}{3}\right) - \left(50\sqrt{2} - \frac{50\sqrt{2}}{3}\right)$$

$$= \frac{400}{3} - \frac{100\sqrt{2}}{3} \quad cm^2$$

$$V = Ah$$

$$= \left(\frac{400}{3} - \frac{100\sqrt{2}}{3}\right) \times 4$$

$$= \frac{1600 - 400\sqrt{2}}{3} \quad cm^3$$

Poorly done

The three methods were equally favoured by students.

Students who performed poorly on this question struggled to find the correct expression for the cross-sectional area or the volume of shampoo.

For example, some students who used the Method 3 could not correctly find the area of the unshaded part as they did not subtract $y = 10$ *from* $y = 20 - \frac{4}{5}x^2$.

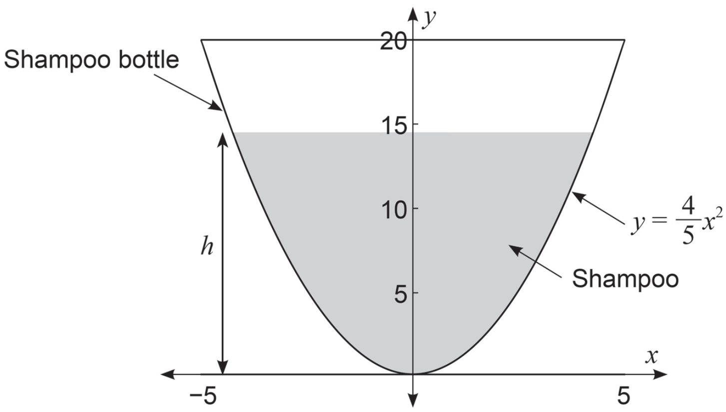
Half marks were deducted if students did not include ± for x.

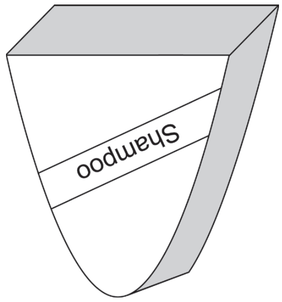
This is a "show" question, so the mark for demonstrating the correct volume could not be awarded if earlier steps were wrong. Therefore, ECF was not accepted here.

Question 32 continues on page D 6

Question 32 (continued)

The bottle is turned upside down and the entire volume of the shampoo allowed to settle, as shown in the diagram below.





(b) Show that the shampoo level h is given by **4**

$$\left(40\sqrt{5} - 10\sqrt{10}\right)^{\frac{2}{3}} \text{ cm}$$

Method 1

$$y = \frac{4}{5}x^2 \quad \Rightarrow x = \pm\frac{\sqrt{5y}}{2}$$

$$A = 2\int_0^h \frac{\sqrt{5y}}{2}\,dy$$

$$= \sqrt{5}\int_0^h \sqrt{y}\,dy$$

$$= \sqrt{5} \times \frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^h$$

$$= \frac{2\sqrt{5}}{3}\left(h^{\frac{3}{2}} - 0\right)$$

$$= \frac{2\sqrt{5}}{3}h^{\frac{3}{2}}$$

The area of the uniform cross-section of shampoo remains the same.

$$\therefore \frac{2\sqrt{5}}{3}h^{\frac{3}{2}} = \frac{400 - 100\sqrt{2}}{3}$$

$$\therefore h^{\frac{3}{2}} = \frac{400 - 100\sqrt{2}}{2\sqrt{5}} = 40\sqrt{5} - 10\sqrt{10}$$

$$\therefore h = \left(40\sqrt{5} - 10\sqrt{10}\right)^{\frac{2}{3}} \ cm$$

Method 2

$$h = \frac{4}{5}x^2 \quad \Rightarrow x = \pm\frac{\sqrt{5h}}{2}$$

$$A = 2\int_0^{\frac{\sqrt{5h}}{2}} \left(h - \frac{4}{5}x^2\right)dx$$

$$= 2\left[hx - \frac{4}{15}x^3\right]_0^{\frac{\sqrt{5h}}{2}}$$

$$= 2\left(h \times \frac{\sqrt{5h}}{2} - \frac{4}{15} \times \left(\frac{\sqrt{5h}}{2}\right)^3 - (0 - 0)\right)$$

$$= \sqrt{5}h^{\frac{3}{2}} - \frac{\sqrt{5}h^{\frac{3}{2}}}{3}$$

$$= \frac{2\sqrt{5}}{3}h^{\frac{3}{2}}$$

The area of the uniform cross-section of shampoo remains the same.

$$\therefore \frac{2\sqrt{5}}{3}h^{\frac{3}{2}} = \frac{400 - 100\sqrt{2}}{3}$$

$$\therefore h^{\frac{3}{2}} = \frac{400 - 100\sqrt{2}}{2\sqrt{5}} = 40\sqrt{5} - 10\sqrt{10}$$

$$\therefore h = \left(40\sqrt{5} - 10\sqrt{10}\right)^{\frac{2}{3}} \ cm$$

Poorly done.

Many students did not fully attempt this question.

Marks were deducted if students did not show the substitution step.

As in Q32(a), the mark allocated for showing the correct shampoo level was not awarded if mistakes were made in earlier steps.

Common errors:

- *Some students incorrectly equated the area to the volume.*
- *Some students used the area between the curve and x-axis to represent the cross-sectional area of shampoo.*
- *Some students who used Method 2 did not calculate the correct boundaries.*

End of Part D Solutions

Part E Solutions 17 marks

Question 33 (4 marks)

The table below shows the future value of annuity with a contribution of \$1 at the end of each period. **4**

Period	0.75%	1.00%	1.50%	2.00%	2.50%	3.00%
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	2.00750	2.01000	2.01500	2.02000	2.02500	2.03000
3	3.02256	3.03010	3.04523	3.06040	3.07563	3.09090
4	4.04523	4.06040	4.09090	4.12161	4.15252	4.18363
5	5.07556	5.10101	5.15227	5.20404	5.25633	5.30914
6	6.11363	6.15202	6.22955	6.30812	6.38774	6.46841
7	7.15948	7.21354	7.32299	7.43428	7.54743	7.66246
8	8.21318	8.28567	8.43284	8.58297	8.73612	8.89234

Kevin opened an account and deposited \$15 000 at the end of each year for 6 years.

At the end of the 7th year, he deposited \$20 000 in the same account and continued to deposit \$20 000 at end of each year until the end of the 10th year.

He made no more deposits after the end of the 10th year.
The money in his account was earning interest at 2.5% per annum compounded annually.

How much money will Kevin have in his account at the end of 15 years?

The account balance at the end of the first 6 years can be determined using the future value table at 2.50% for 6 periods and multiplying that value by the deposit amount of \$15 000.

Therefore, at the end of 6 years the account balance will be

$$6.38774 \times \$15\,000 = \$95\,816.10.$$

This amount will continue to accrue interest for the remaining 9 years at 2.50% per annum, so we multiply it by 1.025^9 to find the total contribution of these deposits towards the balance at the end of 15 years.

Similarly, the total contribution of the \$20 000 deposits will be equal to

$$4.15252 \times \$20\,000 \times 1.025^5.$$

Thus, the final account balance at the end of 15 years is:

$$6.38774 \times \$15\,000 \times 1.025^9 + 4.15252 \times \$20\,000 \times 1.025^5 = \$215\,974.18 \text{ (nearest cent).}$$

1 mark for identifying the appropriate values in the table.

0.5 marks for taking the appropriate products of deposit amount and future value.

1 mark for applying the interest rate in the years after deposits have stopped.

0.5 marks for a correct final answer.

1 mark for providing sufficient reasoning.

Students are reminded that sufficient working must always be shown to get full marks.
Given that this question is worth 4 marks, but only requires 3 marks worth of calculations, it should be recognised that there is a mark assigned to reasoning.

Alternate method (in stages, rounding to the nearest cent)**:**
At the end of 6 years, the account balance will be

$$6.38774 \times \$15\,000 = \$95\,816.10.$$

At the end of 10 years, the account balance will be

$$4.15252 \times \$20\,000 + \$95\,816.10 \times 1.025^4 = \$188\,813.45.$$

At the end of 15 years, the account balance will be

$$\$188\,813.45 \times 1.025^5 = \$213\,625.09.$$

Note: Depending on how granularly the stages are considered, the final result will differ by a few cents due to rounding.
This discrepancy did not affect the marks awarded.

Comments

Generally done ok 2 – many students got most of the marks, but only a minority got all of them.

Common errors included misusing the table, applying the annuity formula incorrectly, and failing to recognise that the 7$^{\text{th}}$ to the 10$^{\text{th}}$ years inclusive is a total of 4 years (not 3).

It was also very disappointing that only a handful of students used the annuity table for the second round of deposits, with almost all students opting to do the 7$^{\text{th}}$ to 10$^{\text{th}}$ years manually (one year at a time).

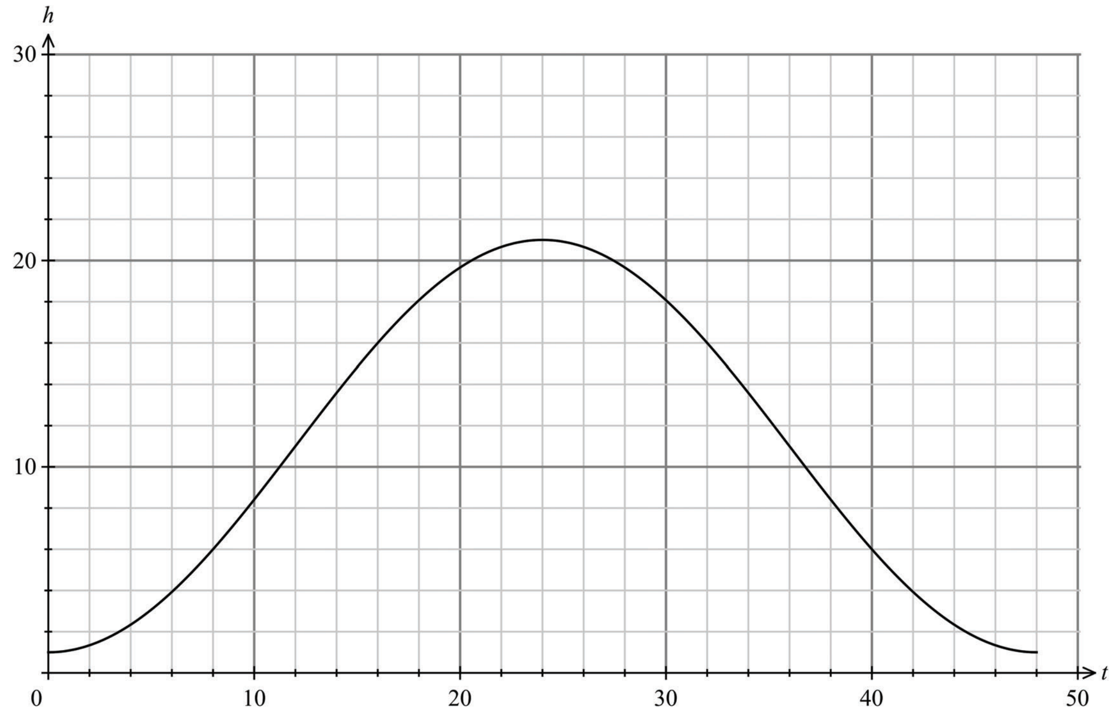
Question 34 (7 marks)

The height of a car on a slow Ferris wheel in Funpark can be modelled by the formula

$h_1(t) = a\sin\left(\dfrac{\pi}{24}(t-c)\right)+11$, where t is the time in seconds and $0 \le t \le 48$.

The height of the car at $t = 0$ is 1 m and it reaches a maximum height of 21 m when $t = 24$, and then it decreases back to 1 m at $t = 48$.

The graph of $h_1(t)$ is as shown.



(a) What are the values of a and c? **2**

From the graph, we can determine that the amplitude of the motion is $a = 10$.

Also from the graph, we can recognise that the centre of the motion occurs at $t = 12$. Thus, $c = 12$.

1 mark for correctly determining a value for a.

1 mark for correctly determining a corresponding value for c.

Note: If a is chosen to be negative (i.e. $a = -10$) then we instead get $c = 36$.

Furthermore, since no restriction on c is provided, adding any multiple of 48 to the value of c will not affect the phase shift.

Comments

Generally done well.

A common error was mistaking c for periodicity, as opposed to the phase shift.

Question 34 continues on page E 4

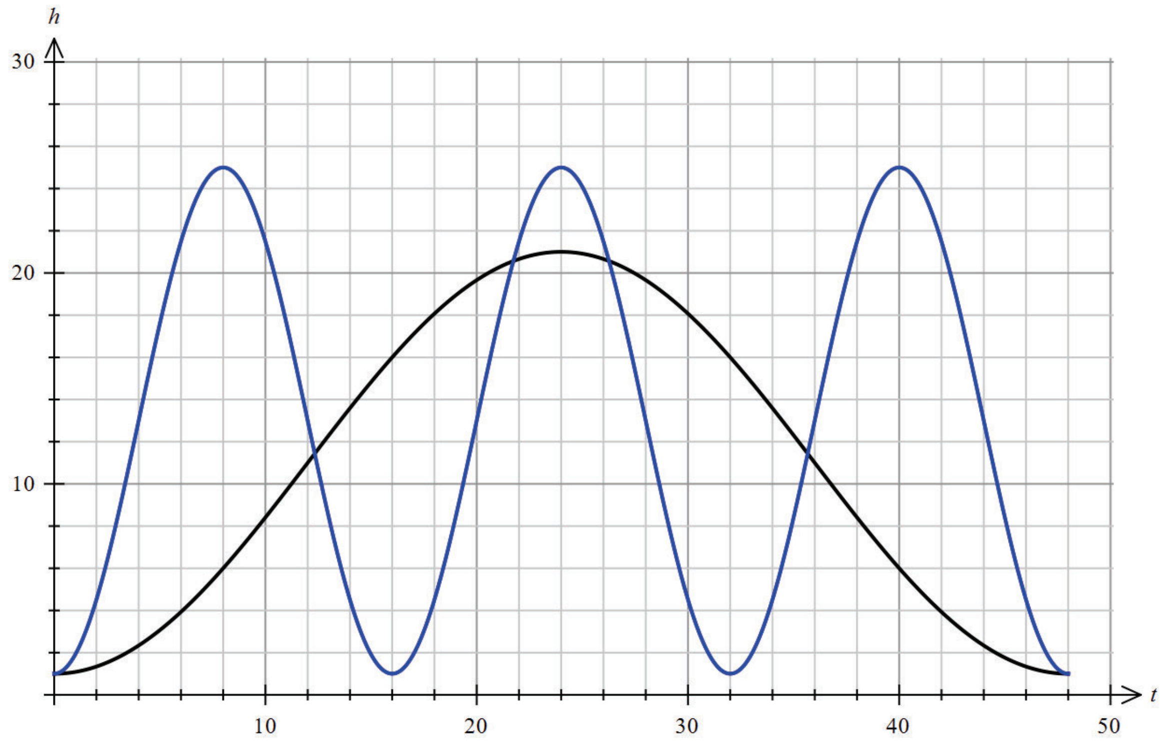
(b) In Funpark there is a faster Ferris wheel than the one in part (a). **3**

The height of a car on this Ferris wheel can be modelled by the formula

$$h_2(t) = 12\sin\left(\frac{\pi}{8}(t-4)\right) + 13.$$

By drawing the graph of $h_2(t)$ on the graph of $h_1(t)$ find the values of the time t in $0 \le t \le 48$ at which the height of both cars is decreasing.

From the equation, we know that $h_2(t)$ has a centre of 13, an amplitude of 12, and a phase shift of 4.



From the graph, we can clearly see that both graphs are decreasing for $t \in (24,32) \cup (40,48)$.

2 marks for the correct graph of $h_2(t)$ (phase, amplitude, and periodicity).

1 mark for identifying all the correct times where both graphs are decreasing.

Note: At the stationary points (peaks and troughs) the cars are neither increasing nor decreasing, hence the exclusion of those values from the intervals.

Comments

Generally done well.

Students are reminded to sketch their graphs at least *somewhat* neatly in case marks are allocated to them.

Question 34 continues on page E 5

(c) Find the rate of change of the height of the car on the slower Ferris wheel when the car on the faster Ferris wheel reaches its minimum height for the first time after its initial starting point. **2**

From the graph, we can identify that the faster Ferris wheel reaches its minimum height for the first time after its initial starting point at $t = 16$.

$$h_1'(t) = \left(\frac{\pi}{24}\right)10\cos\left(\frac{\pi}{24}(t-12)\right)$$

Hence,

$$\begin{aligned} h_1'(16) &= \left(\frac{\pi}{24}\right)10\cos\left(\frac{\pi}{24}(16-12)\right) \\ &= \left(\frac{\pi}{24}\right)10\cos\left(\frac{\pi}{6}\right) \\ &= \left(\frac{\pi}{24}\right)10\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{5\sqrt{3}}{24}\pi \end{aligned}$$

0.5 marks for identifying the correct time value.
1 mark for correctly differentiating the expression for the height of the car (on the slower wheel).
0.5 marks for determining the correct rate of change.

Comments

Generally done well.

The majority of small errors came from differentiating the expression, and an alarming number of errors came from transcription errors.

Students are reminded to read their solutions carefully before going on to the next question.

Question 35 (6 marks)

(a) Find $\dfrac{d}{dx}\left(xe^{-x}\right)$. **1**

$$\frac{d}{dx}\left(xe^{-x}\right)=\frac{d}{dx}(x)e^{-x}+x\frac{d}{dx}\left(e^{-x}\right)$$
$$=e^{-x}-xe^{-x}$$

1 mark for correctly differentiating the expression.

Comments

Generally done very well.

(b) Hence, find $\displaystyle\int xe^{-x}\,dx$. **1**

From the previous part, we have that $xe^{-x}=e^{-x}-\dfrac{d}{dx}\left(xe^{-x}\right)$.

Hence,

$$\int e^{-x}-\frac{d}{dx}\left(xe^{-x}\right)dx=\int e^{-x}\,dx-xe^{-x}$$
$$=-e^{-x}-xe^{-x}+c$$

0.5 marks for appropriate use of the result from part (a).

0.5 marks for correctly integrating the expression.

Comments

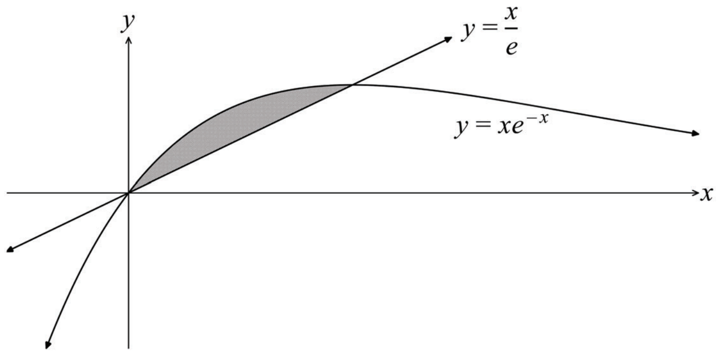
Generally done ok

Most errors came from being unable to properly handle the numerous negative signs introduced.

Question 35 is continues on page E 7

Question 35 (continued)

(c) The diagram below shows the shaded region bounded by the function $y = xe^{-x}$ and the line $y = \frac{x}{e}$. By using part (b), find the exact area of the shaded region. **4**



We can express the shaded area between two curves as the integral of their difference between their points of intersection. Note that $y = xe^{-x}$ is the above curve.

To find the points of intersection, we can equate the two expressions for y:

$$\frac{x}{e} = xe^{-x}$$
$$xe^{-1} = xe^{-x}$$
$$e^{-1} = e^{-x}$$

From this work, we can see that the points of intersection are at $x = 0$ and $x = 1$.

Hence, the shaded area is:

$$\int_0^1 xe^{-x} - \frac{x}{e}\,dx = \left[-e^{-x} - xe^{-x} - \frac{x^2}{2e}\right]_0^1$$
$$= \left(-e^{-1} - e^{-1} - \frac{1}{2e}\right) - (-1 - 0 - 0)$$
$$= 1 - \frac{5}{2e}$$

1 mark for correctly determining the points of intersection.

1 mark for providing a correct expression for the shaded area.

1 mark for correctly integrating the expression.

1 mark for a correct final answer.

Comments

Generally done ok.

There was a significant amount of error carried forward (ECF) in this part, but otherwise well attempted. However, students with obviously negative final areas could not attain full marks for their blunder.

End of Part E Solutions

Part F Solutions 12 marks

Question 36 (3 marks)

A continuous random variable has the probability density function $f(x)$ given by

$$f(x) = \begin{cases} \dfrac{kx}{5-x^2}, & 0 \le x \le 2 \\ 0, & \text{for all other values of } x \end{cases}$$

where k is a positive constant.

(a) Show that the value of k is $\dfrac{2}{\ln 5}$. **2**

$$\int_0^2 f(x)\,dx = 1 \Rightarrow \int_0^2 \frac{kx}{5-x^2}\,dx = 1$$

$$\therefore \frac{k}{-2}\int_0^2 \frac{-2x}{5-x^2}\,dx = 1$$

$$\therefore \frac{k}{-2}\left[\ln\left(5-x^2\right)\right]_0^2 = 1$$

$$\therefore k\left(\ln 1 - \ln 5\right) = -2$$

$$\therefore k = \frac{-2}{-\ln 5} = \frac{2}{\ln 5}$$

Comment: Students who had prepared did very well, otherwise the mark was generally 0.

Question 36 continues on page F 2

(b) Find the cumulative distribution function, $F(x)$. **2**

$$F(x) = \begin{cases} 0, & x < 0 \\ p(x) & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

$$p(x) = \int_0^x \frac{kx}{5 - x^2}\, dx$$

$$= \frac{k}{-2} \int_0^x \frac{-2x}{5 - x^2}\, dx$$

$$= \frac{k}{-2}\left[\ln\left(5 - x^2\right)\right]_0^x$$

$$= -\frac{1}{\ln 5}\left(\ln\left(5 - x^2\right) - \ln 5\right)$$

$$= -\frac{1}{\ln 5}\ln\left(5 - x^2\right) + 1$$

Comments: Less than 40% of students got to the middle function in the piecewise function
Roughly 5% of students got this fully correct.

Common mistakes:

1 $$\int_0^x \frac{\frac{2}{\ln 5}x}{5 - x^2}\, dx = \frac{2}{\ln 5}\left[\ln\left(5 - x^2\right)\right]_0^x$$

2 Some students didn't know $\ln 1 = 0$

$$\int \frac{kx}{5 - x^2}\, dx = -\frac{k}{2}\ln\left(5 - x^2\right) + C$$

$$= -\frac{1}{\ln 5}\ln\left(5 - x^2\right) + C \Rightarrow C = 0$$

3 Most common
Students didn't show the functions for $x < 0$ and for $x > 2$

Question 36 continues on page F 3

(c) Hence, or otherwise, show that the median of X is $\sqrt{5-\sqrt{5}}$. **2**

The median, m, is given by $F(m) = \frac{1}{2}$, where $0 \le m \le 2$.

Method 1

$$-\frac{1}{\ln 5}\ln\left(5 - m^2\right) + 1 = \frac{1}{2}$$

$$\therefore \frac{1}{\ln 5}\ln\left(5 - m^2\right) = \frac{1}{2}$$

$$\therefore \ln\left(5 - m^2\right) = \frac{1}{2}\ln 5 = \ln\sqrt{5}$$

$$\therefore 5 - m^2 = \sqrt{5}$$

$$\therefore m^2 = 5 - \sqrt{5}$$

$$\therefore m = \sqrt{5 - \sqrt{5}} \qquad (\because 0 \le m \le 2)$$

Method 2

$$p\left(\sqrt{5 - \sqrt{5}}\right) = -\frac{1}{\ln 5}\ln\left(5 - m^2\right) + 1$$

$$= -\frac{1}{\ln 5}\ln\left[5 - \left(\sqrt{5 - \sqrt{5}}\right)^2\right] + 1$$

$$= -\frac{1}{\ln 5}\ln\left[5 - \left(5 - \sqrt{5}\right)\right] + 1$$

$$= -\frac{1}{\ln 5}\ln\sqrt{5} + 1$$

$$= -\frac{1}{2} + 1 \qquad \left[\ln\sqrt{5} = \frac{1}{2}\ln 5\right]$$

$$= \frac{1}{2}$$

Comments:
Most students used Method 1.
Those who got part (b) correct did well on this part.

End of Question 36

Question 37 (3 marks)

A continuous random variable, X, has a cumulative distribution function given by: **3**

$$F(x)=\frac{1}{16}x^2\left(5-2\sqrt{x}\right) \text{ for } 0\le x\le 4.$$

Find the (exact) mode of X.

$$F(x)=\frac{1}{16}x^2\left(5-2\sqrt{x}\right)$$

$$=\frac{1}{16}\left(5x^2-2x^{2.5}\right)$$

$$F'(x)=\frac{1}{16}\left(10x-5x^{1.5}\right) \qquad \left[\text{This is the pdf}\right]$$

$$F''(x)=\frac{1}{16}\left(10-7.5x^{\frac{1}{2}}\right)$$

$$F'''(x)=\frac{1}{32}\left(-7.5x^{-\frac{1}{2}}\right)$$

The mode is the greatest value of the pdf in $0\le x\le 4$.

$$F''(x)=0\Rightarrow\frac{1}{16}\left(10-7.5x^{\frac{1}{2}}\right)=0$$

$$\therefore 10-7.5x^{\frac{1}{2}}=0\Rightarrow 7.5x^{\frac{1}{2}}=10$$

$$\therefore x^{\frac{1}{2}}=\frac{10}{7.5}=\frac{4}{3}\Rightarrow x=\frac{16}{9}$$

$$F'''\left(\frac{16}{9}\right)<0$$

$\therefore$ global maximum = mode = $\frac{16}{9}$

Common mistakes:

1 Students found the mode using $F'(x) = 0$

2 Algebra mistakes e.g. $x^2\sqrt{x} = x^{\frac{3}{2}}$.

3 Some students tried to integrate $F(x)$

Most students got between 1.5 – 2 marks.

Question 38 (3 marks)

(a) Find the value of μ such that **1**

$$n(n+1)(n+2)(n+3) = (n^2 + \mu n)(n^2 + \mu n + 2).$$

$$\begin{aligned} n(n+1)(n+2)(n+3) &= n(n+3) \times (n+1)(n+2) \\ &= (n^2 + 3n)\,(n^2 + 3n + 2) \end{aligned}$$

$\therefore \mu = 3$

Comment: No half marks, but most students got this correct

(b) Let n be an integer. **2**

Deduce that $2022 \times 2023 \times 2024 \times 2025 + 1$ is a perfect square.

Let $n = 2022$

$\therefore 2022 \times 2023 \times 2024 \times 2025 + 1 = (n^2 + 3n)\,(n^2 + 3n + 2) + 1$

Let $k = n^2 + 3n \qquad (\in \mathbb{Z})$

$\therefore 2022 \times 2023 \times 2024 \times 2025 + 1 = k(k+2) + 1 = k^2 + 2k + 1 = (k+1)^2 \qquad (\in \mathbb{Z})$

$\therefore 2022 \times 2023 \times 2024 \times 2025 + 1$ is a perfect square.

Alternate Method: Let $n = 2022$ and $k = n^2 + 3n + 1$

$$\begin{aligned} \therefore 2022 \times 2023 \times 2024 \times 2025 + 1 &= (k-1)(k+1) + 1 = k^2 \\ &= (n^2 + 3n + 1)^2 \end{aligned}$$

Comment: Some students tried to use Pascal's Triangle – unsuccessfully.

End of solutions