NSW Education Standards Authority

2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number on the Question 12 Writing Booklet attached

Total marks: 70

Section I – 10 marks (pages 2–9)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 10–18)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 It is given that $\cos\left(\dfrac{23\pi}{12}\right) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

Which of the following is the value of $\cos^{-1}\left(\dfrac{\sqrt{6}+\sqrt{2}}{4}\right)$?

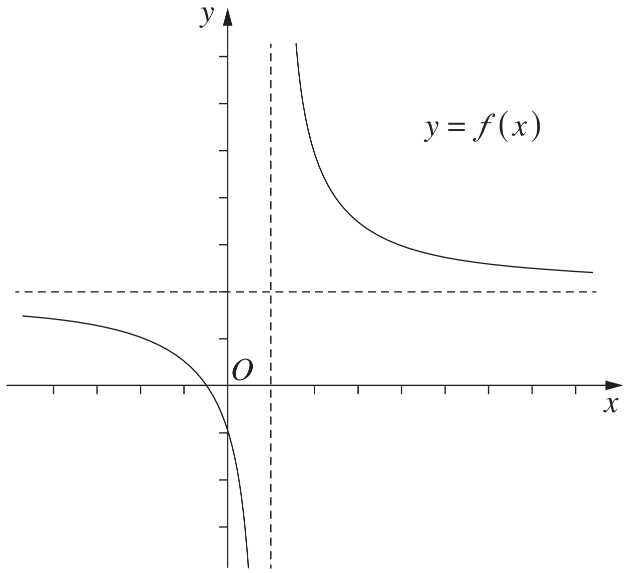
A. $\dfrac{23\pi}{12}$

B. $\dfrac{11\pi}{12}$

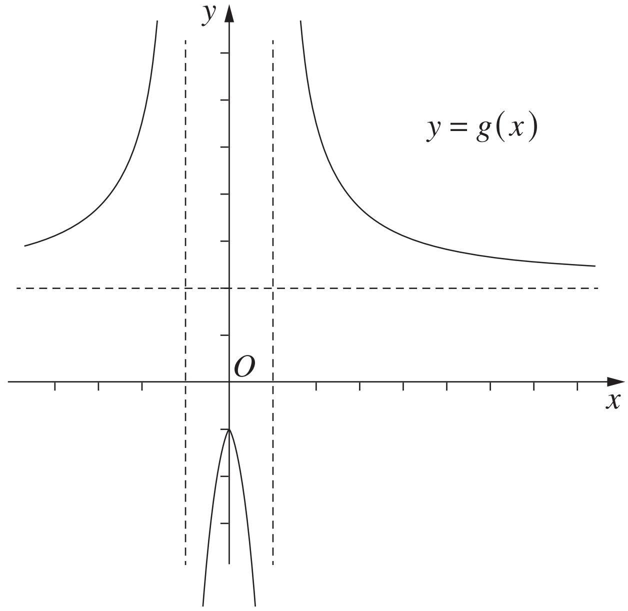
C. $\dfrac{\pi}{12}$

D. $-\dfrac{11\pi}{12}$

2 The graph of $f(x) = \dfrac{3}{x-1} + 2$ is shown.



The graph of $f(x)$ was transformed to get the graph of $g(x)$ as shown.



What transformation was applied?

A. $g(x) = f(|x|)$

B. $g(x) = \sqrt{f(x)}$

C. $g(x) = f(-x)$

D. $g(x) = \dfrac{1}{f(x)}$

3 Let $P(x)$ be a polynomial of degree 5. When $P(x)$ is divided by the polynomial $Q(x)$, the remainder is $2x + 5$.

Which of the following is true about the degree of Q?

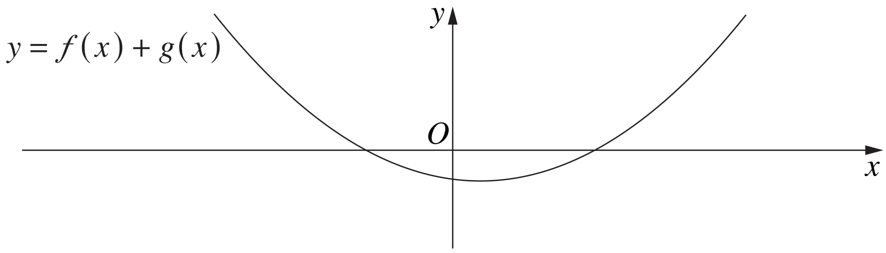
A. The degree must be 1.

B. The degree could be 1.

C. The degree must be 2.

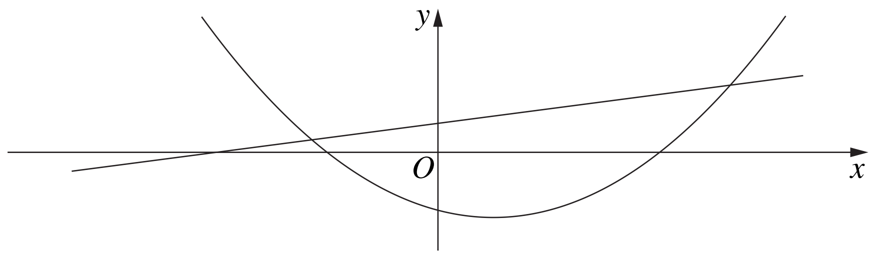
D. The degree could be 2.

4 The diagram shows the graph of the sum of the functions $f(x)$ and $g(x)$.

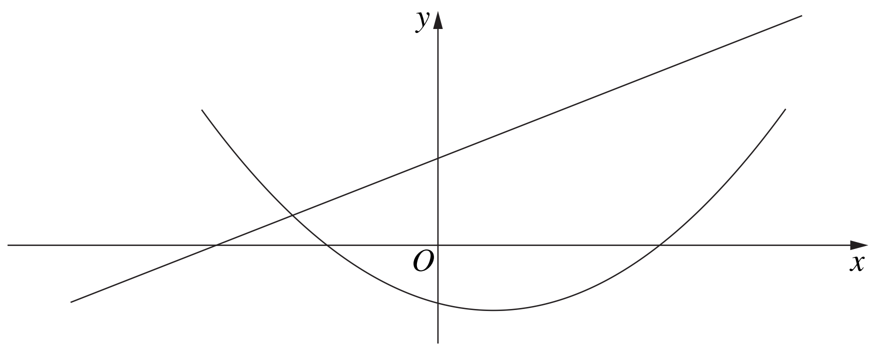


Which of the following best represents the graphs of both $f(x)$ and $g(x)$?

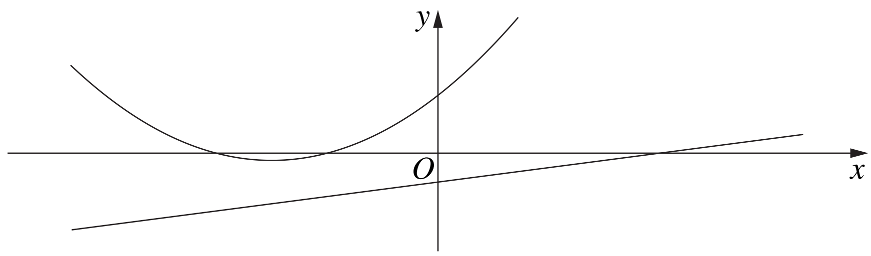
A.



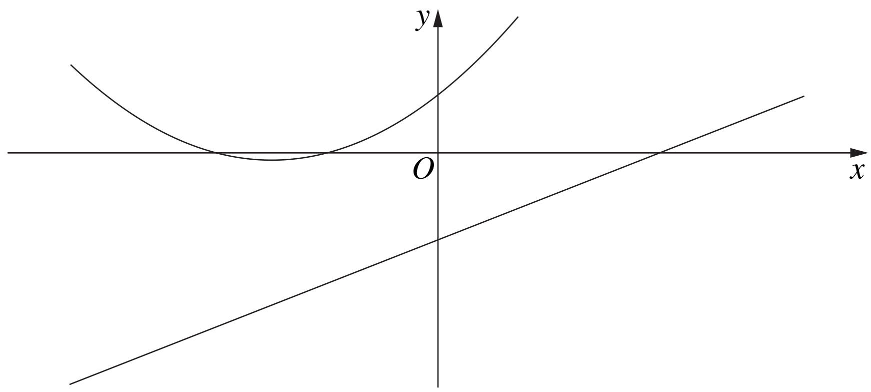
B.



C.



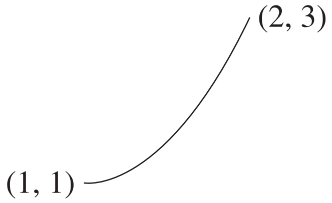
D.



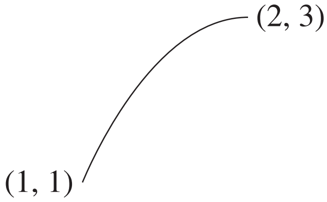
5 A curve is defined in parametric form by $x = 2 + t$ and $y = 3 - 2t^2$ for $-1 \le t \le 0$.

Which diagram best represents this curve?

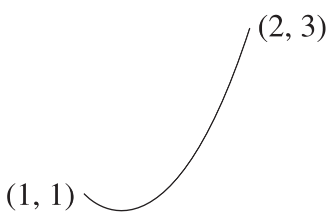
A.



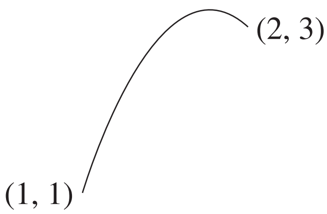
B.



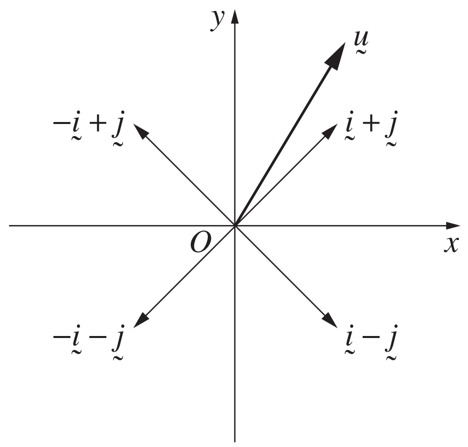
C.



D.



6 The following diagram shows the vector $\underset{\sim}{u}$ and the vectors $\underset{\sim}{i} + \underset{\sim}{j}$, $-\underset{\sim}{i} + \underset{\sim}{j}$, $-\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{i} - \underset{\sim}{j}$.



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Which statement regarding this diagram could be true?

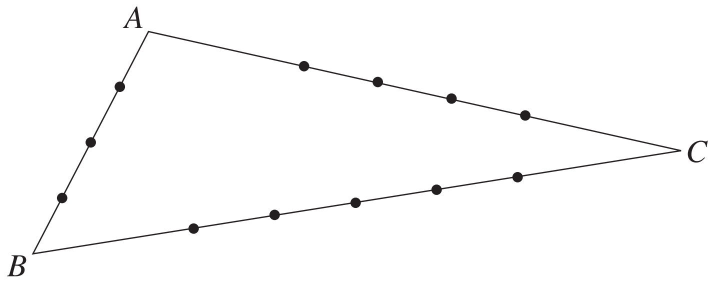
A. The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{i} + \underset{\sim}{j}$ is the vector $1.1\underset{\sim}{i} + 1.8\underset{\sim}{j}$.

B. The projection of $\underset{\sim}{u}$ onto $-\underset{\sim}{i} + \underset{\sim}{j}$ is the vector $-0.4\underset{\sim}{i} + 0.4\underset{\sim}{j}$.

C. The projection of $\underset{\sim}{u}$ onto $-\underset{\sim}{i} - \underset{\sim}{j}$ is the vector $3.2\underset{\sim}{i} + 3.2\underset{\sim}{j}$.

D. The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{i} - \underset{\sim}{j}$ is the vector $0.5\underset{\sim}{i} - 0.5\underset{\sim}{j}$.

7 The diagram shows triangle *ABC* with points chosen on each of the sides. On side *AB*, 3 points are chosen. On side *AC*, 4 points are chosen. On side *BC*, 5 points are chosen.



How many triangles can be formed using the chosen points as vertices?

A. 60

B. 145

C. 205

D. 220

8 The angle between two unit vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is θ and $\left| \underset{\sim}{a} + \underset{\sim}{b} \right| < 1$.

Which of the following best describes the possible range of values of θ?

A. $0 \le \theta < \dfrac{\pi}{3}$

B. $0 \le \theta < \dfrac{2\pi}{3}$

C. $\dfrac{\pi}{3} < \theta \le \pi$

D. $\dfrac{2\pi}{3} < \theta \le \pi$

9 A given function $f(x)$ has an inverse $f^{-1}(x)$.

The derivatives of $f(x)$ and $f^{-1}(x)$ exist for all real numbers x.

The graphs $y = f(x)$ and $y = f^{-1}(x)$ have at least one point of intersection.

Which statement is true for all points of intersection of these graphs?

A. All points of intersection lie on the line $y = x$.

B. None of the points of intersection lie on the line $y = x$.

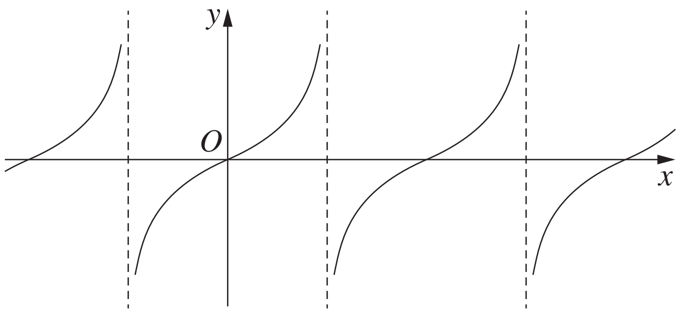
C. At no point of intersection are the tangents to the graphs parallel.

D. At no point of intersection are the tangents to the graphs perpendicular.

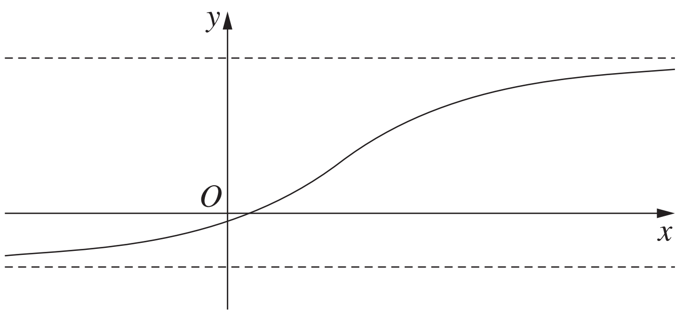
10 Which of the following could be the graph of a solution to the differential equation

$$\frac{dy}{dx} = \sin y + 1?$$

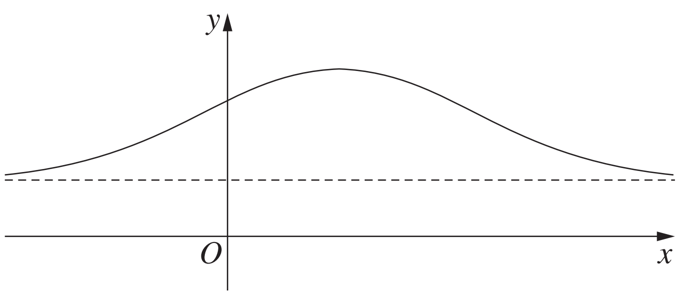
A.



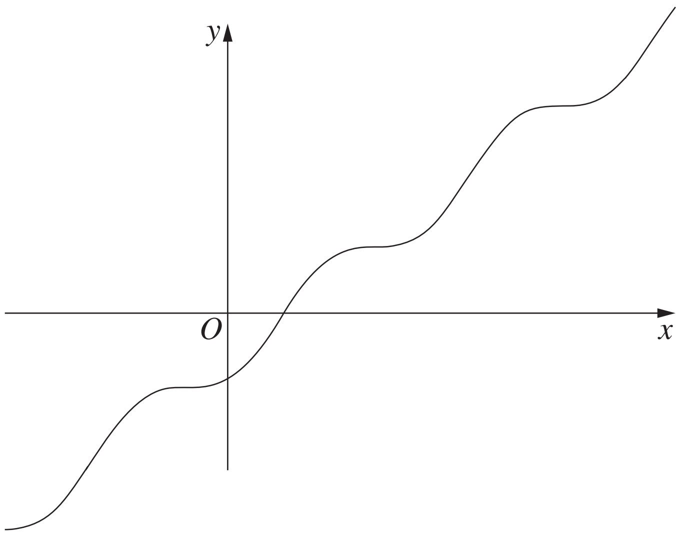
B.



C.



D.



Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) For the vectors $\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j}$, evaluate each of the following.

(i) $\underset{\sim}{u} + 3\underset{\sim}{v}$ **1**

(ii) $\underset{\sim}{u} \cdot \underset{\sim}{v}$ **1**

(b) Find the exact value of $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2+4}}\,dx$ using the substitution $u = x^2 + 4$. **3**

(c) Find the coefficients of x^2 and x^3 in the expansion of $\left(1 - \dfrac{x}{2}\right)^8$. **2**

(d) The vectors $\underset{\sim}{u} = \begin{pmatrix} a \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}$ are perpendicular. **2**

What are the possible values of a?

(e) Express $\sqrt{3}\sin(x) - 3\cos(x)$ in the form $R\sin(x + \alpha)$. **3**

(f) Solve $\dfrac{x}{2-x} \geq 5$. **3**

Question 12 (16 marks) Use the Question 12 Writing Booklet

(a) A direction field is to be drawn for the differential equation **2**

$$\frac{dy}{dx} = \frac{x - 2y}{x^2 + y^2}.$$

On the diagram on page 1 of the Question 12 Writing Booklet, clearly draw the correct slopes of the direction field at the points P, Q and R.

(b) A sports association manages 13 junior teams. It decides to check the age of all players. Any team that has more than 3 players above the age limit will be penalised. **2**

A total of 41 players are found to be above the age limit.

Will any team be penalised? Justify your answer.

(c) Find the equation of the tangent to the curve $y = x\arctan(x)$ at the point with coordinates $\left(1, \frac{\pi}{4}\right)$. Give your answer in the form $y = mx + c$. **3**

Question 12 continues on page 12

(d) In a room with temperature 12°C, coffee is poured into a cup. The temperature of the coffee when it is poured into the cup is 92°C, and it is far too hot to drink.

The temperature, T, in degrees Celsius, of the coffee, t minutes after it is made, can be modelled using the differential equation $\frac{dT}{dt} = k\left(T - T_1\right)$, where k is the constant of proportionality and T_1 is a constant.

(i) It takes 5 minutes for the coffee to cool to a temperature of 76°C. **3**

Using separation of variables, solve the given differential equation to show that $T = 12 + 80e^{\frac{t}{5}\ln\left(\frac{4}{5}\right)}$.

(ii) The optimal drinking temperature for a hot beverage is 57°C. **1**

Find the value of t when the coffee reaches this temperature, giving your answer to the nearest minute.

(e) A game consists of randomly selecting 4 balls from a bag. After each ball is selected it is replaced in the bag. The bag contains 3 red balls and 7 green balls. For each red ball selected, 10 points are earned and for each green ball selected, 5 points are deducted. For instance, if a player picks 3 red balls and 1 green ball, the score will be $3 \times 10 - 1 \times 5 = 25$ points. **2**

What is the expected score in the game?

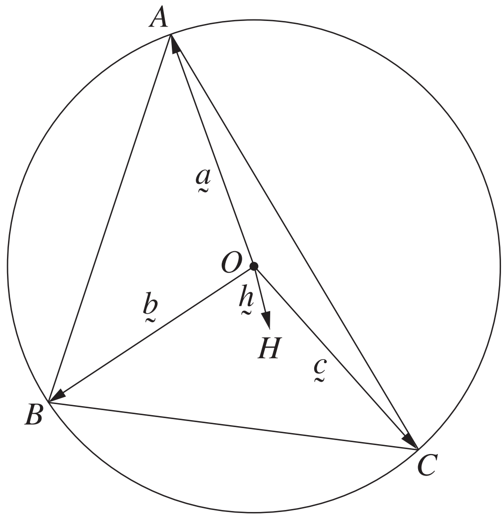
(f) Use mathematical induction to prove that $15^n + 6^{2n+1}$ is divisible by 7 for all integers $n \geq 0$. **3**

End of Question 12

Question 13 (14 marks) Use the Question 13 Writing Booklet

(a) Three different points A, B and C are chosen on a circle centred at O. **3**

Let $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{OB}$ and $\underset{\sim}{c} = \overrightarrow{OC}$. Let $\underset{\sim}{h} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ and let H be the point such that $\overrightarrow{OH} = \underset{\sim}{h}$, as shown in the diagram.



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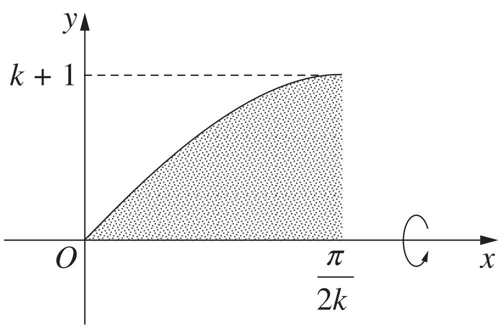
Show that $\overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

(b) A solid of revolution is to be found by rotating the region bounded by the x-axis and the curve $y = (k+1)\sin(kx)$, where $k > 0$, between $x = 0$ and $x = \dfrac{\pi}{2k}$ about the x-axis. **3**



Find the value of k for which the volume is π^2.

Question 13 continues on page 14

Question 13 (continued)

(c) The function f is defined by $f(x) = \sin(x)$ for all real numbers x. Let g be the function defined on $[-1, 1]$ by $g(x) = \arcsin(x)$. **2**

Is g the inverse of f? Justify your answer.

(d) The monic polynomial, P, has degree 3 and roots α, β, γ. **3**

It is given that

$$\alpha^2 + \beta^2 + \gamma^2 = 85 \text{ and}$$
$$P'(\alpha) + P'(\beta) + P'(\gamma) = 87.$$

Find $\alpha\beta + \beta\gamma + \gamma\alpha$.

(e) *You may use the information on page 18 to answer this question.*

A chocolate factory sells 150-gram chocolate bars. There has been a complaint that the bars actually weigh less than 150 grams, so a team of inspectors was sent to the factory to check. They randomly selected 16 bars, weighed them and noted that 8 bars weighed less than 150 grams.

The factory manager claims 80% of the chocolate bars produced by the factory weigh 150 grams or more.

(i) The inspectors used the normal approximation to the binomial distribution to calculate the probability, $\mathcal{P}$, of having at least 8 bars weighing less than 150 grams in a random sample of 16, assuming the factory manager's claim is correct. **2**

Calculate the value of $\mathcal{P}$.

(ii) The factory manager disagrees with the method used by the inspectors as described in part (i). **1**

Explain why the method used by the inspectors might not be valid.

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Find the particular solution to the differential equation $(x-2)\dfrac{dy}{dx} = xy$ that passes through the point $(0, 1)$. **4**

(b) The vectors $\vec{u}$ and $\vec{v}$ are not parallel. The vector $\vec{p}$ is the projection of $\vec{u}$ onto the vector $\vec{v}$. **3**

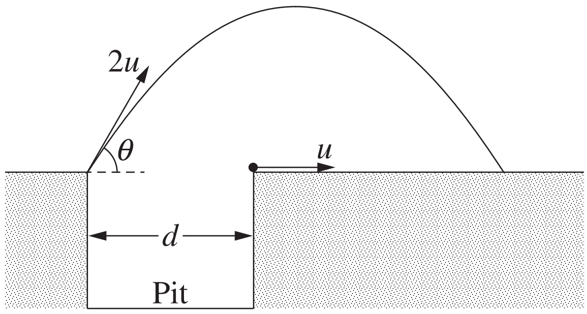
The vector $\vec{p}$ is parallel to $\vec{v}$ so it can be written $\lambda_0 \vec{v}$ for some real number λ_0. (Do NOT prove this.)

Prove that $|\vec{u} - \lambda \vec{v}|$ is smallest when $\lambda = \lambda_0$ by showing that, for all real numbers λ, $|\vec{u} - \lambda_0 \vec{v}| \leq |\vec{u} - \lambda \vec{v}|$.

Question 14 continues on page 16

(c) A video game designer wants to include an obstacle in the game they are developing. The player will reach one side of a pit and must shoot a projectile to hit a target on the other side of the pit in order to be able to cross. However, the instant the player shoots, the target begins to move away from the player at a constant speed that is half the initial speed of the projectile shot by the player, as shown in the diagram below. **4**

The initial distance between the player and the target is d, the initial speed of the projectile is $2u$ and it is launched at an angle of θ to the horizontal. The acceleration due to gravity is g. The launch angle is the ONLY parameter that the player can change.



Taking the position of the player when the projectile is launched as the origin, the positions of the projectile and target at time t after the projectile is launched are as follows.

$$\vec{r}_P = \begin{pmatrix} 2ut\cos\theta \\ 2ut\sin\theta - \frac{g}{2}t^2 \end{pmatrix} \quad \text{Projectile}$$

$$\vec{r}_T = \begin{pmatrix} d + ut \\ 0 \end{pmatrix} \quad \text{Target}$$

(Do NOT prove these.)

Show that, for the player to have a chance of hitting the target, d must be less than 37% of the maximum possible range of the projectile (to 2 significant figures).

Question 14 continues on page 17

Question 14 (continued)

(d) *You may use the information on page 18 to answer this question.* **4**

An airline company that has empty seats on a flight is not maximising its profit.

An airline company has found that there is a probability of 5% that a passenger books a flight but misses it. The management of the airline company decides to allow for overbooking, which means selling more tickets than the number of seats available on each flight.

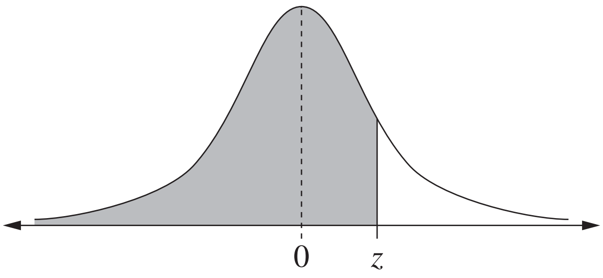
To protect their reputation, management makes the decision that no more than 1% of their flights should have more passengers showing up for the flight than available seats.

Given management's decision and using a suitable approximation, find the maximum number of tickets that can be sold for a flight which has 350 seats.

End of paper

You may use the information below to answer Question 13 (e) and Question 14 (d).

Table of values $P(Z \leq z)$ for the normal distribution $N(0, 1)$



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995

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NSW Education Standards Authority

Centre Number

Student Number

2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Writing Booklet

Question 12

Instructions

- Use this Writing Booklet to answer Question 12.
- Write the number of this booklet and the total number of booklets that you have used for this question (eg: 1 of 3).
- Write your Centre Number and Student Number at the top of this page.
- Write using black pen.
- You may ask for an extra writing booklet if you need more space.
- If you have not attempted the question(s), you must still hand in the writing booklet, with 'NOT ATTEMPTED' written clearly on the front cover.
- You may NOT take any writing booklets, used or unused, from the examination room.

12

of

this booklet

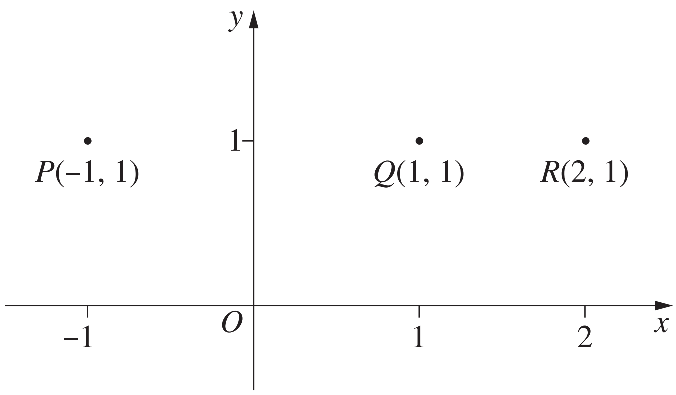
number of booklets for this question

Start here for Question Number: **12**

(a) A direction field is to be drawn for the differential equation

$$\frac{dy}{dx} = \frac{x - 2y}{x^2 + y^2}.$$

Clearly draw the correct slopes of the direction field at the points P, Q and R shown below.



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NSW Education Standards Authority

2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}\left(a + b\right)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$\left(x - h\right)^2 + \left(y - k\right)^2 = r^2$$

Financial Mathematics

$$A = P\left(1 + r\right)^n$$

Sequences and series

$$T_n = a + \left(n - 1\right)d$$

$$S_n = \frac{n}{2}\left[2a + \left(n - 1\right)d\right] = \frac{n}{2}\left(a + l\right)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r} = \frac{a\left(r^n - 1\right)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

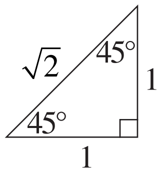
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

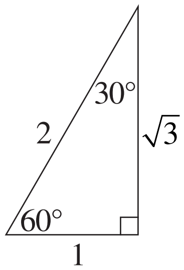
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

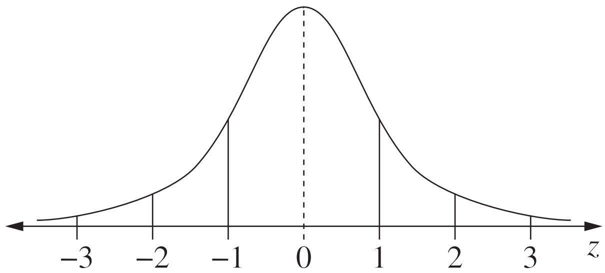
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

2022 HSC Mathematics Extension 1 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	A
3	D
4	A
5	B
6	B
7	C
8	D
9	D
10	B

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j}, \qquad \underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j}$

$$\begin{aligned}\underset{\sim}{u} + 3\underset{\sim}{v} &= \underset{\sim}{i} - \underset{\sim}{j} + 3\left(2\underset{\sim}{i} + \underset{\sim}{j}\right) \\ &= \underset{\sim}{i} - \underset{\sim}{j} + 6\underset{\sim}{i} + 3\underset{\sim}{j} \\ &= 7\underset{\sim}{i} + 2\underset{\sim}{j}\end{aligned}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\begin{aligned}\underset{\sim}{u} \cdot \underset{\sim}{v} &= 1 \times 2 + (-1) \times 1 \\ &= 1\end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains correct integrand, or equivalent merit	2
• Attempts to use the substitution, or equivalent merit	1

Sample answer:

$$\int_0^1 \frac{x}{\sqrt{x^2+4}}\,dx$$

$$= \int_0^1 \frac{2x}{2\sqrt{x^2+4}}\,dx$$

$$= \int_4^5 \frac{1}{2\sqrt{u}}\,du$$

$$= \left[\sqrt{u}\,\right]_4^5$$

$$= \sqrt{5} - 2$$

$u = x^2 + 4$

$du = 2x\,dx$

when $x = 0$, $u = 4$

when $x = 1$, $u = 5$

Question 11 (c)

Criteria	Marks
• Provides correct solution	2
• Obtains a correct expression for one coefficient, or equivalent merit	1

Sample answer:

$$\left(1-\frac{x}{2}\right)^8 = \cdots + \binom{8}{2}\left(1\right)^6\left(-\frac{x}{2}\right)^2 + \binom{8}{3}\left(1\right)^5\left(-\frac{x}{2}\right)^3 + \cdots$$

$$= \cdots + 28 \times \frac{x^2}{4} + 56 \times \frac{-x^3}{8} + \cdots$$

$$= \cdots + 7x^2 - 7x^3 + \cdots$$

$\therefore$ Coefficient of x^2 is 7.

And coefficient of x^3 is –7.

Question 11 (d)

Criteria	Marks
• Provides correct solution	2
• Observes that $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{u} = \begin{pmatrix} a \\ 2 \end{pmatrix}, \qquad \underset{\sim}{v} = \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}$$

$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular $\therefore\ \underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

That is $\quad a(a-7)+2(4a-1)=0$

$$a^2 - 7a + 8a - 2 = 0$$
$$a^2 + a - 2 = 0$$
$$(a+2)(a-1) = 0$$
$$\therefore \quad a = -2 \quad \text{or} \quad 1$$

Question 11 (e)

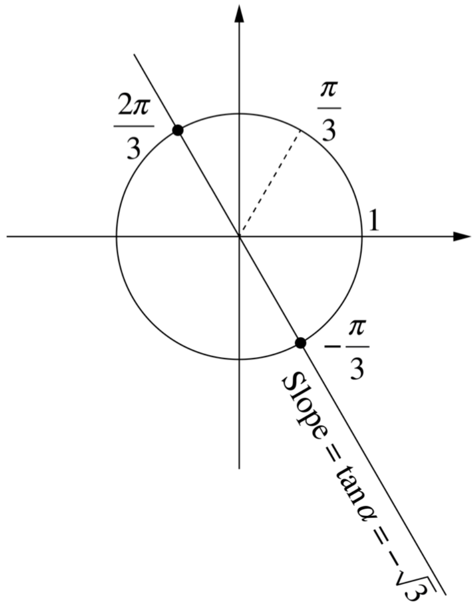
Criteria	Marks
• Provides correct solution	3
• Finds correct argument, or equivalent merit	2
• Finds the correct value of R, or equivalent merit	1

Sample answer:

$$\sqrt{3}\sin(x) - 3\cos(x) = R\sin(x+\alpha)$$
$$= R\sin x\cos\alpha + R\cos x\sin\alpha$$

$\therefore \quad R\sin\alpha = -3 \qquad (1)$

$\quad R\cos\alpha = \sqrt{3} \qquad (2)$



(1) ÷ (2): $\quad \tan\alpha = \dfrac{-3}{\sqrt{3}}$

$\quad = -\sqrt{3}$

$\therefore \quad \alpha = -\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ (See unit circle)

Since $\cos\alpha > 0$ and $\sin\alpha < 0$

$\alpha = -\dfrac{\pi}{3}$

From (1) $\quad R^2\sin^2\alpha = 9 \qquad (3)$

From (2) $\quad R^2\cos^2\alpha = 3 \qquad (4)$

$(3)+(4): \quad R^2\left(\sin^2\alpha + \cos^2\alpha\right) = 12$

$$R^2 = 12$$
$$R = 2\sqrt{3}$$

$$\therefore \sqrt{3}\sin(x) - 3\cos(x) = 2\sqrt{3}\sin\left(x - \frac{\pi}{3}\right)$$

Question 11 (f)

Criteria	Marks
• Provides correct solution	3
• Identifies the correct critical values, or equivalent merit	2
• Excludes $x = 2$ OR attempts to deal with the denominator, or equivalent merit	1

Sample answer:

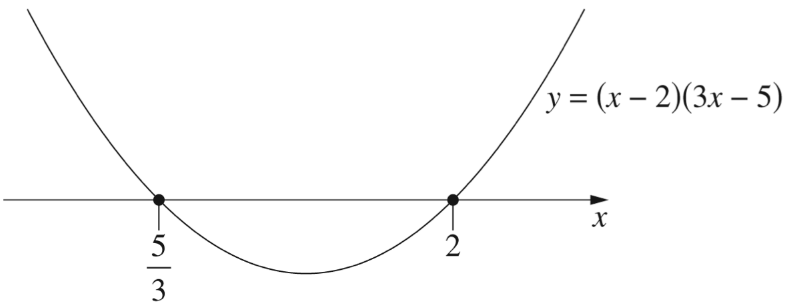
$$\frac{x}{2-x} \geq 5 \qquad x \neq 2$$

$$x(2-x) \geq 5(2-x)^2$$

$$(2-x)\left[5(2-x)-x\right] \leq 0$$

$$(2-x)(-6x+10) \leq 0$$

$$(x-2)(3x-5) \leq 0$$

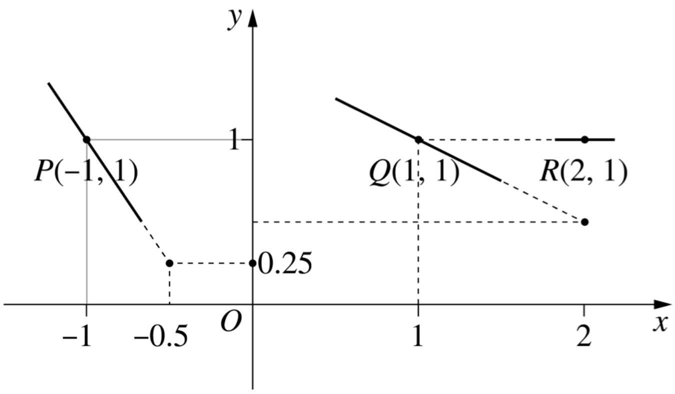


Solutions $= \left[\frac{5}{3}, 2\right)$

Question 12 (a)

Criteria	Marks
• Provides correct drawing	2
• Shows one correct slope	1

Sample answer:



At $P(-1, 1)$ $\quad \dfrac{dy}{dx} = \dfrac{-1-2}{1+1} = -\dfrac{3}{2} = -\dfrac{0.75}{0.5}$

At $Q(1, 1)$ $\quad \dfrac{dy}{dx} = \dfrac{1-2}{1+1} = -\dfrac{1}{2} = -\dfrac{0.5}{1}$

At $R(2, 1)$ $\quad \dfrac{dy}{dx} = \dfrac{2-2}{4+1} = 0$

Question 12 (b)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the pigeonhole principle, or equivalent merit	1

Sample answer:

$$\frac{\text{Number of players above limit}}{\text{Number of teams}} = \frac{41}{13} > 3$$

$\therefore$ Using the Pigeonhole principle, at least one team has more than 3 players above limit. At least one team will be penalised.

Question 12 (c)

Criteria	Marks
• Provides correct solution	3
• Finds the slope of the tangent, or equivalent merit	2
• Attempts to find the derivative of $x \arctan x$, or equivalent merit	1

Sample answer:

$$y = x\tan^{-1}(x)$$

$$y' = x \times \frac{1}{1+x^2} + \tan^{-1}(x) \times 1$$

$$= \frac{x}{1+x^2} + \tan^{-1}(x)$$

At $\left(1, \frac{\pi}{4}\right)$ $\quad y' = \frac{1}{2} + \tan^{-1}(1)$

$$= \frac{1}{2} + \frac{\pi}{4}$$

$$= \frac{\pi + 2}{4}$$

Equation of tangent

$$y - \frac{\pi}{4} = \left(\frac{\pi+2}{4}\right)\left(x-1\right)$$

$$4y - \pi = (\pi+2)(x-1)$$

$$4y = (\pi+2)x - \pi - 2 + \pi$$

$$4y = (\pi+2)x - 2$$

$$\therefore \quad y = \frac{(\pi+2)}{4}x - \frac{1}{2}$$

Question 12 (d) (i)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain T as a function involving an exponential function, or equivalent merit	2
• Separates the variables in the differential equation, or equivalent merit	1

Sample answer:

$$\frac{dT}{dt} = k\left(T - T_1\right)$$

Note that as $t \to +\infty$, $\dfrac{dT}{dt} \to 0$ and T approaches room temperature so T_1 = room temperature = 12.

Also T decreases with time, so $T - T_1 > 0$.

$\dfrac{dT}{T-12} = k\,dt$ (separate variables)

$\ln(T - 12) = kt + c$ where c is a constant

$T - 12 = \mathrm{A}e^{kt}$ where $A = e^c$

$T = 12 + \mathrm{A}e^{kt}$

When $t = 0$: $92 = 12 + A$ so $A = 80$

When $t = 5$: $76 = 12 + 80e^{5k}$

$$e^{5k} = \frac{4}{5}$$

$$5k = \ln\left(\frac{4}{5}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{4}{5}\right)$$

Finally $T = 12 + 80e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t}$ for $t \geq 0$.

Question 12 (d) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

For $T = 57$

$$57 = 12 + 80e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t}$$

$$e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t} = \frac{57-12}{80} = \frac{9}{16}$$

$$\frac{1}{5}\ln\left(\frac{4}{5}\right)t = \ln\left(\frac{9}{16}\right)$$

$$t = \frac{5\ln\left(\frac{9}{16}\right)}{\ln\left(\frac{4}{5}\right)} = 12.89\ldots \qquad \approx 13 \text{ minutes}$$

The coffee will reach a temperature of 57°C after about 13 minutes.

Question 12 (e)

Criteria	Marks
• Provides correct solution	2
• Attempts to use result associated with the binomial distribution, or equivalent merit	1

Sample answer:

Let p be the proportion of red balls and n the number of trials.

Expected number of red balls $= np = 4 \times \dfrac{3}{10}$

Expected number of green balls $= n(1-p) = 4 \times \dfrac{7}{10}$

Expected score $= 10 \times 4 \times \dfrac{3}{10} - 5 \times 4 \times \dfrac{7}{10} = 12 - 14 = -2$.

Answers could include:

- The expected score if you pick <u>one</u> ball is $\dfrac{3 \times 10 + 7 \times (-5)}{10} = -0.5$.
- It is the same for every one of the 4 balls since the balls are replaced so the expected score for the game is $4 \times (-0.5) = -2$.

Question 12 (f)

Criteria	Marks
• Provides correct proof	3
• Proves the inductive step, or equivalent merit	2
• Establishes the base case, or equivalent merit	1

Sample answer:

METHOD 1

Let $a_n = 15^n + 6^{2n+1}$

Base case: $n = 0$

$a_0 = 1 + 6 = 7$ is divisible by 7

so the property holds for $n = 0$

Assume a_k is divisible by 7

$a_k = 15^k + 6^{2k+1} = 7M$, for some integer M.

Prove true for a_{k+1}.

$a_{k+1} = 15^{k+1} + 6^{2k+3} = 7Q$, for some integer Q.

$$\begin{aligned}
LHS &= 15\left(15^k\right) + 6^{2k+3} \\
&= 15\left(7M - 6^{2k+1}\right) + 6^2 \times 6^{2k+1} \qquad \text{(from assumption)} \\
&= 105M - 15 \times 6^{2k+1} + 36 \times 6^{2k+1} \\
&= 105M + 21 \times 6^{2k+1} \\
&= 7\left(15M + 3 \times 6^{2k+1}\right) \\
&= 7Q \\
&= RHS
\end{aligned}$$

This proves, using mathematical induction, that for all integers $n \geq 0$, $15^n + 6^{2n+1}$ is divisible by 7.

Answers could include:

METHOD 2

Let $a_n = 15^n + 6^{2n+1}$

Base case: $n = 0$

$a_0 = 1 + 6 = 7$ is divisible by 7

so the property holds for $n = 0$

Assume a_k is divisible by 7

$$\begin{aligned} a_{k+1} &= 15^{k+1} + 6^{2k+3} \\ &= 15\left(15^k\right) + 6^{2k+3} \end{aligned}$$

Substituting $15^k = a_k - 6^{2k+1}$

$$\begin{aligned} a_{k+1} &= 15\left(a_k - 6^{2k+1}\right) + 6^{2k+3} \\ &= 15a_k + 6^{2k+1}(-15 + 36) \\ &= 15a_k + 21 \times 6^{2k+1} \end{aligned}$$

a_k and 21 are both divisible by 7

so a_{k+1} is also divisible by 7

This proves, using mathematical induction, that for all integers $n \geq 0$, $15^n + 6^{2n+1}$ is divisible by 7.

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Attempts to find the dot product of $\overrightarrow{BH}$ and $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, or equivalent merit	2
• Write $\overrightarrow{BH}$ or $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, or equivalent merit	1

Sample answer:

We have $\overrightarrow{BH} = \underset{\sim}{h} - \underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{c}$, while $\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$. Hence

$$\begin{aligned}\overrightarrow{BH} \cdot \overrightarrow{CA} &= (\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} - \underset{\sim}{c}) \\ &= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{c} \\ &= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{c} \\ &= |\underset{\sim}{a}|^2 - |\underset{\sim}{c}|^2 \\ &= 0 \qquad \text{(as the lengths of } \underset{\sim}{a} \text{ and } \underset{\sim}{c} \text{ are equal, as they are radii of a circle)}\end{aligned}$$

Hence, the two vectors are perpendicular.

Answers could include:

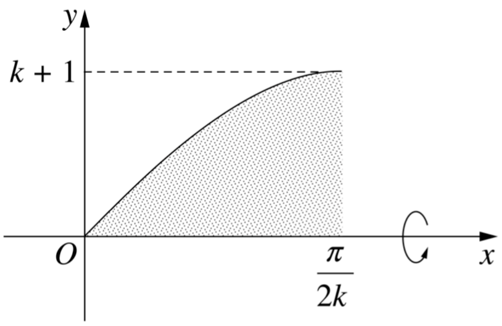
We may observe that $\underset{\sim}{a} + \underset{\sim}{c}$ and $\underset{\sim}{a} - \underset{\sim}{c}$ form the diagonals of a rhombus, as $\underset{\sim}{a}$ and $\underset{\sim}{c}$ have equal length. The diagonals of a rhombus are perpendicular and so $\overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains primitive in terms of $\sin(2kx)$, or equivalent merit	2
• Finds an integral expression for the volume of the solid in terms of $\sin(kx)$, or equivalent merit	1

Sample answer:

$y = (k+1)\sin(kx)$



$$V = \pi\int_0^{\frac{\pi}{2k}} (k+1)^2 \sin^2(kx)\cdot dx$$

$$= \pi(k+1)^2\int_0^{\frac{\pi}{2k}} \frac{1-\cos 2kx}{2}\cdot dx$$

$$= \frac{\pi(k+1)^2}{2}\int_0^{\frac{\pi}{2k}} 1-\cos 2kx\cdot dx$$

$$= \frac{\pi(k+1)^2}{2}\left[x-\frac{\sin 2kx}{2k}\right]_0^{\frac{\pi}{2k}}$$

$$= \frac{\pi(k+1)^2}{2}\left(\frac{\pi}{2k}-\frac{\sin\pi}{2k}\right)$$

$$= \frac{\pi^2(k+1)^2}{4k}$$

$$\cos 2\theta = 1-2\sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\therefore\ \sin^2(kx) = \frac{1}{2}(1-\cos 2kx)$$

For $V = \pi^2$,

$$\frac{(k+1)^2}{4k} = 1$$

$\therefore \quad k = 1 \quad$ (by inspection)

Or solve $k^2 + 2k + 1 = 4k$

$$(k-1)^2 = 0$$

$$k = 1$$

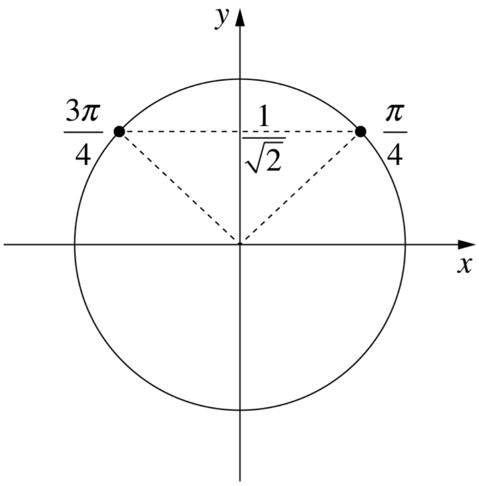
Question 13 (c)

Criteria	Marks
• Provides correct solution	2
• Finds a value of x where $g(f(x)) \neq x$	1

Sample answer:

The domain of $g(f(x))$ is $\mathbb{R}$.
$g(f(x)) = \arcsin(\sin x)$

If g is the inverse function of f, then $g(f(x))$ is equal to x for all x in the domain of $g(f(x))$ by definition of an inverse function.



Let $x = \frac{3\pi}{4}$
$\arcsin\left(\sin\frac{3\pi}{4}\right) = \frac{\pi}{4}$, see unit circle above
So for $x = \frac{3\pi}{4}$, $\arcsin(\sin x) \neq x$

So it is not true that $g(f(x)) = x$ for all x in the domain $g(f(x))$. So, g is NOT the inverse of f.

Question 13 (d)

Criteria	Marks
• Provides correct solution	3
• Evaluates $P'(\alpha)+P'(\beta)+P'(\gamma)$, or equivalent merit	2
• Writes $\alpha^2+\beta^2+\gamma^2$ in term of $\alpha+\beta+\gamma$ and so on OR Defines $P(x)$ and evaluates $P'(\alpha)$ OR Equivalent merit	1

Sample answer:

$\alpha^2+\beta^2+\gamma^2=85$
$P'(\alpha)+P'(\beta)+P'(\gamma)=87$

Let $P(x)=x^3+ax^2+bx+c$
Then $a=-(\alpha+\beta+\gamma)$
$b=\alpha\beta+\beta\gamma+\gamma\alpha$
$c=-\alpha\beta\gamma$

$P'(x)=3x^2+2ax+b$

$$
\begin{aligned}
&P'(\alpha)+P'(\beta)+P'(\gamma)\\
&=3\left(\alpha^2+\beta^2+\gamma^2\right)+2a(\alpha+\beta+\gamma)+3b\\
&=3\left(\alpha^2+\beta^2+\gamma^2\right)-2(\alpha+\beta+\gamma)^2+3(\alpha\beta+\beta\gamma+\gamma\alpha)\\
&=3\left(\alpha^2+\beta^2+\gamma^2\right)-2\left(\alpha^2+\beta^2+\gamma^2\right)-4(\alpha\beta+\beta\gamma+\gamma\alpha)+3(\alpha\beta+\beta\gamma+\gamma\alpha)\\
&=\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\gamma\alpha)
\end{aligned}
$$

$$
\begin{aligned}
87&=85-(\alpha\beta+\beta\gamma+\gamma\alpha)\\
\alpha\beta+\beta\gamma+\gamma\alpha&=85-87\\
&=-2
\end{aligned}
$$

Question 13 (e) (i)

Criteria	Marks
• Provides correct solution	2
• Introduces a suitable normal approximation, or equivalent merit	1

Sample answer:

METHOD 1

Let $\hat{p}$ be the proportion of chocolate bars which weigh less than 150 g.

$\mathcal{P} = P(\hat{p} \geq 0.5)$

Approximate $\hat{p}$ by a normal distribution with the same mean as $\hat{p}$ and the same standard deviation as $\hat{p}$.

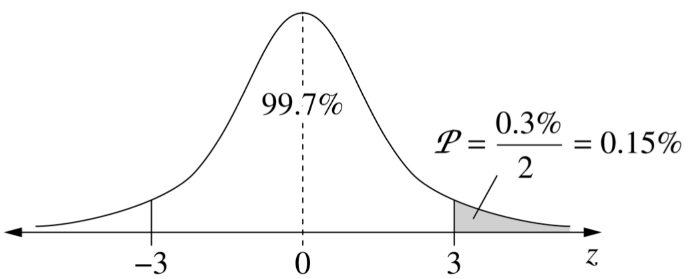
$E(\hat{p}) = 0.2$

$\sigma(\hat{p}) = \sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{0.2 \times 0.8}{16}} = 0.1$

$$\begin{aligned}\mathcal{P} &= P\left(\frac{\hat{p}-0.2}{0.1} \geq \frac{0.5-0.2}{0.1}\right)\\ &\approx P(Z \geq 3)\end{aligned}$$

where Z follows a standard normal distribution.

Using known values



So $\mathcal{P} = P(Z \geq 3) = 0.15\% = 0.0015$

Alternatively using the table of values for the normal distribution, we get the more precise value of $\mathcal{P} = 0.13\% = 0.0013$

Answers could include:

METHOD 2

Let $\hat{p}$ be the proportion of bars in a sample of 16 bars which weigh more than 150 g.

$$\mathcal{P} = P(\hat{p} < 0.5)$$

(If 50% or more of the bars weigh less than 150 g, the remaining ones weigh 150 g or more.)

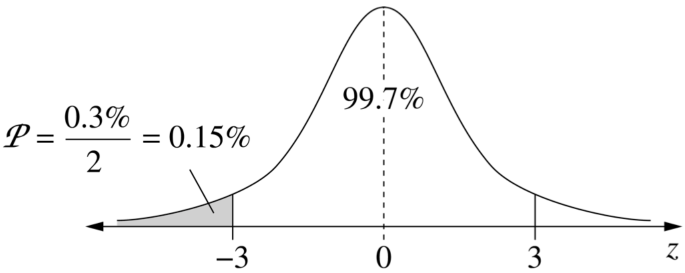
$$E(\hat{p}) = 0.8$$

$$\sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.2 \times 0.8}{16}} = 0.1$$

$$\begin{aligned} \mathcal{P} &= P(\hat{p} < 0.5) \\ &= P\left(\frac{\hat{p} - 0.8}{0.1} < \frac{0.5 - 0.8}{0.1}\right) \\ &\approx P(Z < -3) \end{aligned}$$

When we approximate $\hat{p}$ by a normal distribution.

$\mathcal{P} \simeq P(Z < -3) \approx \dfrac{0.3\%}{2} = 0.15\%$ using known values



(and $\mathcal{P} = 0.13\%$ using the table of normal values)

METHOD 3: (Using counts rather than proportion, without continuity correction)

Let X be the number of chocolate bars in a sample of 16 which weigh less than 150 g.

$\mathcal{P} = P(X \geq 8)$

X follows a binomial distribution $\text{Bin}(n, p) = \text{Bin}(16, 0.2)$, assuming the factory manager's claim is correct.

$$E(X) = np = 16 \times 0.2 = 3.2$$

$$\sigma(X) = \sqrt{np(1-p)} = \sqrt{16 \times 0.2 \times 0.8} = 1.6$$

$$\begin{aligned}\mathcal{P} &= P(X \geq 8) \\ &= P\left(\frac{X - 3.2}{1.6} \geq \frac{8 - 3.2}{1.6}\right) \\ &\approx P(Z \geq 3)\end{aligned}$$

If we approximate X by a normal distribution (without the continuity correction) we get

$\mathcal{P} \approx P(Z \geq 3)$ where Z follows a standard normal distribution

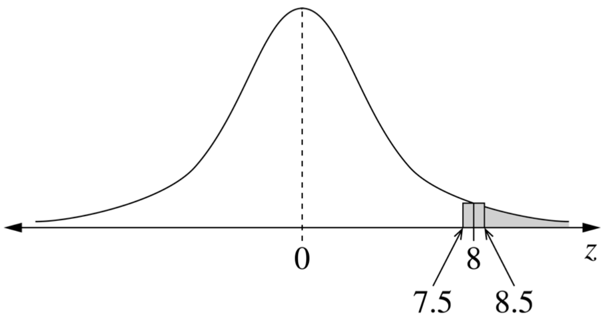
So $\mathcal{P} \approx 0.15\%$ using known values

METHOD 4: (Using counts and the continuity correction.)

Same X as method 3.

We approximate X by a normal distribution Y with the same mean and standard deviation as X.

$$\begin{aligned}\mathcal{P} &= P(X \geq 8)\\ &= P(Y > 7.5)\end{aligned}$$



$$\begin{aligned}\mathcal{P} &= P\left(\frac{Y-3.2}{1.6} > \frac{7.5-3.2}{1.6}\right)\\ &\approx P(Z > 2.6875) && \text{where } Z \text{ is a standard normal}\\ &= 1 - P(Z \leq 2.6875) && \text{using the nearest value from the table}\\ &\approx 1 - 0.9964\\ &= 0.0036\\ &= 0.36\%\end{aligned}$$

Question 13 (e) (ii)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

The number of items checked may not be large enough to assume the data is normally distributed.

Question 14 (a)

Criteria	Marks
• Provides correct solution	4
• Evaluates the constant of integration to find a correct equation for $\ln\lvert y \rvert$	3
• Obtains correct primitive, or equivalent merit	2
• Separates the variables in the differential equation, or equivalent merit	1

Sample answer:

$(x-2)\dfrac{dy}{dx} = xy$ and $(0,\, 1)$

$$\int \frac{1}{y}\,dy = \int \frac{x}{x-2}\,dx$$

$$\ln\lvert y\rvert = \int 1 + \frac{2}{x-2}\,dx$$

$$= x + 2\ln\lvert x-2\rvert + C$$

$$= \ln\lvert x-2\rvert^2 + x + C$$

$$\ln\left(\frac{\lvert y\rvert}{(x-2)^2}\right) = x + C$$

$$\frac{\lvert y\rvert}{(x-2)^2} = e^{x+C}$$

$$\lvert y\rvert = e^{x+C}(x-2)^2$$

$$y = ke^x(x-2)^2 \qquad \text{where} \quad k = \pm e^C$$

When $x = 0,\ y = 1 \qquad \therefore \quad k = \dfrac{1}{4}$

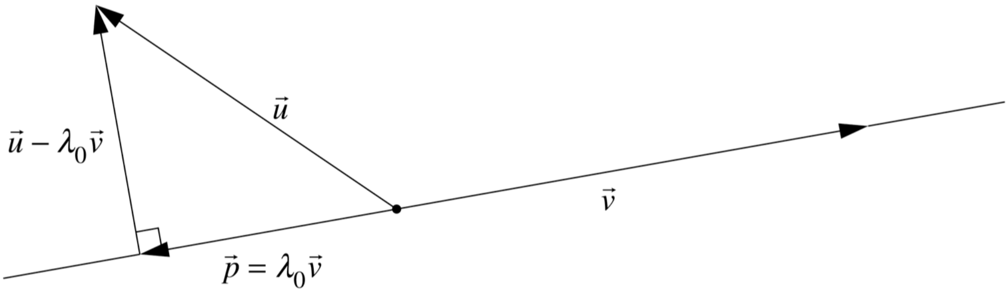
$y = \dfrac{1}{4}e^x(x-2)^2$

Question 14 (b)

Criteria	Marks
• Provides correct solution	3
• Uses an appropriate vector method to obtain a suitable inequality, or equivalent merit	2
• Writes $\underset{\sim}{u} - \lambda \underset{\sim}{v}$ as a sum of vectors parallel and perpendicular to $\underset{\sim}{v}$, or equivalent merit	1

Sample answer:

METHOD 1



Let λ be a real number.

$$|\vec{u} - \lambda \vec{v}|^2 = |\vec{u} - \lambda_0 \vec{v} + \lambda_0 \vec{v} - \lambda \vec{v}|^2$$

$$= \Big|\underbrace{\left(\vec{u} - \lambda_0 \vec{v}\right)}_{\substack{\text{Perpendicular} \\ \text{to } \vec{v}}} + \underbrace{\left(\lambda_0 - \lambda\right)\vec{v}}_{\substack{\text{Parallel} \\ \text{to } \vec{v}}}\Big|^2$$

$$= \left[\left(\vec{u} - \lambda_0 \vec{v}\right) + \left(\lambda_0 - \lambda\right)\vec{v}\right] \cdot \left[\left(\vec{u} - \lambda_0 \vec{v}\right) + \left(\lambda_0 - \lambda\right)\vec{v}\right]$$

$$= |\vec{u} - \lambda_0 \vec{v}|^2 + |\left(\lambda_0 - \lambda\right)\vec{v}|^2 + 0 + 0$$

$$|\left(\lambda_0 - \lambda\right)\vec{v}|^2 \geq 0$$

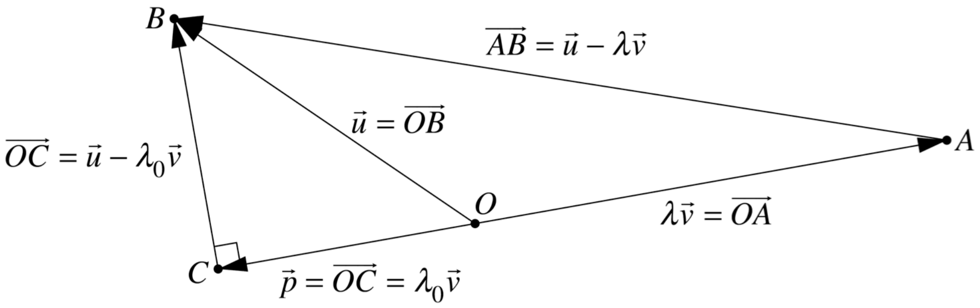
Therefore $|\vec{u} - \lambda \vec{v}|^2 \geq |\vec{u} - \lambda_0 \vec{v}|^2$,

So $|\vec{u} - \lambda \vec{v}| \geq |\vec{u} - \lambda_0 \vec{v}|$.

Hence $|\vec{u} - \lambda \vec{v}|$ is smallest when it equals $|\vec{u} - \lambda_0 \vec{v}|$, and so is smallest when $\lambda = \lambda_0$.

Answers could include:

METHOD 2



Let $\lambda \in \mathbb{R}$.

Let O be a point in the plane.

Let A, B and C be the points defined by

$\overrightarrow{OA} = \lambda\vec{v}$

$\overrightarrow{OB} = \vec{u}$

$\overrightarrow{OC} = \vec{p}$

Because $\vec{p}$ is the projection of $\vec{u}$ onto $\vec{v}$, the triangle ABC has a right angle at C.

In any right-angled triangle the length of the hypotenuse is greater than or equal to the length of any of the sides

so $\left|\overrightarrow{AB}\right| \geq \left|\overrightarrow{BC}\right|$

That is $\left|\vec{u} - \lambda\vec{v}\right| \geq \left|\vec{u} - \lambda_0\vec{v}\right|$.

Hence $\left|\vec{u} - \lambda\vec{v}\right|$ is smallest when it equals $\left|\vec{u} - \lambda_0\vec{v}\right|$, and so is smallest when $\lambda = \lambda_0$.

Question 14 (c)

Criteria	Marks
• Provides correct solution	4
• Finds an expression for d in terms of the maximum range, or equivalent merit	3
• Finds maximum range, or equivalent merit	2
• Attempts to find the time of flight of projectile, or equivalent merit	1

Sample answer:

For the player to hit the target there must be a time t_1 where the positions of the target and of the projectile coincide:

$$\begin{cases} 2ut_1\cos\theta = d + ut_1 & \text{(i)} \\ 2ut_1\sin\theta = \dfrac{g}{2}{t_1}^2 & \text{(ii)} \end{cases}$$

By (i) $\quad t_1 = \dfrac{d}{2u\cos\theta - u}$

t_1 is not 0, so dividing (ii) by t_1 yields

$$2u\sin\theta = \frac{g}{2}t_1 \qquad \text{(iii)}$$

Substitute in t_1 in (iii):

$$2u\sin\theta = \frac{g}{2}\left(\frac{d}{2u\cos\theta - u}\right)$$

$$d = \frac{2u\sin\theta\big(u\times(2\cos\theta-1)\big)\times 2}{g}$$

$$d = \frac{4u^2}{g}\sin\theta(2\cos\theta-1)$$

<u>Maximum range</u>:

It occurs at time t_2 and corresponds to $2ut_2\sin\theta - \frac{g}{2}{t_2}^2 = 0$.

$t_2 \neq 0 \quad$ so $\quad 2u\sin\theta - \dfrac{g}{2}t_2 = 0$

$$t_2 = \frac{4u\sin\theta}{g}$$

The range is $\quad 2ut_2\cos\theta = \dfrac{8u^2}{g}\sin\theta\cos\theta$

$$= \frac{4u^2}{g}\sin 2\theta$$

This is maximum when $\sin 2\theta$ is maximum, that is when $\sin 2\theta = 1$.

The maximum range is $\dfrac{4u^2}{g}$.

Find the maximum value of $\sin\theta(2\cos\theta - 1)$ for θ in $\left(0, \dfrac{\pi}{2}\right)$.

$\sin\theta(2\cos\theta - 1) = \sin 2\theta - \sin\theta$

Let f be the function defined on $\left(0, \dfrac{\pi}{2}\right)$

by $\quad f(\theta) = \sin 2\theta - \sin\theta$

$$f'(\theta) = 2\cos 2\theta - \cos\theta$$

$$= 2\left(2\cos^2\theta - 1\right) - \cos\theta$$

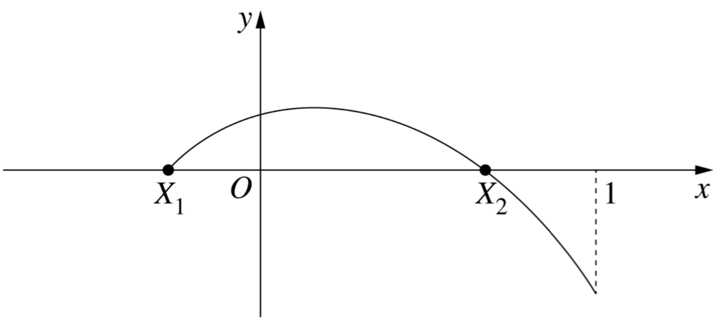
Let $X = \cos\theta$

$f'(\theta) = 2\left(2X^2 - 1\right) - X = 4X^2 - X - 2$

$\Delta = b^2 - 4ac = 1 - 4(-8) = 33$

$X_1 = \dfrac{1 - \sqrt{33}}{8}$ and $X_2 = \dfrac{1 + \sqrt{33}}{8}$

$\approx -0.59 \qquad\qquad \approx 0.84$



$f'(\theta) \geq 0$ if $0 \leq X \leq X_2$

$f'(\theta) \leq 0$ if $X_2 \leq X \leq 1$

So $f(\theta)$ is maximum when $X = X_2$

That is $\cos\theta = \dfrac{1 + \sqrt{33}}{8}$

With $0 < \theta < \dfrac{\pi}{2}$, that means $\theta \approx 0.5678$ which corresponds to $\sin 2\theta - \sin\theta = 0.3690\ldots \simeq 37\%$.

$$d = \underbrace{\frac{4u^2}{g}}_{\text{maximum range}} \times \underbrace{\sin\theta(2\cos\theta - 1)}_{\leq 37\%}$$

So d must be less than 37% of the maximum range.

Question 14 (d)

Criteria	Marks
• Provides correct solution	4
• Obtains a quadratic inequality in N, where N is the number of tickets sold, or equivalent merit	3
• Uses a normal approximation to find an expression for the probability, or equivalent merit	2
• Writes a probability statement similar to $P(X > 350) \leq 0.01$, where X is the number of passengers who turn up, or equivalent merit	1

Sample answer:

METHOD 1:

Let n be the number of tickets sold for a 350-seat flight ($n \geq 350$).

Let X be the number of passengers showing up on a 350-seat flight $X \sim \text{Bin}(n,\ 0.95)$.

The management's decision can be written as:
$P(X > 350) \leq 0.01$
or
$P(X \leq 350) \geq 0.99$

Find n such that $P(X \leq 350) = 0.99$.

Approximating X by the appropriate normal random variable Y, without continuity correction, the condition $P(X \leq 350) \geq 0.99$ can be rewritten.

$P(Y \leq 350) \geq 0.99$.

Need to find n such that $P(Y \geq 350) = 0.99$

This becomes

$$P\left(\frac{Y - 0.95n}{\sqrt{n \times 0.95 \times 0.05}} < \frac{350 - 0.95n}{\sqrt{n \times 0.95 \times 0.05}}\right) = 0.99$$

This happens if

$$\frac{350 - 0.95n}{\sqrt{0.95 \times 0.05n}} \simeq 2.33$$

$$350 - 0.95n = 2.33\sqrt{0.95 \times 0.05n}$$

Let $x = \sqrt{n}$

$$0.95x^2 + 2.33\sqrt{0.95 \times 0.05}x - 350 = 0$$

$$x = \sqrt{n} = \frac{-2.33\sqrt{0.95 \times 0.05} \pm \sqrt{(2.33)^2 \times 0.95 \times 0.05 + 4 \times 0.95 \times 350}}{2 \times 0.95}$$

$\sqrt{n} \approx 18.93$

$n \simeq 358.30$

So $n = 358$ (359 would result in getting too many passengers more than 1% of the time)

Given the management's decision, the optimal number of tickets sold on a 350-seat flight is 358.

Answers could include:

<u>METHOD 2</u>: (with continuity correction)

X, Y and n defined as before.

Approximate the binomial X by the appropriate normal random variable Y, with continuity correction.

$P(Y < 350.5) \geq 0.99$ (i)

$E(Y) = E(X) = n \times 0.95$

$\text{Var}(Y) = n \times 0.95 \times 0.05$

(i) becomes

$$P\left(\frac{Y - 0.95n}{\sqrt{n \times 0.95 \times 0.05}} < \frac{350.5 - 0.95n}{\sqrt{n \times 0.95 \times 0.05}}\right) = 0.99$$

Using the normal distribution table

$$\frac{350.5 - 0.95n}{\sqrt{0.95 \times 0.05n}} \simeq 2.33$$

$$350.5 - 0.95n = 2.33\sqrt{0.95 \times 0.05} \times \sqrt{n}$$

Let $x = \sqrt{n}$

$$0.95x^2 + 2.33\sqrt{0.95 \times 0.05}x - 350.5 = 0$$

$$\Delta = b^2 - 4ac = 0.95 \times 0.05 \times (2.33)^2 - 4 \times 0.95 \times (-350.5)$$

$$\simeq 1332.158$$

$$x = \sqrt{n} = \frac{-2.33\sqrt{0.95 \times 0.05} + \sqrt{1332.158}}{2 \times 0.95}$$

(Can ignore the negative solution since $x = \sqrt{n} > 0$.)

$\sqrt{n} \approx 18.94$

$n \simeq 358.8$

$n = 358$ (359 would result in getting too many passengers more than 1% of the time)

2022 HSC Mathematics Extension 1 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	ME-T1 Inverse Trigonometric Functions	ME11-3
2	1	ME-F1 Further Work with Functions	ME11-1
3	1	ME-F2 Polynomials	ME11-2
4	1	ME-F1 Further Work with Functions	ME11-1
5	1	ME-F1 Further Work with Functions	ME11-2
6	1	ME-V1 Introduction to Vectors	ME12-2
7	1	ME-A1 Working with Combinatorics	ME11-5
8	1	ME-V1 Introduction to Vectors	ME12-2
9	1	ME-C2 Further Calculus Skills	ME12-1
10	1	ME-C3 Applications of Calculus	ME12-4

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	ME-V1 Introduction to Vectors	ME12-2
11 (a) (ii)	1	ME-V1 Introduction to Vectors	ME12-2
11 (b)	3	ME-C2 Further Calculus Skills	ME12-1
11 (c)	2	ME-A1 Working with Combinatorics	ME11-5
11 (d)	2	ME-V1 Introduction to Vectors	ME12-2
11 (e)	3	ME-T3 Trigonometric Equations	ME12-3
11 (f)	3	ME-F1 Further Work with Functions	ME11-1
12 (a)	2	ME-C3 Applications of Calculus	ME12-1
12 (b)	2	ME-A1 Working with Combinatorics	ME11-5
12 (c)	3	ME-C2 Further Calculus Skills	ME12-1
12 (d) (i)	3	ME-C3 Applications of Calculus	ME11-4, ME12-4
12 (d) (ii)	1	ME-C1 Rate of Change	ME11-4
12 (e)	2	ME-S1 The Binomial Distribution	ME12-5
12 (f)	3	ME-P1 Proof by Mathematical Induction	ME12-1
13 (a)	3	ME-V1 Introduction to Vectors	ME12-2

Question	Marks	Content	Syllabus outcomes
13 (b)	3	ME-C3 Applications of Calculus	ME12-4
13 (c)	2	ME-T1 Inverse Trigonometric Functions	ME11-1
13 (d)	3	ME-F2 Polynomials	ME11-1
13 (e) (i)	2	ME-S1 The Binomial Distribution	ME12-5, ME12-7
13 (e) (ii)	1	ME-S1 The Binomial Distribution	ME12-5, ME12-7
14 (a)	4	ME-C3 Applications of Calculus	ME12-4
14 (b)	3	ME-V1 Introduction to Vectors	ME12-2, ME12-7
14 (c)	4	ME-V1 Introduction to Vectors ME-C3 Applications of Calculus	ME12-2, ME12-3
14 (d)	4	ME-S1 The Binomial Distribution ME-S1 The Binomial Distribution	ME12-5, ME12-7