NSW GOVERNMENT

NSW Education Standards Authority

2020 **HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8–14)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which diagram best represents the solution set of $x^2 - 2x - 3 \geq 0$?

A.

B.

C.

D.

2 Given $f(x) = 1 + \sqrt{x}$, what are the domain and range of $f^{-1}(x)$?

A. $x \geq 0, \quad y \geq 0$

B. $x \geq 0, \quad y \geq 1$

C. $x \geq 1, \quad y \geq 0$

D. $x \geq 1, \quad y \geq 1$

3 Which of the following is an anti-derivative of $\dfrac{1}{4x^2+1}$?

A. $2\tan^{-1}\left(\dfrac{x}{2}\right) + c$

B. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + c$

C. $2\tan^{-1}(2x) + c$

D. $\dfrac{1}{2}\tan^{-1}(2x) + c$

4 Maria starts at the origin and walks along all of the vector $2\underset{\sim}{i}+3\underset{\sim}{j}$, then walks along all of the vector $3\underset{\sim}{i}-2\underset{\sim}{j}$ and finally along all of the vector $4\underset{\sim}{i}-3\underset{\sim}{j}$.

How far from the origin is she?

A. $\sqrt{77}$

B. $\sqrt{85}$

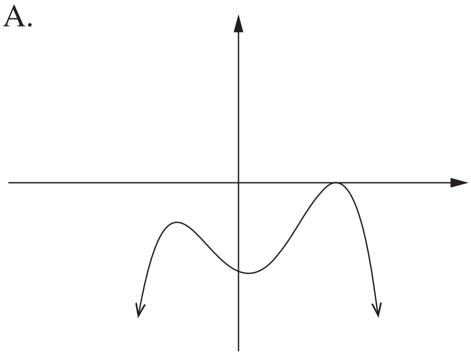
C. $2\sqrt{13}+\sqrt{5}$

D. $\sqrt{5}+\sqrt{7}+\sqrt{13}$

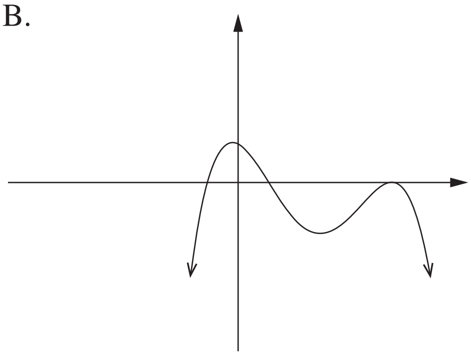
5 A monic polynomial $p(x)$ of degree 4 has one repeated zero of multiplicity 2 and is divisible by x^2+x+1.

Which of the following could be the graph of $p(x)$?

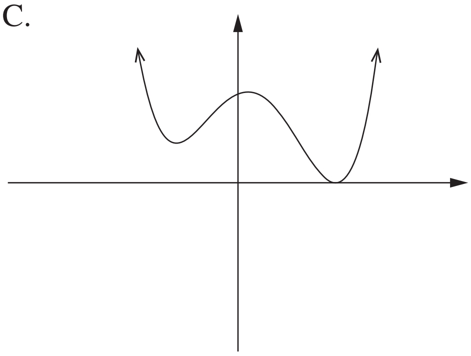
A.



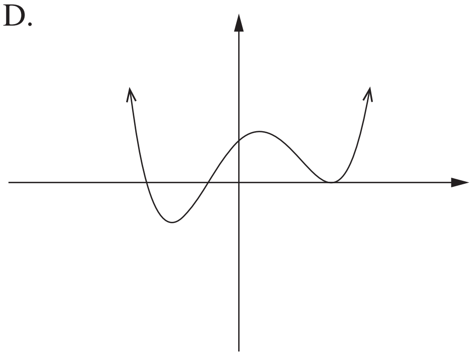
B.



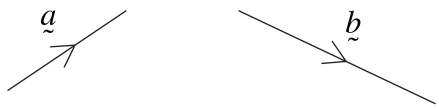
C.



D.

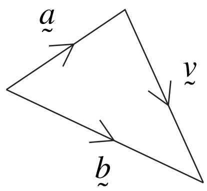


6 The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are shown.

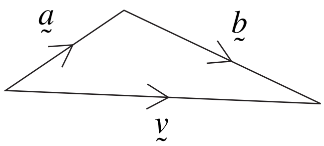


Which diagram below shows the vector $\underset{\sim}{v} = \underset{\sim}{a} - \underset{\sim}{b}$?

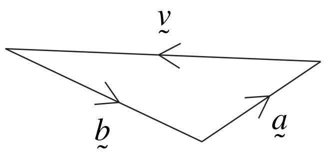
A.



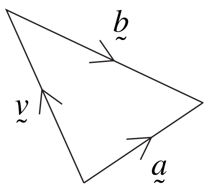
B.



C.

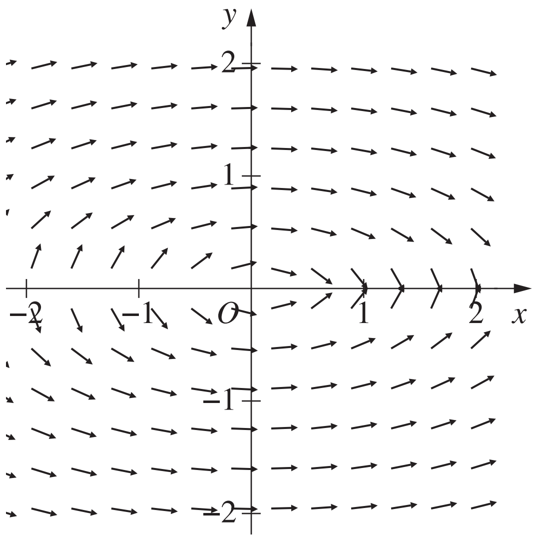


D.

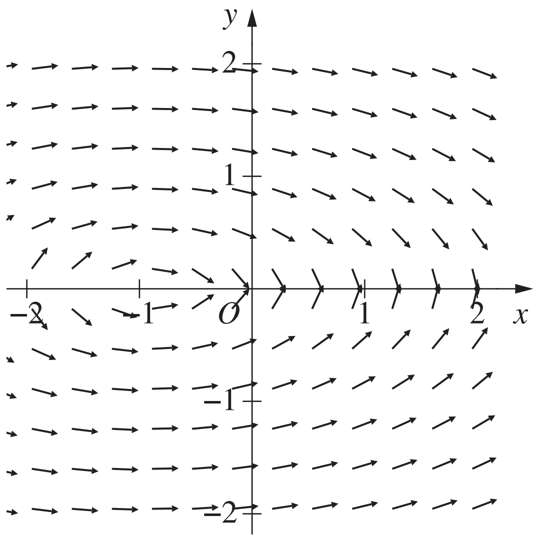


7 Which of the following best represents the direction field for the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{4y}$?

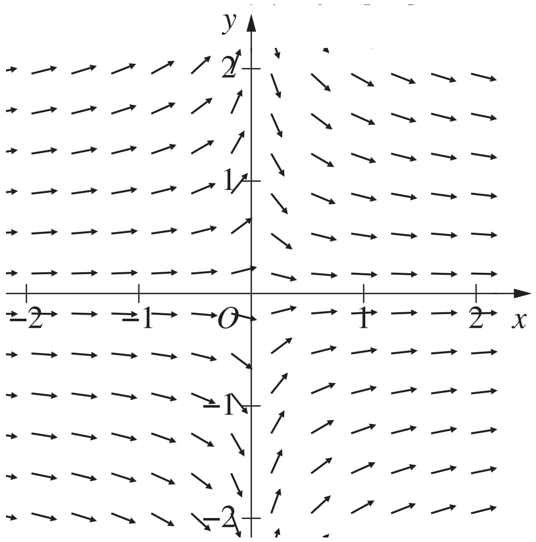
A.



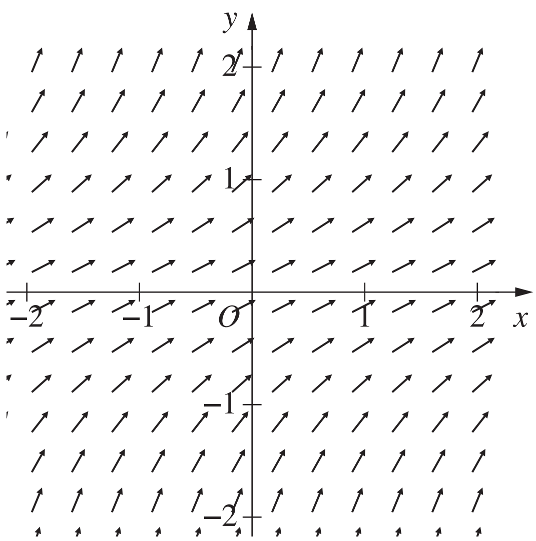
B.



C.



D.



8 Out of 10 contestants, six are to be selected for the final round of a competition. Four of those six will be placed 1st, 2nd, 3rd and 4th.

In how many ways can this process be carried out?

A. $\dfrac{10!}{6!4!}$

B. $\dfrac{10!}{6!}$

C. $\dfrac{10!}{4!2!}$

D. $\dfrac{10!}{4!4!}$

9 The projection of the vector $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ onto the line $y = 2x$ is $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$.

The point (6, 7) is reflected in the line $y = 2x$ to a point A.

What is the position vector of the point A?

A. $\begin{pmatrix} 6 \\ 12 \end{pmatrix}$

B. $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$

C. $\begin{pmatrix} -6 \\ 7 \end{pmatrix}$

D. $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

10 The quantities P, Q and R are connected by the related rates,

$$\frac{dR}{dt} = -k^2$$

$$\frac{dP}{dt} = -l^2 \times \frac{dR}{dt}$$

$$\frac{dP}{dt} = m^2 \times \frac{dQ}{dt}$$

where k, l and m are non-zero constants.

Which of the following statements is true?

A. P is increasing and Q is increasing

B. P is increasing and Q is decreasing

C. P is decreasing and Q is increasing

D. P is decreasing and Q is decreasing

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

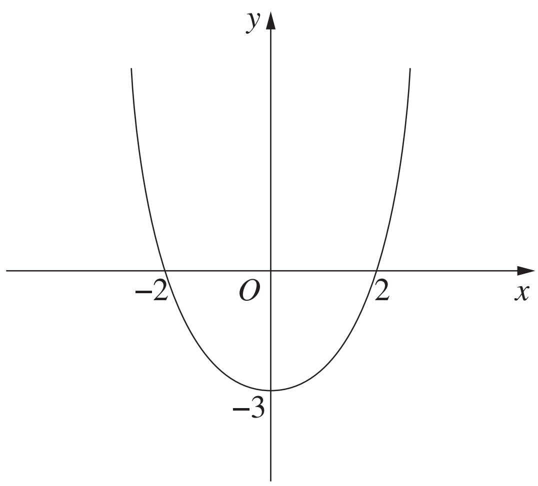
(a) Let $P(x) = x^3 + 3x^2 - 13x + 6$.

(i) Show that $P(2) = 0$. **1**

(ii) Hence, factor the polynomial $P(x)$ as $A(x)B(x)$, where $B(x)$ is a quadratic polynomial. **2**

(b) For what value(s) of a are the vectors $\begin{pmatrix} a \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2a-3 \\ 2 \end{pmatrix}$ perpendicular? **3**

(c) The diagram shows the graph of $y = f(x)$. **3**



Sketch the graph of $y = \dfrac{1}{f(x)}$.

Question 11 continues on page 9

Question 11 (continued)

(d) By expressing $\sqrt{3}\sin x + 3\cos x$ in the form $A\sin(x+\alpha)$, solve $\sqrt{3}\sin x + 3\cos x = \sqrt{3}$, for $0 \le x \le 2\pi$. **4**

(e) Solve $\frac{dy}{dx} = e^{2y}$, finding x as a function of y. **2**

End of Question 11

Please turn over

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) Use the principle of mathematical induction to show that for all integers $n \geq 1$, **3**

$$1 \times 2 + 2 \times 5 + 3 \times 8 + \cdots + n(3n - 1) = n^2(n + 1).$$

(b) When a particular biased coin is tossed, the probability of obtaining a head is $\frac{3}{5}$.

This coin is tossed 100 times.

Let X be the random variable representing the number of heads obtained. This random variable will have a binomial distribution.

(i) Find the expected value, $E(X)$. **1**

(ii) By finding the variance, $\mathrm{Var}(X)$, show that the standard deviation of X is approximately 5. **1**

(iii) By using a normal approximation, find the approximate probability that X is between 55 and 65. **1**

(c) To complete a course, a student must choose and pass exactly three topics. **2**

There are eight topics from which to choose.

Last year 400 students completed the course.

Explain, using the pigeonhole principle, why at least eight students passed exactly the same three topics.

(d) Find $\displaystyle\int_0^{\frac{\pi}{2}} \cos 5x \sin 3x \, dx$. **3**

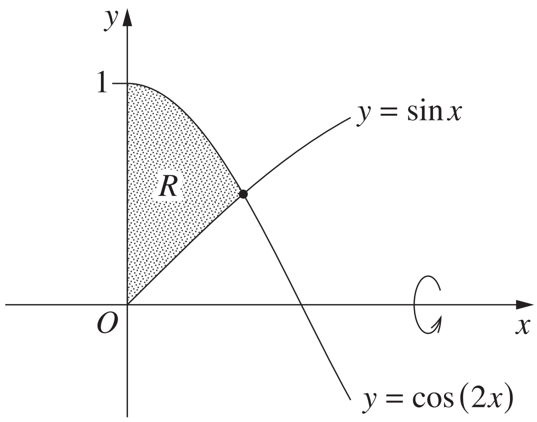
(e) Find the curve which satisfies the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and passes through the point $(1, 0)$. **3**

Question 13 (16 marks) Use the Question 13 Writing Booklet

(a) (i) Find $\dfrac{d}{d\theta}\left(\sin^3\theta\right)$. **1**

(ii) Use the substitution $x = \tan\theta$ to evaluate $\displaystyle\int_0^1 \frac{x^2}{\left(1+x^2\right)^{\frac{5}{2}}}\,dx$. **4**

(b) The region R is bounded by the y-axis, the graph of $y = \cos(2x)$ and the graph of $y = \sin x$, as shown in the diagram. **4**



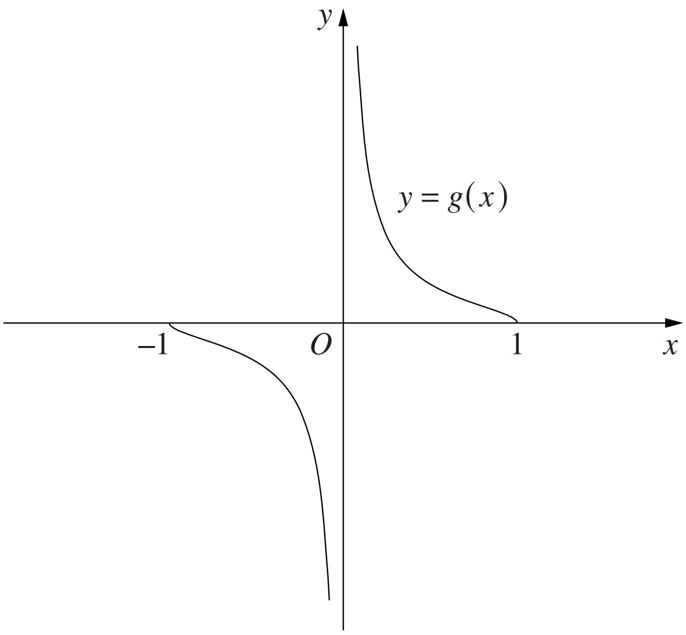
Find the volume of the solid of revolution formed when the region R is rotated about the x-axis.

Question 13 continues on page 12

Question 13 (continued)

(c) Suppose $f(x) = \tan\left(\cos^{-1}(x)\right)$ and $g(x) = \dfrac{\sqrt{1-x^2}}{x}$.

The graph of $y = g(x)$ is given.



(i) Show that $f'(x) = g'(x)$. **4**

(ii) Using part (i), or otherwise, show that $f(x) = g(x)$. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) (i) Use the identity $(1+x)^{2n} = (1+x)^n (1+x)^n$ **2**

to show that

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2,$$

where n is a positive integer.

(ii) A club has $2n$ members, with n women and n men. **2**

A group consisting of an even number $(0, 2, 4, \ldots, 2n)$ of members is chosen, with the number of men equal to the number of women.

Show, giving reasons, that the number of ways to do this is $\binom{2n}{n}$.

(iii) From the group chosen in part (ii), one of the men and one of the women are selected as leaders. **2**

Show, giving reasons, that the number of ways to choose the even number of people and then the leaders is

$$1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \cdots + n^2\binom{n}{n}^2.$$

(iv) The process is now reversed so that the leaders, one man and one woman, are chosen first. The rest of the group is then selected, still made up of an equal number of women and men. **2**

By considering this reversed process and using part (ii), find a simple expression for the sum in part (iii).

Question 14 continues on page 14

Question 14 (continued)

(b) (i) Show that $\sin^3\theta - \frac{3}{4}\sin\theta + \frac{\sin(3\theta)}{4} = 0.$ **2**

(ii) By letting $x = 4\sin\theta$ in the cubic equation $x^3 - 12x + 8 = 0.$ **2**

Show that $\sin(3\theta) = \frac{1}{2}.$

(iii) Prove that $\sin^2\frac{\pi}{18} + \sin^2\frac{5\pi}{18} + \sin^2\frac{25\pi}{18} = \frac{3}{2}.$ **3**

End of paper

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2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

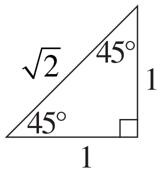
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

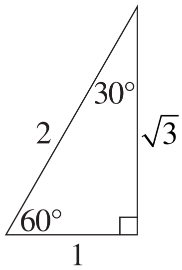
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

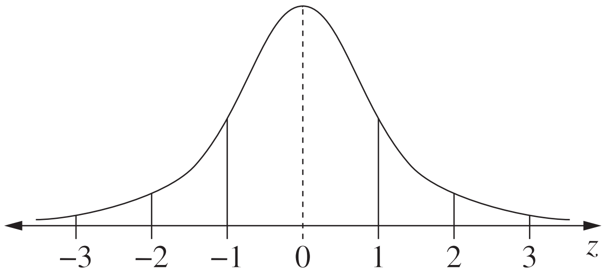
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

2020 HSC Mathematics Extension 1 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	A
2	C
3	D
4	B
5	C
6	D
7	A
8	C
9	B
10	A

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\begin{aligned} P(2) &= 2^3 + 3(2)^2 - 13(2) + 6 \\ &= 0 \end{aligned}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Attempts to divide by $x - 2$, or equivalent merit	1

Sample answer:

$$\begin{array}{r l} & x^2 + 5x - 3 \\ x - 2 \big) & \overline{x^3 + 3x^2 - 13x + 6} \\ & \underline{x^3 - 2x^2} \\ & \quad 5x^2 - 13x \\ & \quad \underline{5x^2 - 10x} \\ & \qquad -3x + 6 \\ & \qquad \underline{-3x + 6} \\ & \qquad\qquad 0 \end{array}$$

$$\therefore\ P(x) = (x - 2)(x^2 + 5x - 3)$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Evaluates the dot product and sets it equal to 0, or equivalent value	2
• Writes a dot product $= 0$, or equivalent merit	1

Sample answer:

Since the vectors are perpendicular,

$$0 = \begin{pmatrix} a \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2a-3 \\ 2 \end{pmatrix}$$

$$0 = 2a^2 - 3a - 2$$

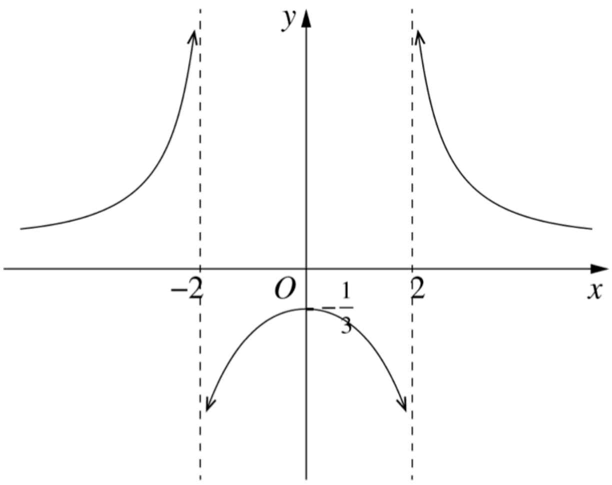
$$0 = (2a+1)(a-2)$$

$$a = -\frac{1}{2} \quad \text{or} \quad 2$$

Question 11 (c)

Criteria	Marks
• Provides correct sketch	3
• Provides a sketch with some correct features	2
• Marks asymptotes at $x = -2$ or 2, or marks local maximum at $y = -\frac{1}{3}$, or equivalent merit	1

Sample answer:



Question 11 (d)

Criteria	Marks
• Provides correct solution	4
• Correctly writes $\sqrt{3}\sin x + 3\cos x$ in the form $A\sin(x+\alpha)$ and finds one solution	3
• Finds A and α, or equivalent merit	2
• Finds the value of A, or equivalent merit	1

Sample answer:

$$\begin{aligned}\sqrt{3}\sin x + 3\cos x &= A\sin(x+\alpha)\\ &= A\sin x\cos\alpha + A\cos x\sin\alpha\end{aligned}$$

$$\begin{aligned}\therefore\ A\cos\alpha &= \sqrt{3} \qquad (1)\\ A\sin\alpha &= 3 \qquad (2)\end{aligned}$$

dividing, $\tan\alpha = \dfrac{3}{\sqrt{3}} = \sqrt{3} \quad \Rightarrow \quad \alpha = \dfrac{\pi}{3}.$

$$\therefore\ A\cos\frac{\pi}{3} = \sqrt{3} \quad \Rightarrow \quad A = 2\sqrt{3}$$

$$\therefore\ \sqrt{3}\sin x + 3\cos x = 2\sqrt{3}\sin\left(x+\frac{\pi}{3}\right)$$

$$\begin{aligned}\text{so}\quad 2\sqrt{3}\sin\left(x+\frac{\pi}{3}\right) &= \sqrt{3}\\ \sin\left(x+\frac{\pi}{3}\right) &= \frac{1}{2}\\ x+\frac{\pi}{3} &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6},\\ x &= \frac{\pi}{2}, \frac{11\pi}{6} \quad \text{given } 0 \le x \le 2\pi.\end{aligned}$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	2
• Rewrites as $\frac{dx}{dy} = e^{-2y}$, or equivalent merit	1

Sample answer:

$$\frac{dy}{dx} = e^{2y}$$

$$\frac{dx}{dy} = e^{-2y}$$

$$x = \int e^{-2y}\, dy$$

$$= -\frac{1}{2}e^{-2y} + C$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	3
• Proves inductive step by assuming true for k (or equivalent) and using that assumption to show true for $k + 1$, or equivalent merit	2
• Verifies base case, $n = 1$, or equivalent merit	1

Sample answer:

$$(1\times 2)+(2\times 5)+(3\times 8)+\cdots+n(3n-1)=n^2(n+1)$$

For $n = 1$ LHS $= 1\ (2)$ $= 2$ RHS $= 1\ (2)$ $= 2$

$\therefore$ Statement is true for $n = 1$

Assume statement true for $n = k$, that is,

$$(1\times 2)+(2\times 5)+(3\times 8)+\cdots+k(3k-1)=k^2(k+1)$$

Prove true for $n = k + 1$

That is, we show that

$$(1\times 2)+(2\times 5)+\cdots+k(3k-1)+(k+1)\big(3(k+1)-1\big)=(k+1)^2(k+2)$$

$$\begin{aligned}\text{LHS} &= k^2(k+1)+(k+1)(3k+2)\\ &= \big(k+1\big)\big(k^2+3k+2\big)\\ &= (k+1)(k+1)(k+2)\\ &= (k+1)^2(k+2)\\ &= \text{RHS}\end{aligned}$$

$\therefore$ By the principle of mathematical induction the statement is true for $n \geq 1$.

Question 12 (b) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$E(X) = np$$
$$= 100 \times \frac{3}{5}$$
$$= 60$$

Question 12 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\sigma = \sqrt{np(1-p)}$$
$$= \sqrt{100 \times \frac{3}{5} \times \frac{2}{5}}$$
$$= \sqrt{24}$$
$$\doteqdot 5$$

Question 12 (b) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$P(55 \le X \le 65) = P(-1 \le Z \le 1)$$
$$\simeq 68\%$$

Question 12 (c)

Criteria	Marks
• Provides correct explanation	2
• Evaluates $\binom{8}{3}$, or obtains the expression $\dfrac{400}{\binom{8}{3}}$, or equivalent merit	1

Sample answer:

There are $\binom{8}{3} = 56$ possible choices of 3 topics and $\dfrac{400}{56} = 7.14$.

As there are 56 possible combinations, we can have at most 392 students without exceeding 7 students per combination. But we have 400 students, so at least one combination has 8 or more students.

Question 12 (d)

Criteria	Marks
• Provides correct solution	3
• Finds correct primitive, or equivalent merit	2
• Uses product to sum result, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} \cos 5x \, \sin 3x \, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(\sin 8x - \sin 2x)\, dx$$

$$= \left[\frac{1}{2}\left(-\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(-\frac{1}{8} - \frac{1}{2}\right) - \frac{1}{2}\left(-\frac{1}{8} + \frac{1}{2}\right)$$

$$= -\frac{1}{2}$$

Question 12 (e)

Criteria	Marks
• Provides correct solution	3
• Obtains $x^2 + y^2 =$ constant, or equivalent merit	2
• Separates the variable, or equivalent merit	1

Sample answer:

$$\frac{dy}{dx} = \frac{-x}{y}$$

Separating variables, $\int y\,dy = \int -x\,dx$

$$\therefore \quad \frac{y^2}{2} = \frac{-x^2}{2} + c$$

$$x^2 + y^2 = d, \text{ where } d = 2c$$

The curve passes through (1, 0)

So $d = 1^2 + 0^2 = 1$

Hence the equation of D is

$$x^2 + y^2 = 1 \qquad \text{(unit circle)}$$

Question 13 (a) (i)

Criteria	Marks
• Provides correct derivative	1

Sample answer:

$$\frac{d}{d\theta}\left(\sin^3\theta\right) = 3\sin^2\theta\cos\theta$$

Question 13 (a) (ii)

Criteria	Marks
• Provides correct solution	4
• Obtains correctly simplified integrand in terms of $\sin\theta$ and $\cos\theta$, or equivalent merit	3
• Correctly substitutes and attempts to simplify (ignoring limits), or equivalent merit	2
• Uses given substitution, or equivalent merit	1

Sample answer:

Let $x = \tan\theta$

$$\frac{dx}{d\theta} = \sec^2\theta$$

At $x = 0$, $\theta = 0$

at $x = 1$, $\theta = \frac{\pi}{4}$

$$\therefore \int_0^1 \frac{x^2}{\left(1+x^2\right)^{\frac{5}{2}}}\,dx = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\left(1+\tan^2\theta\right)^{\frac{5}{2}}}\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\sec^3\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^2\theta}\cos^3\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^2\theta\cos\theta\,d\theta$$

$$= \left[\frac{1}{3}\sin^3\theta\right]_0^{\frac{\pi}{4}} \qquad \text{by part (i)}$$

$$= \frac{1}{3}\times\left(\frac{\sqrt{2}}{2}\right)^3$$

$$= \frac{2\sqrt{2}}{3\times 2^3}$$

$$= \frac{\sqrt{2}}{12}$$

Question 13 (b)

Criteria	Marks
• Provides correct solution	4
• Obtains a correct expression for the volume and uses a double-angle formula, or equivalent merit	3
• Finds point of intersection and writes volume as a difference of volumes OR • Finds point of intersection and finds volume generated by $y = \sin x$, or equivalent merit	2
• Finds point of intersection, or writes volume as a difference of two volumes, or equivalent merit	1

Sample answer:

The curves intersect when

$$\begin{aligned} \cos 2x &= \sin x \\ 1 - 2\sin^2 x &= \sin x \\ 2\sin^2 x + \sin x - 1 &= 0 \end{aligned}$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$$\therefore\ \sin x = \frac{1}{2},\ (\sin x \neq -1).$$

$$\therefore\ x = \frac{\pi}{6}$$

The volume is given by

$$\begin{aligned} V &= \pi \int_0^{\frac{\pi}{6}} \cos^2 2x\, dx - \pi \int_0^{\frac{\pi}{6}} \sin^2 x\, dx \\ &= \pi \left(\int_0^{\frac{\pi}{6}} \frac{1 + \cos 4x}{2}\, dx - \int_0^{\frac{\pi}{6}} \frac{1 - \cos 2x}{2}\, dx \right) \\ &= \pi \left[\frac{\sin 4x}{8} + \cancel{\frac{x}{2}} - \cancel{\frac{x}{2}} + \frac{\sin 2x}{4} \right]_0^{\frac{\pi}{6}} \\ &= \pi \left(\frac{\sin \frac{2\pi}{3}}{8} + \frac{\sin \frac{\pi}{3}}{4} \right) \\ &= \frac{\pi}{8} \left(\frac{\sqrt{3}}{2} + \sqrt{3} \right) \\ &= \frac{3\pi\sqrt{3}}{16} \end{aligned}$$

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	4
• Obtains one correct derivative and makes some progress towards obtaining the other, or equivalent merit	3
• Obtains one correct derivative or attempts both, or equivalent merit	2
• Attempts one derivative, or equivalent merit	1

Sample answer:

$$f'(x) = \sec^2\left(\cos^{-1} x\right) \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{1}{\cos^2\left(\cos^{-1} x\right)} \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-1}{x^2\sqrt{1-x^2}}$$

$$g'(x) = \frac{-\frac{x}{\sqrt{1-x^2}} \times x - \sqrt{1-x^2} \times 1}{x^2}$$

$$= \frac{-\cancel{x^2} - \left(1 - \cancel{x^2}\right)}{x^2\sqrt{1-x^2}}$$

$$= \frac{-1}{x^2\sqrt{1-x^2}}$$

$\therefore\ f'(x) = g'(x)$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution for both parts of the domain	3
• Provides a correct solution for one part of the domain, or equivalent merit	2
• Observes that $f(x) - g(x)$ is a constant, or equivalent merit	1

Sample answer:

$$f'(x) = \frac{-1}{x^2\sqrt{1-x^2}} = g'(x)$$

And so $f'(x) - g'(x) = 0$

$\Rightarrow f(x) - g(x)$ is a constant

For $x < 0$, say $x = -1$

$$f(-1) - g(-1) = \tan\left(\cos^{-1}(-1)\right) - \frac{\sqrt{1-(-1)^2}}{(-1)}$$

$$= \tan(\pi) + 0$$

$$= 0$$

So for $x < 0$, $f(x) = g(x)$.

For $x > 0$, say $x = 1$,

$$f(1) - g(1) = \tan\left(\cos^{-1}(1)\right) - \frac{\sqrt{1-(1)^2}}{(1)}$$

$$= \tan(0) + 0$$

$$= 0$$

So for $x > 0$, $f(x) = g(x)$

So for all x in the domain

$f(x) = g(x)$

Question 14 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Identifies coefficient of x^n on the left hand side, or equivalent merit	1

Sample answer:

The left hand side is $\binom{2n}{n}$ which is the coefficient of x^n in the expansion of $(1+x)^{2n}$.

This means that $\binom{n}{0}\binom{n}{0}+\binom{n}{1}\binom{n}{1}+\cdots+\binom{n}{n}\binom{n}{n}$ should be the coefficient of x^n in the expansion of $(1+x)^n(1+x)^n$.

The first term, $\binom{n}{0}$ is the constant term in the expansion of $(1+x)^n$ and so should be multiplied by an x^n term. But $\binom{n}{0}=\binom{n}{n-0}=\binom{n}{n}$ is equal to the coefficient of x^n in the expansion $(1+x)^n$.

Thus $\binom{n}{0}^2$ is the coefficient of the x^n term that comes from the constant in the expansion of $(1+x)^n$ times the coefficient of the x^n term in the expansion of $(1+x)^n$.

Similarly $\binom{n}{1}^2$ is the coefficient of the x^n term that comes from the coefficient of x term in the expansion of $(1+x)^n$ times the coefficient of the x^{n-1} term in the expansion of $(1+x)^n$.

Therefore, the right hand side is the coefficient of the x^n term in the expansion of $(1+x)^n(1+x)^n$.

OR

Question 14 (a) (i) (continued)

The coefficient of x^n in expansion of $(1+x)^{2n}$ is $\binom{2n}{n}$.

The coefficient of x^n in $(1+x)^n(1+x)^n$ is

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{k}\binom{n}{n-k} + \cdots + \binom{n}{n}\binom{n}{0}$$

But $\binom{n}{n-k} = \binom{n}{k}$

So the coefficient is $\binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \cdots + \binom{n}{n}\binom{n}{n}$

Hence $\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Explains why one of the terms is $\binom{n}{k} \times \binom{n}{k}$, or equivalent merit	1

Sample answer:

We can choose:

0 men and 0 women in $\binom{n}{0}\binom{n}{0} = \binom{n}{0}^2$ ways

or 1 man and 1 woman in $\binom{n}{1}^2$ ways

.
.
.

or n men and n women in $\binom{n}{n}^2$ ways

Hence the total number of ways is

$$\binom{n}{0}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \qquad \text{from part (i)}$$

Question 14 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Explains why one of the terms is $k^2\binom{n}{k}^2$, or equivalent merit	1

Sample answer:

We can choose 1 woman and 1 leader in $\binom{n}{1}\times 1$ ways and similarly for the men. This gives

$\binom{n}{1}\times 1\times\binom{n}{1}\times 1=\binom{n}{1}^2\times 1^2$ for this case.

We can choose 2 women and 1 leader in $\binom{n}{2}\binom{2}{1}=\binom{n}{2}\times 2$ ways and similarly for the men.

This gives $\binom{n}{2}\times 2\times\binom{n}{2}\times 2=\binom{n}{2}^2\times 2^2$ for this case.

$$\vdots$$

We can choose n women and 1 leader in $\binom{n}{n}\binom{n}{1}=\binom{n}{n}\times n$ ways and similarly for the men.

This gives $\binom{n}{n}\times n\times\binom{n}{n}\times n=\binom{n}{n}^2\times n^2$ for this case.

And so the total is $1^2\binom{n}{1}^2+2^2\binom{n}{2}^2+\cdots+n^2\binom{n}{n}^2$.

Question 14 (a) (iv)

Criteria	Marks
• Provides correct solution	2
• Recognises that choosing the leaders first reduces the problem to using part (ii) with $n - 1$ women and $n - 1$ men, or equivalent merit	1

Sample answer:

There are n ways to choose a woman leader, leaving $(n - 1)$ women from which to choose.

There are n ways to choose a man leader, leaving $(n - 1)$ men from which to choose.

By part (ii) there are $\binom{2(n-1)}{(n-1)}$ ways to choose the $(n - 1)$ women and $(n - 1)$ men.

Hence $1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \cdots + n^2\binom{n}{n}^2 = n^2\binom{2n-2}{n-1}$.

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Expands $\sin(2\theta + \theta)$, or equivalent merit	1

Sample answer:

$$\begin{aligned}
\sin 3\theta &= \sin(2\theta + \theta) \\
&= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\
&= 2\sin\theta \cos^2\theta + \left(1 - 2\sin^2\theta\right)\sin\theta \\
&= 2\sin\theta\left(1 - \sin^2\theta\right) + \sin\theta - 2\sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

$$\therefore 4\sin^3\theta - 3\sin\theta + \sin(3\theta) = 0$$

$$\therefore \sin^3\theta - \frac{3}{4}\sin\theta + \frac{\sin(3\theta)}{4} = 0$$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains $\sin^3\theta - \frac{3}{4}\sin\theta + \frac{8}{64} = 0$, or equivalent merit	1

Sample answer:

$$x^3 - 12x + 8 = 0$$

$$x = 4\sin\theta, \text{ so}$$

$$64\sin^3\theta - 48\sin\theta + 8 = 0$$

$$\therefore \sin^3\theta - \frac{3}{4}\sin\theta + \frac{8}{64} = 0$$

comparing with (i),

$$\frac{8}{64} = \frac{\sin 3\theta}{4} \quad \Rightarrow \quad \sin 3\theta = \frac{32}{64} = \frac{1}{2}.$$

Question 14 (b) (iii)

Criteria	Marks
• Provides correct proof	3
• Obtains all the roots of the cubic, or equivalent merit	2
• Obtains $4\sin\frac{\pi}{18}$ as a solution of the cubic in part (b) (ii), or equivalent merit	1

Sample answer:

From $\quad \sin 3\theta = \frac{1}{2}$

we have $\quad 3\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6} + 2\pi, \frac{5\pi}{6} + 2\pi, \frac{\pi}{6} + 4\pi, \ldots$

$$\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \ldots$$

So $4\sin\frac{\pi}{18},\ 4\sin\frac{5\pi}{18},\ 4\sin\frac{13\pi}{18}$ etc are roots of the cubic, but they are not all distinct.

The angles $\frac{\pi}{18}$ and $\frac{5\pi}{18}$ are distinct angles in the first quadrant and so have different positive sine values. The angle $\frac{25\pi}{18}$ is in the third quadrant and so has a negative sine value. Hence we can take $\alpha = 4\sin\frac{\pi}{18},\ \beta = 4\sin\frac{5\pi}{18},\ \gamma = 4\sin\frac{25\pi}{18}$ as 3 distinct roots,

Now $\quad \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$$= 0^2 + 2 \times 12$$

$$= 24$$

$$\therefore\ 16\left(\sin^2\frac{\pi}{18} + \sin^2\frac{5\pi}{18} + \sin^2\frac{25\pi}{18}\right) = 24$$

$$\therefore\ \sin^2\frac{\pi}{18} + \sin^2\frac{5\pi}{18} + \sin^2\frac{25\pi}{18} = \frac{24}{16} = \frac{3}{2}.$$

2020 HSC Mathematics Extension 1 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	ME-F1 Further Work with Functions	ME11–2
2	1	ME-F1 Further Work with Functions	ME11–1
3	1	ME-C2 Further Calculus Skills	ME12–1
4	1	ME-V1 Introduction to Vectors	ME12–2
5	1	ME-F2 Polynomials	ME11–1
6	1	ME-V1 Introduction to Vectors	ME12–2
7	1	ME-C3 Applications of Calculus	ME12–4
8	1	ME-A1 Working with Combinatorics	ME11–5
9	1	ME-V1 Introduction to Vectors	ME12–2
10	1	ME-C1 Rates of Change	ME11–4

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	ME-F2 Polynomials	ME11–2
11 (a) (ii)	2	ME-F2 Polynomials	ME11–2
11 (b)	3	ME-V1 Introduction to Vectors	ME12–2
11 (c)	3	ME-F1 Further Work with Functions	ME11–2
11 (d)	4	ME-T3 Trigonometric Equations	ME12–3
11 (e)	2	ME-C3 Applications of Calculus	ME12–4
12 (a)	3	ME-P1 Proof by Mathematical Induction	ME12–1
12 (b) (i)	1	ME-S1 The Binomial Distribution	ME12–5
12 (b) (ii)	1	ME-S1 The Binomial Distribution	ME12–5
12 (b) (iii)	1	ME-S1 The Binomial Distribution	ME12–5
12 (c)	2	ME-A1 Working with Combinatorics	ME11–5, ME12-7
12 (d)	3	ME-T2 Further Trigonometric Identities ME-C2 Further Calculus Skills	ME12–1
12 (e)	3	ME-C3 Applications of Calculus	ME12–4
13 (a) (i)	1	ME-C2 Further Calculus Skills	ME12–1

Question	Marks	Content	Syllabus outcomes
13 (a) (ii)	4	ME-C2 Further Calculus Skills	ME12–1
13 (b)	4	ME-C3 Applications of Calculus	ME12–4
13 (c) (i)	4	ME-C2 Further Calculus Skills	ME12–4
13 (c) (ii)	3	ME-C2 Further Calculus Skills	ME12–4, ME 12 -7
14 (a) (i)	2	ME-A1 Working with Combinatorics	ME11–5
14 (a) (ii)	2	ME-A1 Working with Combinatorics	ME11–5
14 (a) (iii)	2	ME-A1 Working with Combinatorics	ME11–5
14 (a) (iv)	2	ME-A1 Working with Combinatorics	ME11–5
14 (b) (i)	2	ME-T3 Trigonometric Equations	ME12–3
14 (b) (ii)	2	ME-T3 Trigonometric Equations	ME12–3
14 (b) (iii)	3	ME-F2 Polynomials ME-T3 Trigonometric Equations	ME12–3