SYDNEY GRAMMAR SCHOOL

NAME _____ MATHEMATICS MASTER _____

2021 Term III Annual Examination

Form V Mathematics Advanced

Friday 3rd September 2021
12:55pm

General Instructions

- Working time — 2 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 80

Section I (8 marks) Questions 1 – 8

- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (72 marks) Questions 9 – 14

- Because of the nature of this task, greater weight than normal will be placed on working. Clear reasoning and full calculations are required.
- Write your name and master on each page.

Your sheets must be ORDERED
then scanned and uploaded in a
SINGLE PDF FILE
to the Schoology page of your Mathematics class

5A: WJM	5B: SDP	5C: LYL	5D: NJL
5E: FMW	5F: RPL	5G: BLR	5H: CWG
5I: AEA	5P: JAZ	5Q: TCW	

Checklist

- Reference Sheet
- Multiple-choice answer sheet

Writer: FMW

	Marks
Multiple Choice	
Question 9	
Question 10	
Question 11	
Question 12	
Question 13	
Question 14	
TOTAL	

Section I

QUESTION ONE (1 mark)

What are the solutions of the equation $x^2 - 6x + 5 = 0$?

(A) $x = -5$, $x = -1$

(B) $x = 5$, $x = 1$

(C) $x = -2$, $x = -3$

(D) $x = 2$, $x = 3$

QUESTION TWO (1 mark)

A fair coin is tossed twice. What is the probability of tossing two tails?

(A) $\frac{1}{4}$

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) $\frac{2}{3}$

QUESTION THREE (1 mark)

What is the value of the gradient of the tangent to the curve $y = 3x^2 + 2x$ at the point $(1, 5)$?

(A) 1

(B) 4

(C) 6

(D) 8

QUESTION FOUR (1 mark)

Which of the following describes the relation $y = |x - 3|$?

(A) one-to-one

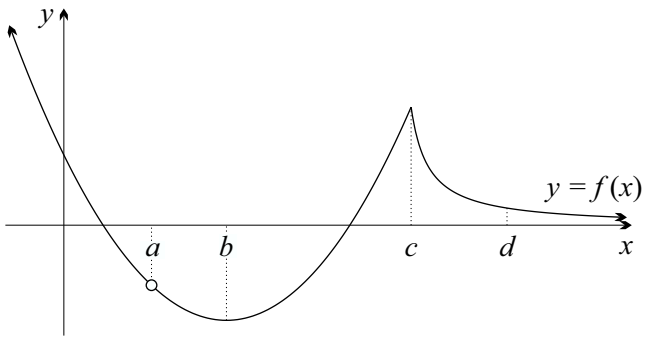
(B) one-to-many

(C) many-to-one

(D) many-to-many

QUESTION FIVE (1 mark)

The diagram below shows the graph of $y = f(x)$.



At which value of x is the graph continuous but not differentiable?

(A) a

(B) b

(C) c

(D) d

QUESTION SIX (1 mark)

What is the equation of the line parallel to $y = 2x - 1$ with x-intercept $(-3, 0)$?

(A) $2x - y - 3 = 0$

(B) $x + 2y + 3 = 0$

(C) $x - 2y + 6 = 0$

(D) $2x - y + 6 = 0$

QUESTION SEVEN (1 mark)

Given the functions $y = \tan x$ and $y = -\cos x$, which of the following is a true statement?

(A) Both functions are EVEN.

(B) Both functions are ODD.

(C) $y = \tan x$ is EVEN and $y = -\cos x$ is ODD.

(D) $y = \tan x$ is ODD and $y = -\cos x$ is EVEN.

QUESTION EIGHT (1 mark)

A point lies on the circle with equation $x^2 + y^2 - 24x - 10y + 48 = 0$. What is the furthest distance, in units, the point could be from the origin?

(A) 29

(B) 24

(C) 13

(D) 11

End of Section I

The paper continues in the next section

Section II

QUESTION NINE (12 marks) **Marks**

(a) Evaluate $\log_2 16$. 1

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(b) Rationalise the denominator of $\dfrac{3}{\sqrt{5}-2}$. 1

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(c) Find the distance between the points $(-5,\ 2)$ and $(4,\ -1)$, leaving your answer in simplified surd form. 2

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QUESTION NINE (Continued)

(d) Simplify $\dfrac{y^2+3y}{y^2-9}$. [2]

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(e) (i) Sketch the curve $y = 2^{-x}$, clearly showing all important features. [2]

(ii) Use your graph above to explain why $y = 2^{-x}$ is a function. [1]

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QUESTION NINE (Continued)

(f) Explain in words how to transform the graph of $y = (x-1)^3$ to obtain each of the following functions.

(i) $y = -(x-1)^3$ [1]

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(ii) $y = (x+2)^3 + 4$ [2]

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QUESTION TEN (12 marks) **Marks**

(a) Differentiate with respect to x:

(i) $y = 7 - 9x^5$ 1

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(ii) $y = \sqrt{x}$ 1

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(iii) $y = \dfrac{3}{x^4}$ 1

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(iv) $y = (x - 3)(x + 5)$ 1

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(v) $y = e^{\frac{1}{2}x+1}$ 1

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QUESTION TEN (Continued)

(b) What is the domain of the function $y = \sqrt{x+2}$? [1]

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(c) Find the equation of the perpendicular bisector of the interval joining the points $(2, 6)$ and $(-1, 5)$, writing your final answer in general form. [3]

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QUESTION TEN (Continued)

(d) Write down the exact value of $\tan \dfrac{2\pi}{3}$. [1]

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(e) Solve $\sin x = 0$ for $0 \leq x \leq 2\pi$. [2]

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QUESTION ELEVEN (12 marks) **Marks**

(a) Sketch the graphs of the following functions on separate number planes, showing all intercepts with the axes and all asymptotes. You should show at least two specific points for each graph.

(i) $y = \dfrac{2}{x+1}$ 2

(ii) $y = \log_e x + 1$ 2

QUESTION ELEVEN (Continued)

(b) (i) Differentiate $y = (2x - 3)^3$. [2]

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(ii) Hence, find all the points on the curve $y = (2x - 3)^3$ where the gradient of the tangent is equal to 24. [3]

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QUESTION ELEVEN (Continued)

(c) A triangle has sides 13 cm, 7 cm and $5\sqrt{3}$ cm. Find the size of the smallest angle of the triangle. Give your answer in degrees, approximating correct to the nearest minute if necessary. 3

QUESTION TWELVE (12 marks) **Marks**

(a) Solve $|4x - 3| = 7$. 2

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(b) Differentiate with respect to x:

(i) $y = \dfrac{x}{x-1}$ 2

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(ii) $y = 5x(2-x)^7$. Give your answer in fully factorised form. 2

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QUESTION TWELVE (Continued)

(c) (i) Express $x^2 - 4x + 8$ in the form $(x - p)^2 + q$. [2]

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(ii) Hence, explain why $x^2 - 4x + 8 = 0$ has no real solutions. [1]

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(d) Consider the functions $f(x) = \dfrac{1}{x}$ and $g(x) = 1 - x$.
Find:

(i) $g(-2a)$ [1]

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(ii) $f\big(g(x)\big)$ and state the value of x for which the composite function is not defined. [2]

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QUESTION THIRTEEN (12 marks) **Marks**

(a) Consider the function $f(x) = x^2 - 3x + 7$.

(i) Write down an expression for $f(x + h)$. 1

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(ii) Hence, differentiate $f(x)$ from first principles. 2

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QUESTION THIRTEEN (Continued)

(b) Ravi has three floral shirts and four striped shirts. On each of the three days Monday, Tuesday and Wednesday, he selects one shirt at random to wear. He only wears a shirt for one day and does not wear it again in the same week.

(i) Draw a probability tree diagram to illustrate the sample space. 1

(ii) What is the probability that Ravi wears a floral shirt on Monday and a striped shirt on each of the other days? 1

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QUESTION THIRTEEN (Continued)

(iii) What is the probability that Ravi wears the same type of shirt on all three days? [1]

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(iv) Given that he wears a striped shirt on Monday, what is the probability that Ravi wears a striped shirt on both the other two days? [1]

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(c) A scientist is experimenting with a colony of bacteria and has discovered that the number of bacteria N after t minutes is given by $N = N_0 e^{kt}$ where k is a positive constant. There are initially 2000 bacteria in the colony and after $1\frac{1}{2}$ hours this has grown to 4000 bacteria.

(i) Show that $k = \frac{1}{90} \log_e 2$. [1]

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QUESTION THIRTEEN (Continued)

(ii) What is the rate of change of the bacteria, in bacteria per minute, after 3 hours? Give your answer to 4 significant figures. [2]

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(iii) How long will it take for the bacteria colony to reach 100 000? Give your answer in hours and minutes. Round your answer to the nearest minute. [2]

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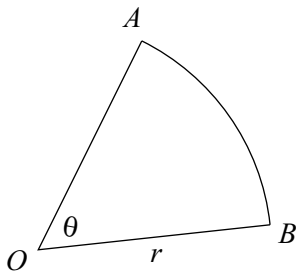
QUESTION FOURTEEN (12 marks) **Marks**

(a) Find $\dfrac{d}{dx}\left(\dfrac{3x}{\sqrt{1-3x}}\right)$, expressing your answer as a single fraction with no negative indices. [2]

QUESTION FOURTEEN (Continued)

(b) 3



The diagram above shows a sector AOB of a circle of radius r and centre O with $\angle AOB$ in radians. If $\angle AOB = \theta$ find the value of θ if the area of the sector is equal to twice the square of the length of arc AB.

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QUESTION FOURTEEN (Continued)

(c) The line l is the tangent to the curve $y = 1 - \dfrac{1}{x}$ at the point where $x = a$, where a is a constant.

(i) Show that l has equation $x - a^2y = 2a - a^2$. 2

QUESTION FOURTEEN (Continued)

(ii) If the line l passes through the origin, find the value of a and hence the equation of l. [2]

QUESTION FOURTEEN (Continued)

(d) Find any points on the curve $y = e^x + 2e^{-x}$ where the tangent is horizontal. Give your answer/s in exact form. [3]

END OF PAPER

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- Multiple-choice answer sheet

Writer: FMW

	Marks
Multiple Choice	
Question 9	
Question 10	
Question 11	
Question 12	
Question 13	
Question 14	
TOTAL	

Section I

QUESTION ONE

What are the solutions of the equation $x^2 - 6x + 5 = 0$?

$(x-5)(x-1) = 0$
$x = 5$ or $x = 1$

(A) $x = -5$, $x = -1$

(B) $x = 5$, $x = 1$

(C) $x = -2$, $x = -3$

(D) $x = 2$, $x = 3$

QUESTION TWO

A fair coin is tossed twice. What is the probability of tossing two tails?

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(A) $\frac{1}{4}$

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) $\frac{2}{3}$

QUESTION THREE

What is the value of the gradient of the tangent to the curve $y = 3x^2 + 2x$ at the point $(1, 5)$?

$y' = 6x + 2$
$= 6(1) + 2$
$= 8$

(A) 1

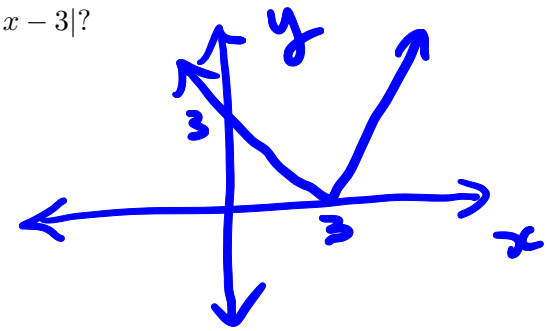
(B) 4

(C) 6

(D) 8

QUESTION FOUR

Which of the following describes the relation $y = |x - 3|$?



(A) one-to-one

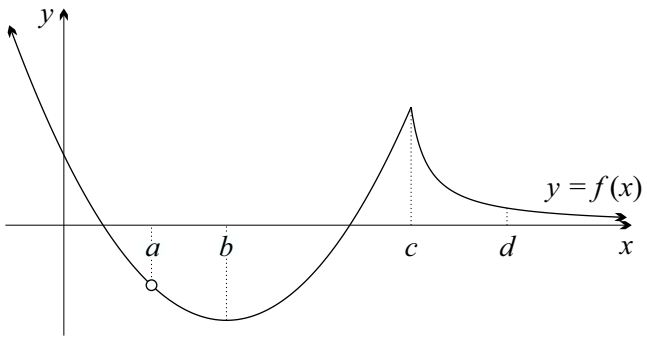
(B) one-to-many

(C) many-to-one

(D) many-to-many

QUESTION FIVE

The diagram below shows the graph of $y = f(x)$.



At which value of x is the graph continuous but not differentiable?

(A) a

(B) b

(C) c

(D) d

QUESTION SIX

What is the equation of the line parallel to $y = 2x - 1$ with x-intercept $(-3, 0)$?

(A) $2x - y - 3 = 0$

(B) $x + 2y + 3 = 0$

(C) $x - 2y + 6 = 0$

(D) $2x - y + 6 = 0$

$y - 0 = 2(x + 3)$

$y = 2x + 6$

$2x - y + 6 = 0$

QUESTION SEVEN

Given the functions $y = \tan x$ and $y = -\cos x$, which of the following is a true statement?

(A) Both functions are EVEN.

(B) Both functions are ODD.

(C) $y = \tan x$ is EVEN and $y = -\cos x$ is ODD.

(D) $y = \tan x$ is ODD and $y = -\cos x$ is EVEN.

QUESTION EIGHT

A point lies on the circle with equation $x^2 + y^2 - 24x - 10y + 48 = 0$. What is the furthest distance, in units, the point could be from the origin?

(A) 29

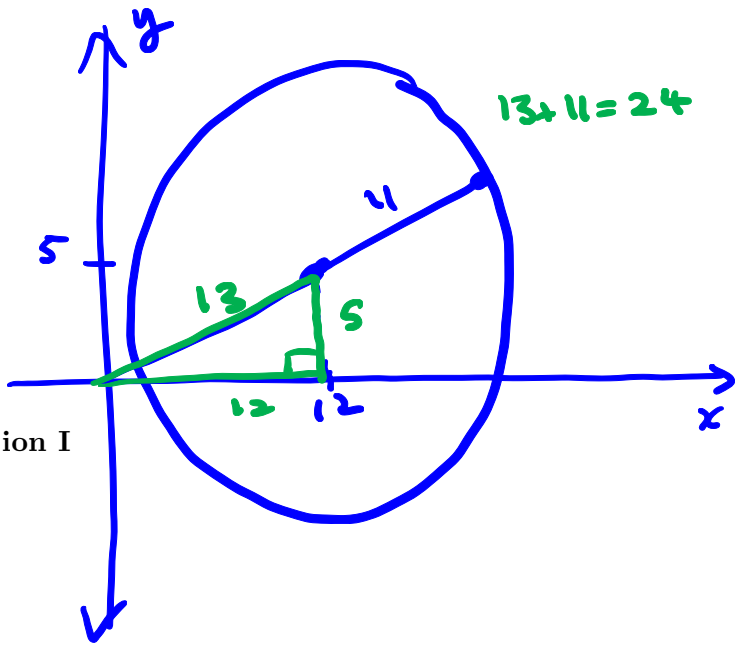
(B) 24

(C) 13

(D) 11

$x^2 - 24x + 144 + y^2 - 10y + 25 = -48 + 144 + 25$

$(x-12)^2 + (y-5)^2 = 121$



End of Section I

The paper continues in the next section

Section II

QUESTION NINE (12 marks) **Marks**

(a) Evaluate $\log_2 16$. [1]

$2^4 = 16$ so $\log_2 16 = 4$ ✓

(b) Rationalise the denominator of $\frac{3}{\sqrt{5}-2}$. [1]

$= \frac{3}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2}$

$= \frac{3\sqrt{5}+6}{5-4}$

$= 3\sqrt{5}+6$ ✓

(c) Find the distance between the points $(-5, 2)$ and $(4, -1)$, leaving your answer in simplified surd form. [2]

$d^2 = (2+1)^2 + (-5-4)^2$ ✓

$= 3^2 + (-9)^2$

$= 9 + 81$

$= 90$

$d = \sqrt{90}$

$= 3\sqrt{10}$ ✓

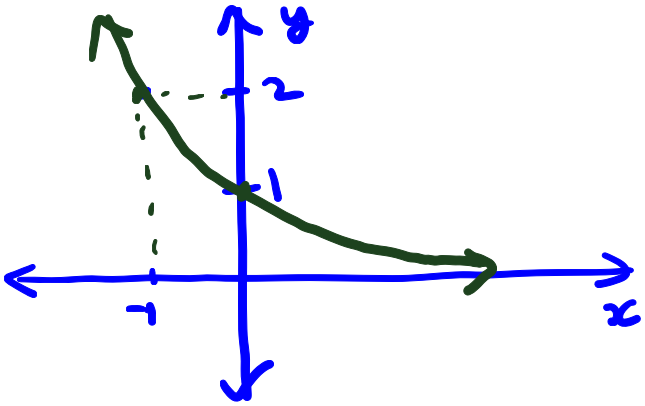
QUESTION NINE (Continued)

(d) Simplify $\dfrac{y^2+3y}{y^2-9}$. [2]

$= \dfrac{y(y+3)}{(y-3)(y+3)}$ ✓

$= \dfrac{y}{y-3}$ ✓

(e) (i) Sketch the curve $y = 2^{-x}$, clearly showing all important features. [2]



✓ correct shape with $y=0$ clearly an asymptote

✓ two sensible points including the y-intercept

(ii) Use your graph above to explain why $y = 2^{-x}$ is a function. [1]

The graph passes the vertical line test.

OR: The graph is one-to-one

OR Similar ✓

QUESTION NINE (Continued)

(f) Explain in words how to transform the graph of $y = (x-1)^3$ to obtain each of the following functions.

(i) $y = -(x-1)^3$. [1]

reflect across the x-axis. ✓

(ii) $y = (x+2)^3 + 4$. [2]

shift 3 units to the left ✓
and four units up. ✓

QUESTION TEN (12 marks) **Marks**

(a) Differentiate with respect to x:

(i) $y = 7 - 9x^5$ [1]

$\frac{dy}{dx} = -45x^4$ ✓

(ii) $y = \sqrt{x}$ [1]

$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ ✓

$= \frac{1}{2\sqrt{x}}$

(iii) $y = \frac{3}{x^4}$ [1]

$y = 3x^{-4}$

$\frac{dy}{dx} = -12x^{-5}$ ✓

$= -\frac{12}{x^5}$

(iv) $y = (x-3)(x+5)$ [1]

$y = x^2 + 2x - 15$ ✓

$\frac{dy}{dx} = 2x + 2$

(v) $y = e^{\frac{1}{2}x+1}$ [1]

$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x+1}$ ✓

QUESTION TEN (Continued)

(b) What is the domain of the function $y = \sqrt{x+2}$? [1]

$x + 2 \geq 0$ ✓

$x \geq -2$

(c) Find the equation of the perpendicular bisector of the interval joining the points $(2, 6)$ and $(-1, 5)$, writing your final answer in general form. [3]

$M = \left(\frac{2-1}{2}, \frac{6+5}{2}\right)$ ✓ $m_1 = \frac{6-5}{2+1}$

$= \left(\frac{1}{2}, \frac{11}{2}\right)$ ✓ $= \frac{1}{3}$

$m_2 = -3$

equation is:

$y - \frac{11}{2} = -3\left(x - \frac{1}{2}\right)$ ✓

$2y - 11 = -6x + 3$

$6x + 2y - 14 = 0$ ✓

$3x + y - 7 = 0$

must be in general form

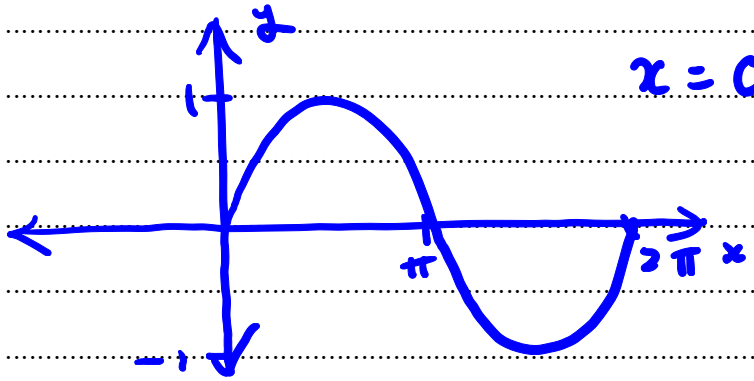
QUESTION TEN (Continued)

(d) Write down the exact value of $\tan \frac{2\pi}{3}$. [1]

$\tan \frac{2\pi}{3} = -\tan \frac{\pi}{3}$ ✓

$= -\sqrt{3}$ ✓

(e) Solve $\sin x = 0$ for $0 \leq x \leq 2\pi$. [2]

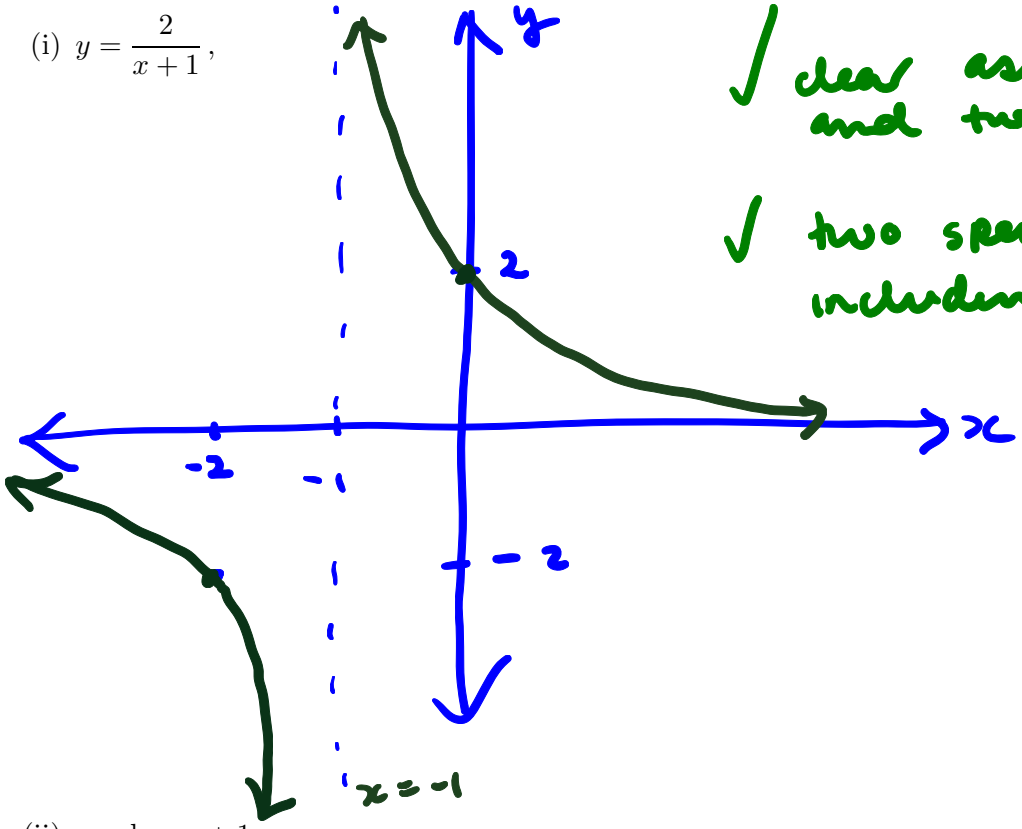


$x = 0, \pi$ or 2π ✓✓

QUESTION ELEVEN (12 marks) **Marks**

(a) Sketch the graphs of the following functions on separate number planes, showing all intercepts with the axes and all asymptotes. You should show at least two specific points for each graph.

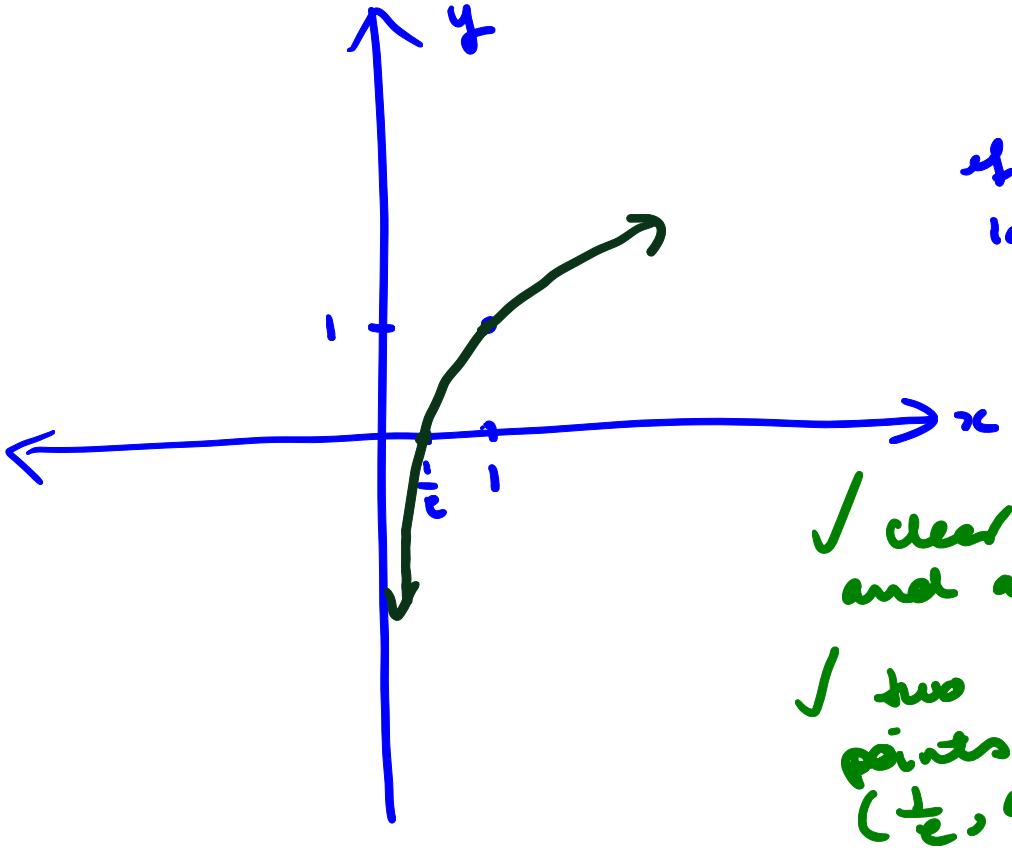
(i) $y = \dfrac{2}{x+1}$, 2



✓ clear asymptotes and two parts

✓ two specific points including (0, 2)

(ii) $y = \log_e x + 1$. 2



if $y = 0$

$\log_e x + 1 = 0$

$\log_e x = -1$

$x = \frac{1}{e}$

✓ clear shape and asymptote

✓ two specific points including $(\frac{1}{e}, 0)$

QUESTION ELEVEN (Continued)

(b) (i) Differentiate $y = (2x-3)^3$. [2]

$\frac{dy}{dx} = 3(2x-3)^2 \times 2$ ✓✓

$= 6(2x-3)^2$

(ii) Hence find all the points on the curve $y = (2x-3)^3$ where the gradient of the tangent is equal to 24. [3]

$6(2x-3)^2 = 24$ ✓

$(2x-3)^2 = 4$

$2x-3 = 2$ or $2x-3 = -2$ ✓

$2x = 5$ $\quad$ $2x = 1$

$x = \frac{5}{2}$ $\quad$ $x = \frac{1}{2}$

$y = 8$ $\quad$ $y = -8$

the points $(\frac{5}{2}, 8)$ and $(\frac{1}{2}, -8)$

OR $(2x-3)^2 = 4$

$4x^2 - 12x + 9 = 4$

$4x^2 - 12x + 5 = 0$

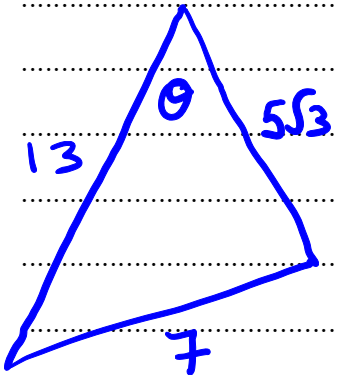
$4x^2 - 2x - 10x + 5 = 0$

$2x(2x-1) - 5(2x-1) = 0$

$(2x-1)(2x-5) = 0$ etc.

QUESTION ELEVEN (Continued)

(c) A triangle has sides 13 cm, 7 cm and $5\sqrt{3}$ cm. Find the size of the smallest angle of the triangle. Give your answer in degrees, approximating correct to the nearest minute if necessary. [3]



note the smallest angle is opposite the smallest side.

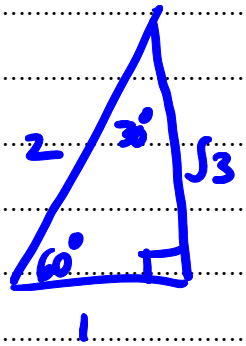
$$\cos\theta = \frac{13^2 + (5\sqrt{3})^2 - 7^2}{2 \times 13 \times 5\sqrt{3}}$$

$$= \frac{169 + 75 - 49}{130\sqrt{3}}$$

$$= \frac{195}{130\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{195\sqrt{3}}{390}$$

$$= \frac{\sqrt{3}}{2}$$



$$\theta = 30°$$

QUESTION TWELVE (12 marks) **Marks**

(a) Solve $|4x - 3| = 7$. [2]

$4x - 3 = 7$ ✓ $\quad 4x - 3 = -7$

$4x = 10$ ✓ $\quad 4x = -4$

$x = \frac{5}{2}$ $\quad x = -1$

(b) Differentiate with respect to x:

(i) $y = \dfrac{x}{x-1}$, [2]

$\dfrac{dy}{dx} = \dfrac{(x-1)(1) - x(1)}{(x-1)^2}$ ✓ $\quad u = x \quad v = x - 1$

$u' = 1 \quad v' = 1$

$= \dfrac{x - 1 - x}{(x-1)^2}$

$= -\dfrac{1}{(x-1)^2}$ ✓

(ii) $y = 5x(2-x)^7$. Give your answer in fully factorised form. [2]

$y' = 5x \times 7(2-x)^6 \times (-1) + (2-x)^7 \times 5$ ✓

$= 5(2-x)^6(-7x + 2 - x)$

$= 5(2-x)^6(2 - 8x)$

$= 10(2-x)^6(1 - 4x)$ ✓

QUESTION TWELVE (Continued)

(c) (i) Express $x^2 - 4x + 8$ in the form $(x-p)^2 + q$. [2]

$x^2 - 4x + 4 + 8 - 4$

$= (x-2)^2 + 4$ ✓ ✓

(ii) Hence explain why $x^2 - 4x + 8 = 0$ has no real solutions. [1]

must use (i) ✓

If $(x-1)^2 + 4 = 0$

$(x-1)^2 = -4$ which has no solution

(OR: the minimum value of x^2-4x+8 is 4

OR: the graph is concave up with vertex at $(2,4)$)

(d) Consider the functions $f(x) = \dfrac{1}{x}$ and $g(x) = 1 - x$.

Find:

(i) $g(-2a)$ [1]

$g(-2a) = 1 - (-2a)$

$= 1 + 2a$ ✓

(ii) $f(g(x))$ and state the value of x for which the composite function is not defined. [2]

$f(g(x)) = \dfrac{1}{1-x}$ ✓ which is not

defined at $1 - x = 0$

$x = 1$ ✓

QUESTION THIRTEEN (12 marks) **Marks**

(a) Consider the function $f(x) = x^2 - 3x + 7$.

(i) Write down an expression for $f(x+h)$. [1]

$f(x+h) = (x+h)^2 - 3(x+h) + 7$ ✓

(ii) Hence, differentiate $f(x)$ from first principles. [2]

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) + 7 - (x^2 - 3x + 7)}{h}$$ ✓

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 7 - x^2 + 3x - 7}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h}$$

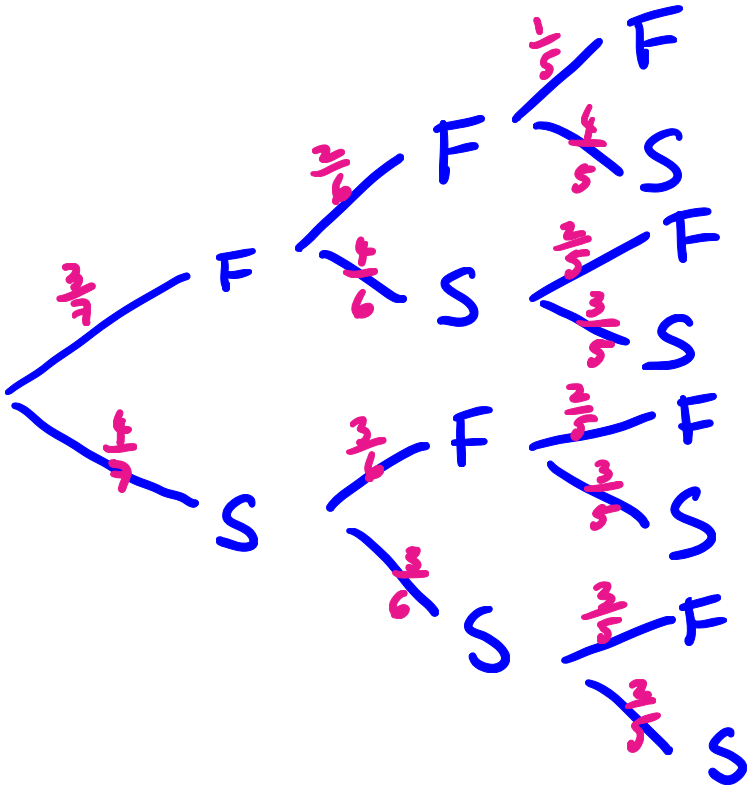
$$= \lim_{h \to 0} 2x + h - 3$$ ✓

$$= 2x - 3$$

QUESTION THIRTEEN (Continued)

(b) Ravi has three floral shirts and four striped shirts. On each of the three days Monday, Tuesday and Wednesday, he selects one shirt at random to wear. He only wears a shirt for one day and does not wear it again in the same week.

(i) Draw a probability tree diagram to illustrate the sample space. [1]



Must have all branches ✓

(ii) What is the probability that Ravi wears a floral shirt on Monday and a striped shirt on the other days? [1]

$P(E) = \frac{3}{7} \times \frac{4}{6} \times \frac{3}{5}$ ✓

$= \frac{6}{35}$

QUESTION THIRTEEN (Continued)

(iii) What is the probability that Ravi wears the same type of shirt on all three days? [1]

$P(E) = \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} + \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5}$

$= \frac{1}{35} + \frac{4}{35}$

$= \frac{5}{35}$

$= \frac{1}{7}$ ✓

(iv) Given that he wears a striped shirt on Monday, what is the probability that Ravi wears a striped shirt on both the other two days? [1]

$P(A/B) = \dfrac{\frac{4}{7} \times \frac{3}{6} \times \frac{2}{5}}{\frac{4}{7}}$

$= \frac{1}{5}$

or from the tree

$\frac{3}{6} \times \frac{2}{5} = \frac{1}{5}$

(c) A scientist is experimenting with a colony of bacteria and has discovered that the number of bacteria N after t minutes is given by $N = N_0 e^{kt}$ where k is a positive constant. There are initially 2000 bacteria in the colony and after $1\frac{1}{2}$ hours this has grown to 4000 bacteria.

(i) Show that $k = \frac{1}{90} \log_e 2$. [1]

Note, $N_0 = 2000$ and $1\frac{1}{2}$ hours $= 90$ minutes

$4000 = 2000\, e^{90k}$

$2 = e^{90k}$ ✓

$90k = \log_e 2$

$k = \frac{1}{90} \log_e 2$

QUESTION THIRTEEN (Continued)

(ii) What is the rate of change of the bacteria, in bacteria per minute, after 3 hours? Give your answer to 4 significant figures. [2]

$3h = 180 min$

$\frac{dN}{dt} = kN$

$= k \times 2000 e^{180k}$

$= \frac{1}{90} \log_e 2 \times 2000 e^{180 \times \frac{1}{90} \log_e 2}$

$\doteq 61.61$ bacteria/min ✓

(iii) How long will it take for the bacteria colony to reach 100 000? Give your answer in hours and minutes. Round your answer to the nearest minute. [2]

$100000 = 2000 e^{kt}$ ✓

$50 = e^{kt}$

$kt = \log_e 50$

$t = \frac{1}{k} \log_e 50$

$= 507.94...$

$= 8.465.. h$

$\doteq 8 h 28 min$ ✓

QUESTION FOURTEEN (12 marks) **Marks**

(a) Find $\frac{d}{dx}\left(\frac{3x}{\sqrt{1-3x}}\right)$, expressing your answer as a single fraction with no negative indices. 2

$u = 3x \qquad v = (1-3x)^{\frac{1}{2}}$

$u' = 3 \qquad v' = \frac{1}{2}(1-3x)^{-\frac{1}{2}} \times -3$

$= \dfrac{(1-3x)^{\frac{1}{2}} \times 3 - 3x \times -\frac{3}{2}(1-3x)^{-\frac{1}{2}}}{1-3x}$

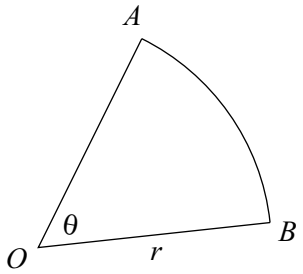
$= \dfrac{\frac{3}{2}(1-3x)^{-\frac{1}{2}}\left(2(1-3x)+3x\right)}{1-3x}$

$= \dfrac{3(2-6x+3x)}{2\sqrt{1-3x}\,(1-3x)}$

$= \dfrac{3(2-3x)}{2\sqrt{1-3x}\,(1-3x)}$

QUESTION FOURTEEN (Continued)

(b) 3



The diagram above shows a sector AOB of a circle of radius r and centre O with $\angle AOB$ in radians. If $\angle AOB = \theta$ find the value of θ if the area of the sector is equal to twice the square of the length of arc AB.

using $l = r\theta$ and $A = \frac{1}{2}r^2\theta$

$\frac{1}{2}r^2\theta = 2l^2$

$\frac{1}{2}r^2\theta = 2r^2\theta^2$ ✓

$r^2\theta = 4r^2\theta^2$ ✓

$r^2\theta - 4r^2\theta^2 = 0$

$r^2\theta(1 - 4\theta) = 0$

$\theta = 0$ or $4\theta = 1$

trivial $\theta = \frac{1}{4}$ ✓

QUESTION FOURTEEN (Continued)

(c) The line l is the tangent to the curve $y = 1 - \frac{1}{x}$ at the point where $x = a$, where a is a constant.

(i) Show that l has equation $x - a^2y = 2a - a^2$. 2

$y = 1 - x^{-1}$

$y' = x^{-2}$

$= \frac{1}{x^2}$

at $x = a$, $y' = \frac{1}{a^2}$ ✓ and $y = 1 - \frac{1}{a}$

tangent is

$y - (1 - \frac{1}{a}) = \frac{1}{a^2}(x - a)$

$a^2y - a^2(1 - \frac{1}{a}) = x - a$ ✓

$a^2y - a^2 + a = x - a$

$x - a^2y^2 = 2a - a^2$

as required

QUESTION FOURTEEN (Continued)

(ii) If the line l passes through the origin, find the value of a and hence the equation of l. [2]

if $(0,0)$ is on l

then

$0 - 0 = 2a - a^2$

$a(a-2) = 0$

$a = 0$ or $a = 2$ ✓

$a \neq 0$

as the curve is not defined at $x = 0$

so l has equation

$x - 2^2 y^2 = 2(2) - 2^2$

$x - 4y = 4 - 4$

$x - 4y = 0$ ✓

QUESTION FOURTEEN (Continued)

(d) Find any points on the curve $y = e^x + 2e^{-x}$ where the tangent is horizontal. Give your answer/s in exact form. [3]

$\frac{dy}{dx} = e^x - 2e^{-x}$

if $\frac{dy}{dx} = 0$, $e^x - 2e^{-x} = 0$ ✓

$e^{-x}(e^{2x} - 2) = 0$

$e^{-x} \neq 0$ or $e^{2x} = 2$

$2x = \log_e 2$

$x = \frac{1}{2}\log_e 2$ ✓

if $x = \frac{1}{2}\log_e 2$

$y = e^{\frac{1}{2}\log 2} + 2e^{-\frac{1}{2}\log_e 2}$

$= e^{\log_e \sqrt{2}} + 2e^{\log_e \frac{1}{\sqrt{2}}}$

$= \sqrt{2} + 2 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \sqrt{2} + \sqrt{2}$ ✓

$= 2\sqrt{2}$

the point is $(\frac{1}{2}\log_e 2, 2\sqrt{2})$

END OF PAPER

SYDNEY GRAMMAR SCHOOL

NAME _____ MATHEMATICS MASTER _____

2021 Term III Annual Examination

Form V Mathematics Advanced

Friday 3rd September 2021

- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ○ B ● C ○ D ○

Question Two

A ● B ○ C ○ D ○

Question Three

A ○ B ○ C ○ D ●

Question Four

A ○ B ○ C ● D ○

Question Five

A ○ B ○ C ● D ○

Question Six

A ○ B ○ C ○ D ●

Question Seven

A ○ B ○ C ○ D ●

Question Eight

A ○ B ● C ○ D ○