NSW GOVERNMENT

NSW Education Standards Authority

Centre Number

Student Number

2023 **HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 70

Section I – 10 marks (pages 2–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8–16)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 The temperature $T(t)$°C of an object at time t seconds is modelled using Newton's Law of Cooling,

$$T(t) = 15 + 4e^{-3t}.$$

What is the initial temperature of the object?

A. -3

B. 4

C. 15

D. 19

2 A standard six-sided die is rolled 12 times.

Let $\hat{p}$ be the proportion of the rolls with an outcome of 2.

Which of the following expressions is the probability that at least 9 of the rolls have an outcome of 2?

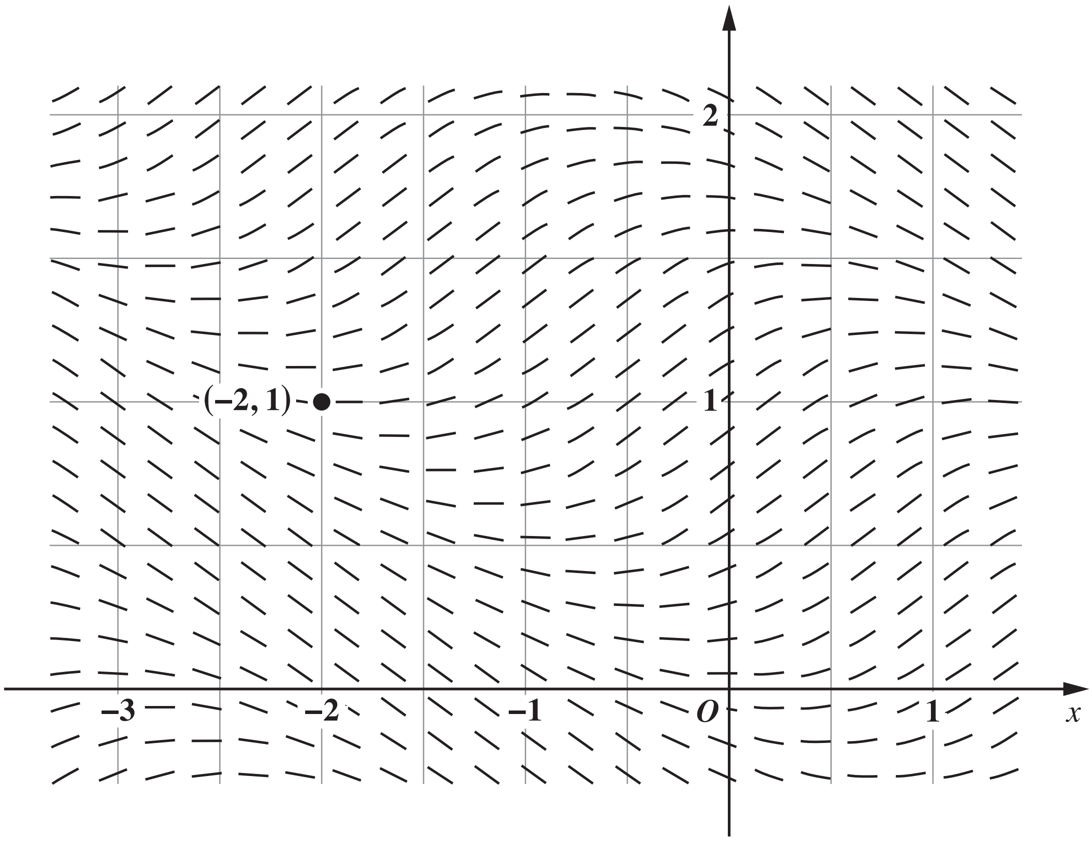
A. $P\left(\hat{p} \geq \frac{3}{4}\right)$

B. $P\left(\hat{p} \geq \frac{1}{6}\right)$

C. $P\left(\hat{p} \leq \frac{3}{4}\right)$

D. $P\left(\hat{p} \leq \frac{1}{6}\right)$

3 The diagram shows the direction field of a differential equation. A particular solution to the differential equation passes through $(-2, 1)$.



Where does the solution that passes through $(-2, 1)$ cross the y-axis?

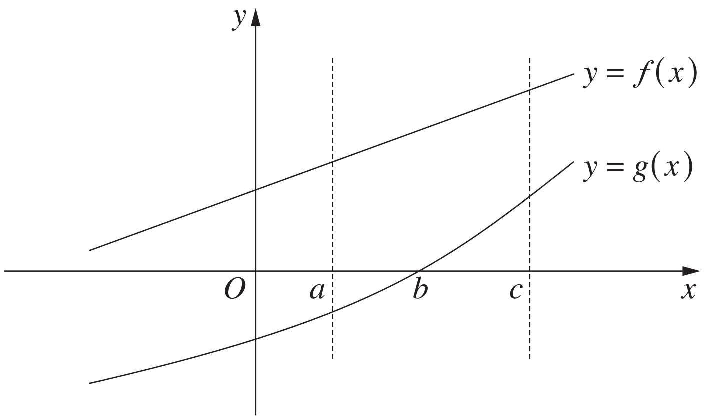
A. $y = 1.12$

B. $y = 1.34$

C. $y = 1.56$

D. $y = 1.78$

4 The diagram shows the graphs of the functions $f(x)$ and $g(x)$.



It is known that

$$\int_a^c f(x)\,dx = 10$$

$$\int_a^b g(x)\,dx = -2$$

$$\int_b^c g(x)\,dx = 3\,.$$

What is the area between the curves $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = c$?

A. 5

B. 7

C. 9

D. 11

5 Which of the following is the value of $\sin^{-1}(\sin a)$ given that $\pi < a < \dfrac{3\pi}{2}$?

A. $a - \pi$

B. $\pi - a$

C. a

D. $-a$

6 Given the two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, let $\underset{\sim}{c}$ be the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

What is the projection of $10\underset{\sim}{a}$ onto $2\underset{\sim}{b}$?

A. $2\underset{\sim}{c}$

B. $5\underset{\sim}{c}$

C. $10\underset{\sim}{c}$

D. $20\underset{\sim}{c}$

7 Which statement is always true for real numbers a and b where $-1 \le a < b \le 1$?

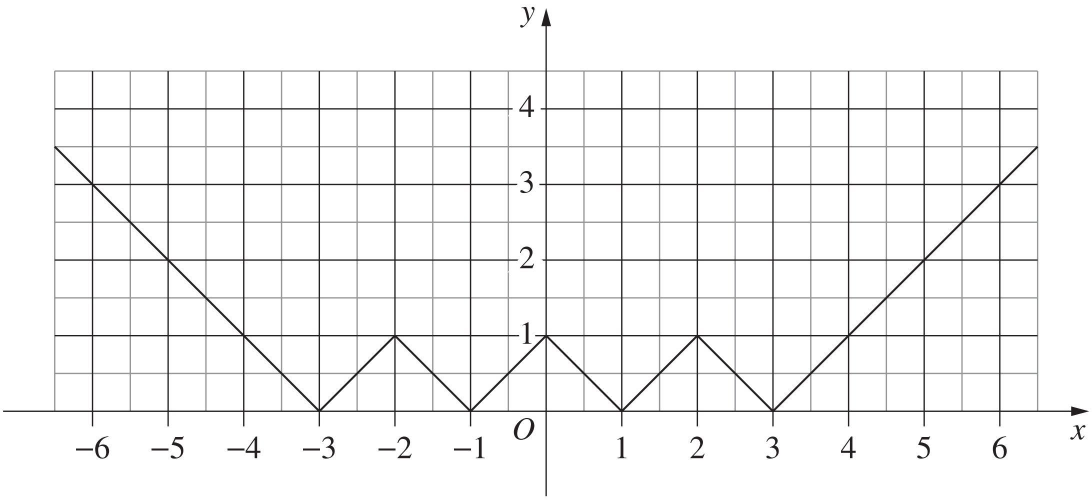
A. $\sec a < \sec b$

B. $\sin^{-1} a < \sin^{-1} b$

C. $\arccos a < \arccos b$

D. $\cos^{-1} a + \sin^{-1} a < \cos^{-1} b + \sin^{-1} b$

8 The diagram shows the graph of a function.



Which of the following is the equation of the function?

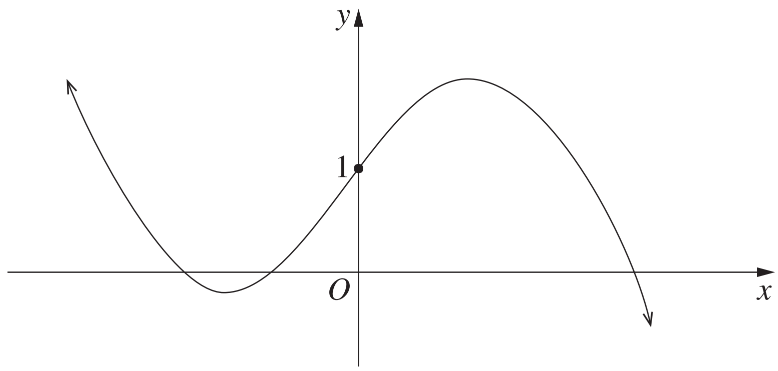
A. $y = \left|1 - \left||x| - 2\right|\right|$

B. $y = \left|2 - \left||x| - 1\right|\right|$

C. $y = \left|1 - |x - 2|\right|$

D. $y = \left|2 - |x - 1|\right|$

9 The graph of a cubic function, $y = f(x)$, is given below.



Which of the following functions has an inverse relation whose graph has more than 3 points with an x-coordinate of 1?

A. $y = \sqrt{f(x)}$

B. $y = \dfrac{1}{f(x)}$

C. $y = f(|x|)$

D. $y = |f(x)|$

10 A group with 5 students and 3 teachers is to be arranged in a circle.

In how many ways can this be done if no more than 2 students can sit together?

A. $4! \times 3!$

B. $5! \times 3!$

C. $2! \times 5! \times 3!$

D. $2! \times 2! \times 2! \times 3!$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) The parametric equations of a line are given below. **2**

$$x = 1 + 3t$$
$$y = 4t$$

Find the Cartesian equation of this line in the form $y = mx + c$.

(b) In how many different ways can all the letters of the word CONDOBOLIN be arranged in a line? **2**

(c) Consider the polynomial **3**

$$P(x) = x^3 + ax^2 + bx - 12,$$

where a and b are real numbers.

It is given that $x + 1$ is a factor of $P(x)$ and that, when $P(x)$ is divided by $x - 2$, the remainder is -18.

Find a and b.

Question 11 continues on page 9

(d) Find $\int \frac{1}{\sqrt{4-9x^2}}\,dx$. **2**

(e) Solve $\cos\theta + \sin\theta = 1$ for $0 \le \theta \le 2\pi$. **3**

(f) A recent census found that 30% of Australians were born overseas.

A sample of 900 randomly selected Australians was surveyed.

Let $\hat{p}$ be the sample proportion of surveyed people who were born overseas.

A normal distribution is to be used to approximate $P(\hat{p} \le 0.31)$.

(i) Show that the variance of the random variable $\hat{p}$ is $\frac{7}{30\,000}$. **2**

(ii) Use the standard normal distribution and the information on page 16 to approximate $P(\hat{p} \le 0.31)$, giving your answer correct to two decimal places. **2**

End of Question 11

Please turn over

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Evaluate $\displaystyle\int_3^4 (x+2)\sqrt{x-3}\,dx$ using the substitution $u = x - 3$. **3**

(b) Use mathematical induction to prove that **3**

$$\left(1 \times 2\right) + \left(2 \times 2^2\right) + \left(3 \times 2^3\right) + \cdots + \left(n \times 2^n\right) = 2 + (n-1)2^{n+1}$$

for all integers $n \geq 1$.

(c) A gym has 9 pieces of equipment: 5 treadmills and 4 rowing machines.

On average, each treadmill is used 65% of the time and each rowing machine is used 40% of the time.

(i) Find an expression for the probability that, at a particular time, exactly 3 of the 5 treadmills are in use. **2**

(ii) Find an expression for the probability that, at a particular time, exactly 3 of the 5 treadmills are in use and no rowing machines are in use. **1**

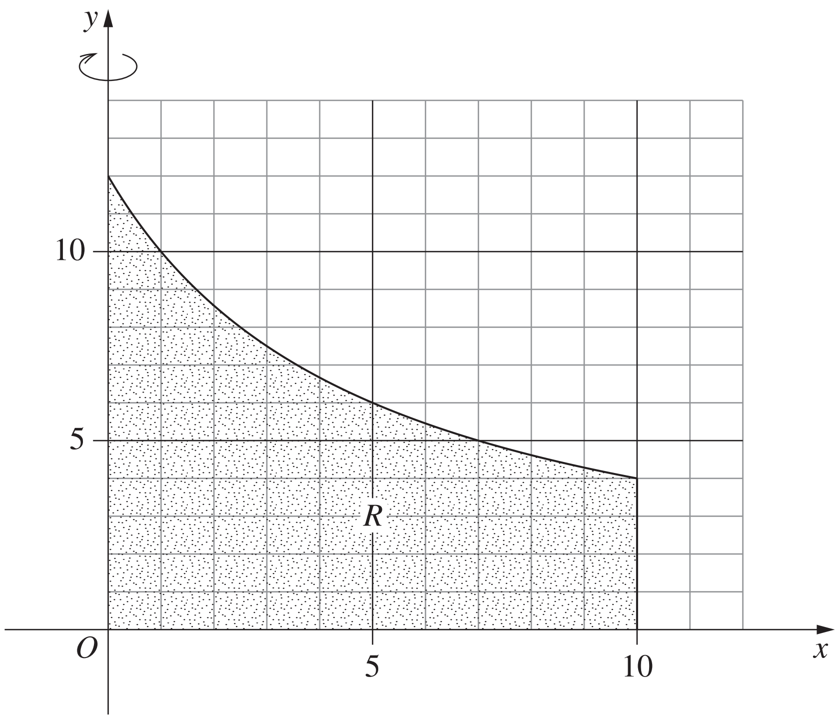
(d) It is known that ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$ for all integers such that $1 \leq r \leq n-1$. (Do NOT prove this.) **2**

Find ONE possible set of values for p and q such that

$${}^{2022}C_{80} + {}^{2022}C_{81} + {}^{2023}C_{1943} = {}^{p}C_{q}.$$

Question 12 continues on page 11

(e) The region, R, bounded by the hyperbola $y = \dfrac{60}{x + 5}$, the line $x = 10$ and the coordinate axes is shown. **4**



Find the volume of the solid of revolution formed when the region R is rotated about the y-axis. Leave your answer in exact form.

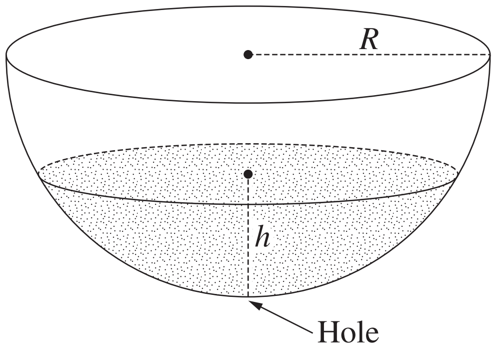
End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) A hemispherical water tank has radius R cm. The tank has a hole at the bottom which allows water to drain out.

Initially the tank is empty. Water is poured into the tank at a constant rate of $2kR$ cm^3 s^{-1}, where k is a positive constant.

After t seconds, the height of the water in the tank is h cm, as shown in the diagram, and the volume of water in the tank is V cm^3.



It is known that $V = \pi\left(Rh^2 - \dfrac{h^3}{3}\right)$.   (Do NOT prove this.)

While water flows into the tank and also drains out of the bottom, the rate of change of the volume of water in the tank is given by $\dfrac{dV}{dt} = k(2R - h)$.

(i) Show that $\dfrac{dh}{dt} = \dfrac{k}{\pi h}$. **2**

(ii) Show that the tank is full of water after $T = \dfrac{\pi R^2}{2k}$ seconds. **2**

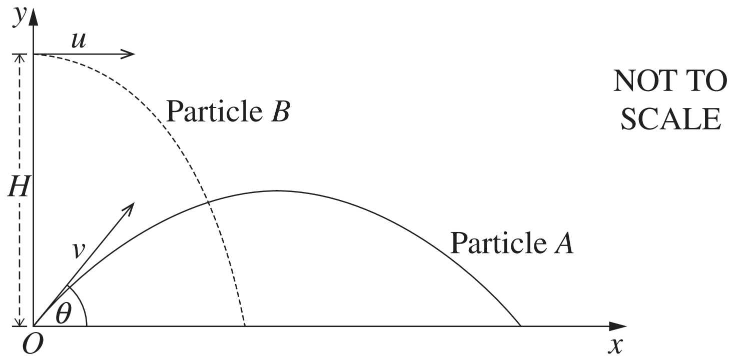
(iii) The instant the tank is full, water stops flowing into the tank, but it continues to drain out of the hole at the bottom as before. **3**

Show that the tank takes 3 times as long to empty as it did to fill.

Question 13 continues on page 13

Question 13 (continued)

(b) Particle A is projected from the origin with initial speed $v \text{ m s}^{-1}$ at an angle θ with the horizontal plane. At the same time, particle B is projected horizontally with initial speed $u \text{ m s}^{-1}$ from a point that is H metres above the origin, as shown in the diagram.



The position vector of particle A, t seconds after it is projected, is given by

$$\mathbf{r}_A(t) = \begin{pmatrix} vt\cos\theta \\ vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

The position vector of particle B, t seconds after it is projected, is given by

$$\mathbf{r}_B(t) = \begin{pmatrix} ut \\ H - \frac{1}{2}gt^2 \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

The angle θ is chosen so that $\tan\theta = 2$.

The two particles collide.

(i) By first showing that $\cos\theta = \dfrac{1}{\sqrt{5}}$, verify that $v = \sqrt{5}u$. **2**

(ii) Show that the particles collide at time $T = \dfrac{H}{2u}$. **1**

When the particles collide, their velocity vectors are perpendicular.

(iii) Show that $H = \dfrac{2u^2}{g}$. **3**

(iv) Prior to the collision, the trajectory of particle A was a parabola. (Do NOT prove this.) **2**

Find the height of the vertex of that parabola above the horizontal plane. Give your answer in terms of H.

End of Question 13

Question 14 (14 marks) Use the Question 14 Writing Booklet

(a) Let $f(x) = 2x + \ln x$, for $x > 0$.

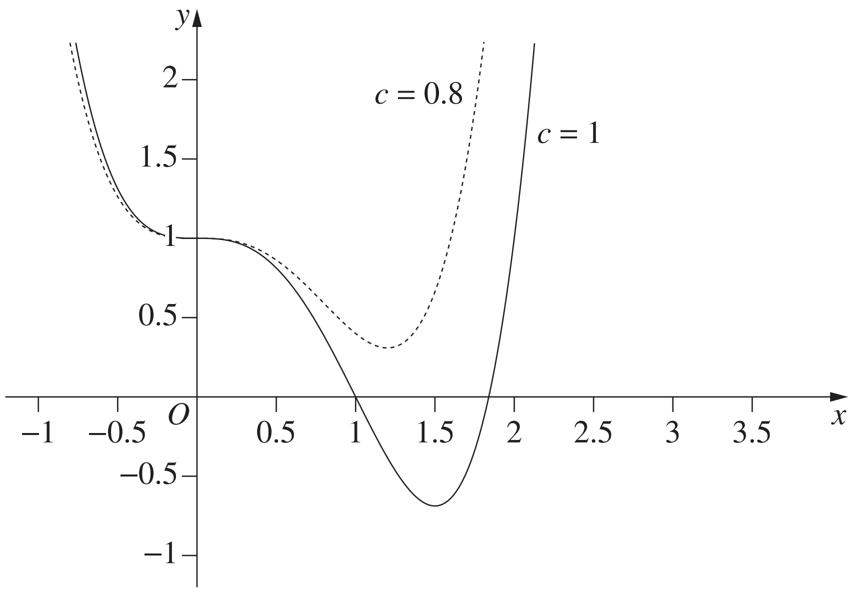
(i) Explain why the inverse of $f(x)$ is a function. **1**

(ii) Let $g(x) = f^{-1}(x)$. By considering the value of $f(1)$, or otherwise, evaluate $g'(2)$. **2**

(b) Consider the hyperbola $y = \dfrac{1}{x}$ and the circle $(x - c)^2 + y^2 = c^2$, where c is a constant.

(i) Show that the x-coordinates of any points of intersection of the hyperbola and circle are zeros of the polynomial $P(x) = x^4 - 2cx^3 + 1$. **1**

(ii) The graphs of $y = x^4 - 2cx^3 + 1$ for $c = 0.8$ and $c = 1$ are shown. **3**



By considering the given graphs, or otherwise, find the exact value of $c > 0$ such that the hyperbola $y = \dfrac{1}{x}$ and the circle $(x - c)^2 + y^2 = c^2$ intersect at only one point.

Question 14 continues on page 15

Question 14 (continued)

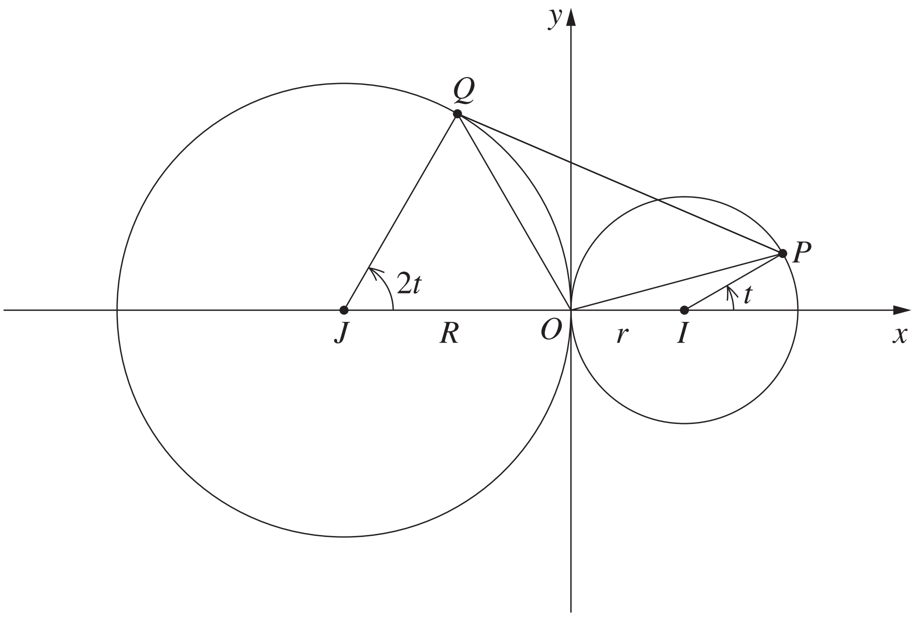
(c) (i) Given a non-zero vector $\begin{pmatrix} p \\ q \end{pmatrix}$, it is known that the vector $\begin{pmatrix} q \\ -p \end{pmatrix}$ is perpendicular to $\begin{pmatrix} p \\ q \end{pmatrix}$ and has the same magnitude. (Do NOT prove this.) **3**

Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, respectively.

Using the given information, or otherwise, show that the area of triangle OAB is $\frac{1}{2}\left|a_1b_2 - a_2b_1\right|$.

(ii) The point P lies on the circle centred at $I(r, 0)$ with radius $r > 0$, such that $\overrightarrow{IP}$ makes an angle of t to the horizontal. **4**

The point Q lies on the circle centred at $J(-R, 0)$ with radius $R > 0$, such that $\overrightarrow{JQ}$ makes an angle of $2t$ to the horizontal.



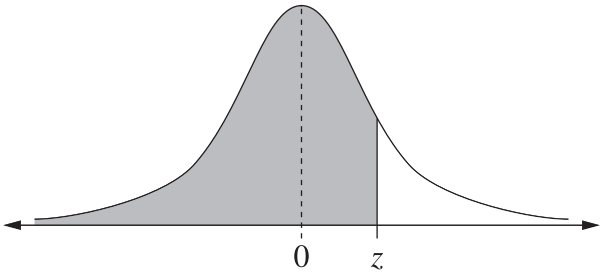
Note that $\overrightarrow{OP} = \overrightarrow{OI} + \overrightarrow{IP}$ and $\overrightarrow{OQ} = \overrightarrow{OJ} + \overrightarrow{JQ}$.

Using part (i), or otherwise, find the values of t, where $-\pi \le t \le \pi$, that maximise the area of triangle OPQ.

End of paper

Use the information below to answer Question 11 (f) (ii).

Table of values $P(Z \le z)$ for the normal distribution $N(0,1)$



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995

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2023 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

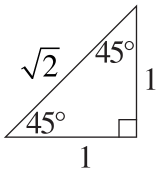
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

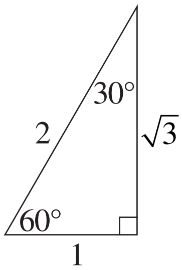
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

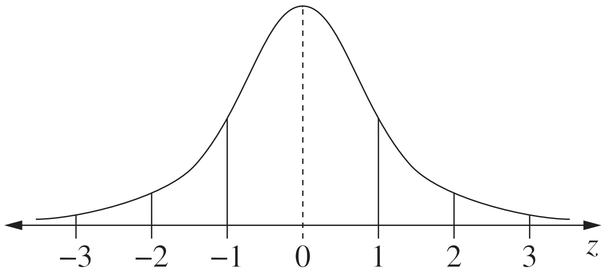
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X-\mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1-p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

2023 HSC Mathematics Extension 1 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	D
2	A
3	C
4	C
5	B
6	C
7	B
8	A
9	D
10	B

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Obtains *t* in terms of *x*, or equivalent merit	1

Sample answer:

$$x = 1 + 3t$$

$$\therefore \quad t = \frac{x-1}{3}$$

$$\therefore \quad y = 4\left(\frac{x-1}{3}\right)$$

$$y = \frac{4}{3}x - \frac{4}{3}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Divides 10! by a relevant number, or equivalent merit	1

Sample answer:

10 letters, 3 Os and 2 Ns

$$\therefore \quad \frac{10!}{3!2!} = 302\,400$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Obtains two equations for *a* and *b*, or equivalent merit	2
• Observes that $P(-1) = 0$ or $P(2) = -18$, or equivalent merit	1

Sample answer:

$P(-1) = 0$ by factor theorem

$$\begin{aligned}(-1)^3 + a(-1)^2 + b(-1) - 12 &= 0\\ -1 + a - b - 12 &= 0\\ a - b &= 13 \qquad (1)\end{aligned}$$

$P(2) = -18$ by remainder theorem

$$\begin{aligned}2^3 + a(2)^2 + b(2) - 12 &= -18\\ 8 + 4a + 2b - 12 &= -18\\ 4a + 2b &= -14\\ 2a + b &= -7 \qquad (2)\end{aligned}$$

(1) + (2): $\quad 3a = 6$
$\qquad\qquad a = 2$

Using this in (1) $\quad 2 - b = 13$
$\qquad\qquad\qquad\qquad b = -11$

Question 11 (d)

Criteria	Marks
• Provides correct solution	2
• Attempts to rewrite the integrand in a suitable form, or equivalent merit	1

Sample answer:

$$\begin{aligned}\int \frac{1}{\sqrt{4-9x^2}}\,dx &= \frac{1}{3}\int \frac{3}{\sqrt{2^2-(3x)^2}}\,dx\\ &= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C\end{aligned}$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	3
• Obtains an equation or graph involving a single trigonometric function OR • Obtains a quadratic equation in *t*, and attempts to solve it OR • equivalent merit	2
• Uses the *t*-substitution, attempts to write $\cos\theta + \sin\theta$ as a single trigonometric function, or equivalent merit	1

Sample answer:

Let $t = \tan\dfrac{\theta}{2}$

$$\therefore \quad \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = 1$$

$$1 - t^2 + 2t = 1 + t^2$$

$$2t^2 - 2t = 0$$

$$2t(t-1) = 0$$

$\therefore \quad t = 0$ or $t = 1$

$\tan\dfrac{\theta}{2} = 0$

$\dfrac{\theta}{2} = 0,\ \pi,\ \cancel{2\pi},\ \ldots$

$\theta = 0,\ 2\pi$

$\tan\dfrac{\theta}{2} = 1$

$\dfrac{\theta}{2} = \dfrac{\pi}{4},\ \cancel{\dfrac{5\pi}{4}},\ \ldots$

$\theta = \dfrac{\pi}{2}$

$$\therefore \quad \theta = 0,\ \frac{\pi}{2},\ 2\pi$$

Question 11 (e) (continued)

Alternative solution

$$\cos\theta + \sin\theta = 1 \qquad \theta \in [0, 2\pi]$$

Let $\cos\theta + \sin\theta = R\sin(\theta + \alpha)$

$$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

$$\therefore \quad \left.\begin{aligned} R\sin\alpha &= 1 \\ R\cos\alpha &= 1 \end{aligned}\right\}$$

$$\therefore \quad R = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\tan\alpha = 1 \qquad \therefore \quad \alpha = \frac{\pi}{4}$$

$$\therefore \quad \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = 1$$

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\theta + \frac{\pi}{4} = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{9\pi}{4},\ \ldots$$

$$\theta = 0,\ \frac{\pi}{2},\ 2\pi$$

Question 11 (f) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains p = 0.3, or equivalent merit	1

Sample answer:

$P(\text{born overseas}) = 0.3$

Let $p = 0.3 \quad \therefore\ 1 - p = 0.7$

$n = 900$

$\therefore$ Distribution of $\hat{p}$ is approximately normal with mean $\mu = 0.3$ and

$$\begin{aligned}\sigma^2 &= \frac{0.3 \times 0.7}{900} \\ &= \frac{0.21}{900} \\ &= \frac{7}{30\,000}\end{aligned}$$

Question 11 (f) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains the correct z-score, or equivalent merit	1

Sample answer:

$$\begin{aligned}P(\hat{p} \le 0.31) &\approx P\left(Z \le \frac{0.31 - 0.3}{\sqrt{\frac{7}{30\,000}}}\right) \\ &= P(X \le 0.6546\ldots) \\ &\approx P(Z \le 0.65) \\ &= 0.7422 \qquad \text{from table} \\ &= 0.74 \qquad \text{2 decimal places}\end{aligned}$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	3
• Obtains correct anti-derivative in terms of *u*, or equivalent merit	2
• Obtains correct integrand in terms of *u*, or equivalent merit	1

Sample answer:

$$\int_3^4 (x+2)\sqrt{x-3}\,dx \qquad u = x-3 \quad \begin{cases} x=4 \to u=1 \\ x=3 \to u=0 \end{cases}$$

$$du = dx$$

$$= \int_0^1 (u+5)\sqrt{u}\,du$$

$$= \int_0^1 u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\,du$$

$$= \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{5\times 2}{3}u^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{5} + \frac{10}{3}$$

$$= \frac{56}{15}$$

Question 12 (b)

Criteria	Marks
• Provides correct solution	3
• Establishes the inductive step, or equivalent merit	2
• Establishes the base case, or equivalent merit	1

Sample answer:

When $n = 1$

$$\text{LHS} = 1 \times 2 = 2$$

$$\text{RHS} = 2 + (1-1)2^{1+1} = 2$$

$$\text{LHS} = \text{RHS}$$

$\therefore$ True when $n = 1$

Assume true when $n = k$

ie $\quad (1 \times 2) + \left(2 \times 2^2\right) + \cdots + \left(k \times 2^k\right) = 2 + (k-1)2^{k+1}$

Consider $n = k + 1$

$$\begin{aligned} \text{RHS} &= 2 + (k+1-1)2^{k+1+1} \\ &= 2 + k \,.\, 2^{k+2} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= (1 \times 2) + \left(2 \times 2^2\right) + \cdots + \left(k \times 2^k\right) + \left((k+1) \times 2^{k+1}\right) \\ &= 2 + (k-1)2^{k+1} + (k+1)2^{k+1} \qquad \text{by assumption} \\ &= 2 + 2^{k+1}(k-1+k+1) \\ &= 2 + 2^{k+1}(2k) \\ &= 2 + k \,.\, 2^{k+2} \\ &= \text{RHS} \end{aligned}$$

Therefore, by mathematical induction, it is true for all integers $n \geq 1$.

Question 12 (c) (i)

Criteria	Marks
• Provides correct answer	2
• Demonstrates some understanding of binomial probability, or equivalent merit	1

Sample answer:

$${}^5C_3(0.65)^3(0.35)^2$$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$${}^5C_3(0.65)^3(0.35)^2 \times (0.6)^4$$

Question 12 (d)

Criteria	Marks
• Provides correct solution	2
• Uses given result to combine the first two terms on the left-hand side, or equivalent merit	1

Sample answer:

$${}^{2022}C_{80} + {}^{2022}C_{81} + {}^{2023}C_{1943}$$

$$= {}^{2023}C_{81} + {}^{2023}C_{1943}$$

$$= {}^{2023}C_{81} + {}^{2023}C_{80}$$

$$= {}^{2024}C_{81}$$

So $p = 2024$ and $q = 81$ is a possible solution

Question 12 (e)

Criteria	Marks
• Provides correct solution	4
• Obtains the volume formed by revolving the hyperbola about the *y*-axis, or equivalent merit	3
• Obtains correct integral expression for the volume formed by revolving the hyperbola about the *y*-axis, or equivalent merit	2
• Writes *x* in terms of y, or recognises that the volume is the sum of two simpler volumes, or equivalent merit	1

Sample answer:

The total volume is the sum of the volume when the given hyperbola is revolved about the y-axis, V_1, and the volume of a cylinder, V_2.

$$y = \frac{60}{x+5}$$
$$x + 5 = \frac{60}{y}$$
$$x = \frac{60}{y} - 5$$

$$V_1 = \pi \int_4^{12} x^2\, dy$$
$$= \pi \int_4^{12} \left(\frac{60}{y} - 5\right)^2 dy$$
$$= \pi \int_4^{12} \frac{3600}{y^2} - \frac{600}{y} + 25\, dy$$
$$= \pi \left[-\frac{3600}{y} - 600 \ln y + 25y \right]_4^{12}$$
$$= \pi \left[\left(-\frac{3600}{12} - 600 \ln 12 + 25 \times 12 \right) - \left(-\frac{3600}{4} - 600 \ln 4 + 25 \times 4 \right) \right]$$
$$= \pi(-600 \ln 12 + 800 + 600 \ln 4)$$
$$= \pi(800 - 600 \ln 3)$$

$$V_2 = \pi \times 10^2 \times 4$$
$$= 400\pi$$

Total = $V_1 + V_2$

$\therefore$ Total = $\pi(1200 - 600 \ln 3)$ units^3

Question 13 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains $\frac{dV}{dh}$, or equivalent merit	1

Sample answer:

$$V = \pi\left(Rh^2 - \frac{h^3}{3}\right)$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

So $$\frac{dV}{dt} = \pi\left(2Rh - h^2\right)\frac{dh}{dt}$$

$$= \pi h(2R - h)\frac{dh}{dt}$$

But we are given

$$\frac{dV}{dt} = k(2R - h)$$

So $$\pi h(2R - h)\frac{dh}{dt} = k(2R - h)$$

$$\pi h\frac{dh}{dt} = k \qquad \text{as } 0 \le h \le R \text{ so } 2R - h \ne 0$$

$$\frac{dh}{dt} = \frac{k}{\pi h}$$

Question 13 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Separates the variables from the differential equation in part (i), or equivalent merit	1

Sample answer:

$$\frac{dh}{dt} = \frac{k}{\pi h}$$

When $t = 0,\ h = 0$
$t = T,\ h = R$

So $$\int_0^R \pi h\,dh = \int_0^T k\,dt$$

$$\left[\frac{\pi h^2}{2}\right]_0^R = \left[kt\right]_0^T$$

$$\frac{\pi R^2}{2} - 0 = kT - 0$$

$$T = \frac{\pi R^2}{2k}$$

Question 13 (a) (iii)

Criteria	Marks
• Provides correct solution	3
• Finds correct expression for $\frac{dh}{dt}$, or equivalent merit	2
• Models the situation to find the new differential equation $\frac{dV}{dt} = -kh$, or equivalent merit	1

Sample answer:

When the tank is full the inflow of water stops. This means that the rate of change in volume is only dependent on *h*.

ie $$-kh = \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

So $$-kh = \pi h(2R - h)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-k}{\pi(2R - h)}$$

When $t = 0, \quad h = R$
$t = T_E, \quad h = 0$

$$\int_R^0 \pi(2R - h)\,dh = \int_0^{T_E} -k\,dt$$

$$\left[\pi\left(2Rh - \frac{h^2}{2}\right)\right]_R^0 = -kT_E$$

$$-\pi\left(2R^2 - \frac{R^2}{2}\right)^2 = -kT_E$$

$$\pi\frac{3R^2}{2} = kT_E$$

$$T_E = 3\frac{\pi R^2}{2k} = 3T$$

So the tank takes 3 times as long to empty.

Question 13 (b) (i)

Criteria	Marks
• Provides correct proof	2
• Equates *x*-coordinates, or equivalent merit	1

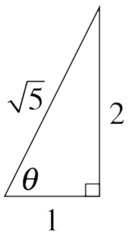
Sample answer:

The particles collide when their x-coordinates are the same at the same time.

$$vt\cos\theta = ut$$

So $\quad u = v\cos\theta$

θ in 1st quad. and $\tan\theta = 2$



So $\quad \cos\theta = \dfrac{1}{\sqrt{5}} \quad$ and $\quad \sin\theta = \dfrac{2}{\sqrt{5}}$

$$u = v\cos\theta$$

$$u = \frac{v}{\sqrt{5}}$$

$$v = \sqrt{5}\,u$$

Question 13 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Particles collide at time T so y-coordinates are the same.

$$vT\sin\theta - \frac{1}{2}gT^2 = H - \frac{1}{2}gT^2$$

$$\sqrt{5}uT\frac{2}{\sqrt{5}} = H$$

$$T = \frac{H}{2u}$$

Question 13 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Evaluates the dot product of the velocity vectors and equates to 0, or equivalent merit	2
• Observes that the dot product of the velocity vectors is 0, or equivalent merit	1

Sample answer:

At $t = T$ the vectors $\underset{\sim}{v}_A$ and $\underset{\sim}{v}_B$ are perpendicular, so $\underset{\sim}{v}_A \cdot \underset{\sim}{v}_B = 0$.

$$\begin{pmatrix} v\cos\theta \\ v\sin\theta - gT \end{pmatrix} \cdot \begin{pmatrix} u \\ -gT \end{pmatrix} = 0$$

$$uv\cos\theta - gT(v\sin\theta - gT) = 0$$

$$u\sqrt{5}u\frac{1}{\sqrt{5}} - \sqrt{5}ugT\frac{2}{\sqrt{5}} + g^2T^2 = 0$$

$$u^2 - 2ugT + g^2T^2 = 0$$

$$(u - gT)^2 = 0$$

$$u = gT = \frac{gH}{2u}$$

$$H = \frac{2u^2}{g}$$

Question 13 (b) (iv)

Criteria	Marks
• Provides correct solution	2
• Identifies that $\dot{y} = 0$ for particle *A*, or equivalent merit	1

Sample answer:

Vertex of $y_A(t) = vt\sin\theta - \frac{1}{2}gt^2$ occurs when $\dot{y} = 0$

$$t = \frac{-v\sin\theta}{2\left(-\frac{1}{2}g\right)} \qquad \text{[axis of symmetry]}$$

$$= \frac{\sqrt{5}u\frac{2}{\sqrt{5}}}{g}$$

$$= \frac{2u}{g}$$

Height of particle *A* at that time is

$$y_A = vt\sin\theta - \frac{1}{2}gt^2$$

$$= \sqrt{5}u \times \frac{2u}{g} \times \frac{2}{\sqrt{5}} - \frac{g}{2} \times \frac{4u^2}{g^2}$$

$$= \frac{4u^2}{g} - \frac{2u^2}{g}$$

$$= \frac{2u^2}{g}$$

$$= H$$

Question 14 (a) (i)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

$$f'(x) = 2 + \frac{1}{x} > 0 \text{ for } x > 0$$

f is always increasing.

Therefore since f is one-to-one for $x > 0$, f has an inverse function.

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds $g'(x)$ or $f'(1)$, or equivalent merit	1

Sample answer:

$f(1) = 2$ therefore $g(2) = 1$

$$g'(x) = \frac{1}{f'(g(x))}$$

So $\quad g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{3}$

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

The points of intersection of the hyperbola and the circle satisfy

$$\begin{cases} y = \dfrac{1}{x} & (1) \\ (x-c)^2 + y^2 = c^2 & (2) \end{cases}$$

Substitute (1) into (2)

$$(x-c)^2 + \frac{1}{x^2} = c^2 \qquad \times x^2$$

$$x^2\left(x^2 - 2cx + c^2\right) + 1 = c^2x^2$$

$$x^4 - 2cx^3 + 1 = 0$$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds the *x*-coordinate of the double root in terms of *c*, or equivalent merit	2
• Recognises that the circle and hyperbola are tangent when the polynomial has a double root and so is tangent to the *x*-axis, or equivalent merit	1

Sample answer:

Let $f(x) = x^4 - 2cx^3 + 1$

The hyperbola and the circle have exactly one point of intersection if there exists a value of x such that $f(x) = 0$ and $f'(x) = 0$.

$$f'(x) = 4x^3 - 6cx^2$$

$$f'(x) = 2x^2(2x - 3c)$$

$$f'(x) = 0 \Leftrightarrow x = 0 \quad \text{or} \quad x = \frac{3c}{2}$$

Note that $f(0) \neq 0$

$$f\left(\frac{3c}{2}\right) = 0 \quad \Leftrightarrow \left(\frac{3c}{2}\right)^4 - 2c\left(\frac{3c}{2}\right)^3 + 1 = 0$$

$\times 2^4$

$$\Leftrightarrow 3^4c^4 - 3^3 \times 2^2c^4 + 2^4 = 0$$

$$\Leftrightarrow -27c^4 = -16$$

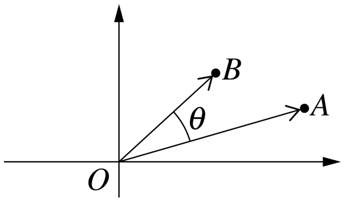
$$\Leftrightarrow c = \frac{2}{\sqrt[4]{27}}$$

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Recognise that the dot product of $\overline{OB}$ with the vector perpendicular to $\overline{OA}$, using the given information, is related to the area, or vice versa, or equivalent merit	2
• Attempts to use a vector perpendicular to $\overline{OA}$ or $\overline{OB}$, or equivalent merit	1

Sample answer:

Let θ be the angle between $\overline{OA}$ and $\overline{OB}$



Area of triangle = $\frac{1}{2}ab\sin\theta$, where $a = \left|\overline{OA}\right|$ and $b = \left|\overline{OB}\right|$

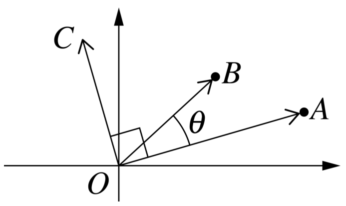
Let $\overrightarrow{OC} = \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}$ be the vector perpendicular to $\overline{OA}$ using the given information.

We are told that $\overline{OC}$ is perpendicular to $\overline{OA}$ and with the same magnitude.

$$\overrightarrow{OB} \cdot \overrightarrow{OC} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} = a_2 b_1 - a_1 b_2 \qquad \text{———— (1)}$$

We can also express $\overline{OB} \cdot \overline{OC}$ in terms of θ

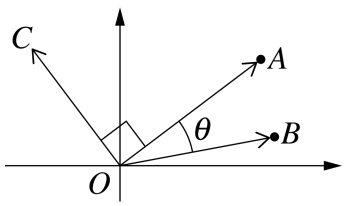
$$\begin{aligned} \overline{OB} \cdot \overline{OC} &= \left|\overline{OB}\right| \cdot \left|\overline{OC}\right| \cos(\angle BOC) \\ &= b \times a \times \cos(\angle BOC) \end{aligned}$$



Case 1:

$\angle BOC = \frac{\pi}{2} - \theta$

so $\overline{OB} \cdot \overline{OC} = ab\sin\theta$



Case 2:

$\angle BOC = \frac{\pi}{2} + \theta$

so $\overline{OB} \cdot \overline{OC} = -ab\sin\theta$

In both cases $\left|\overline{OB} \cdot \overline{OC}\right| = \left|ab\sin\theta\right|$ = 2 × (Area of triangle)

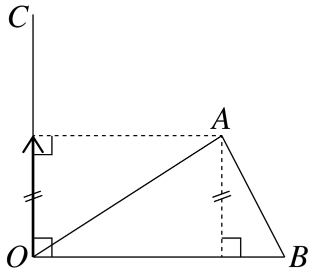
Therefore, area of triangle = $\frac{1}{2}\left|\overrightarrow{OB} \cdot \overrightarrow{OC}\right| = \frac{1}{2}\left|a_2 b_1 - a_1 b_2\right|$ [From (1)]

$$= \frac{1}{2}\left| a_1b_2 - a_2b_1 \right|$$

Question 14 (c) (i) (continued)

Alternative solution

Projection of $\left|\overline{AC}\right|$ onto $\left|\overline{OC}\right|$



Area triangle $= \frac{1}{2}\left|\overrightarrow{OB}\right| \times$ perpendicular height of triangle OAB

$= \frac{1}{2}\left|\overrightarrow{OB}\right| \times \left|\text{projection of } \overrightarrow{OA} \text{ onto } \overrightarrow{OC}\right|$

where $\overline{OC} \perp \overline{OB}$ and $\left|\overline{OC}\right| = \left|\overline{OB}\right|$

$$\overrightarrow{OC} = \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix}$$

$$\therefore \quad \left|\overrightarrow{OA} \text{ projected onto } \overrightarrow{OC}\right| = \frac{\left|\overrightarrow{OA} \cdot \overrightarrow{OC}\right|}{\left|\overrightarrow{OC}\right|}$$

$$= \frac{\left|a_1b_2 - a_2b_1\right|}{\left|\overrightarrow{OB}\right|}$$

$$\therefore \quad \text{Area of triangle } = \frac{1}{2}\left|\overrightarrow{OB}\right| \times \frac{\left|a_1b_2 - a_2b_1\right|}{\left|\overrightarrow{OB}\right|}$$

$$= \frac{1}{2}\left| a_1b_2 - a_2b_1 \right|$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	4
• Finds the values of *t* at the stationary points of the area, or equivalent merit	3
• Obtains the area of triangle *OPQ* in terms of *t*, or equivalent merit	2
• Obtains the components of $\overline{OP}$ or $\overline{OQ}$ in terms of *t*, or equivalent merit	1

Sample answer:

$$\overrightarrow{OP} = \overrightarrow{OI} + \overrightarrow{IP} = \begin{pmatrix} r + r\cos t \\ r\sin t \end{pmatrix}$$

$$\overrightarrow{OQ} = \overrightarrow{OJ} + \overrightarrow{JQ} = \begin{pmatrix} -R + R\cos 2t \\ R\sin 2t \end{pmatrix}$$

Using part (i), we get that the area of triangle *OPQ* is

$$\begin{aligned}
\text{Area of triangle } &= \frac{1}{2}\left|(r + r\cos t)(R\sin 2t) - (r\sin t)(-R + R\cos 2t)\right| \\
&= \frac{1}{2}\left|Rr(1+\cos t)\sin 2t - Rr\sin t(-1+\cos 2t)\right| \\
&= \frac{1}{2}Rr\left|(\sin 2t\cos t - \sin t\cos 2t) + \sin 2t + \sin t\right| \\
&= \frac{1}{2}Rr\left|\sin t + \sin 2t + \sin t\right|
\end{aligned}$$

$$\text{Area of triangle } = \frac{1}{2}Rr\left|\sin 2t + 2\sin t\right|$$

Let $f(t) = \sin 2t + 2\sin t$, so Area triangle $= Rr\left|f(t)\right|$

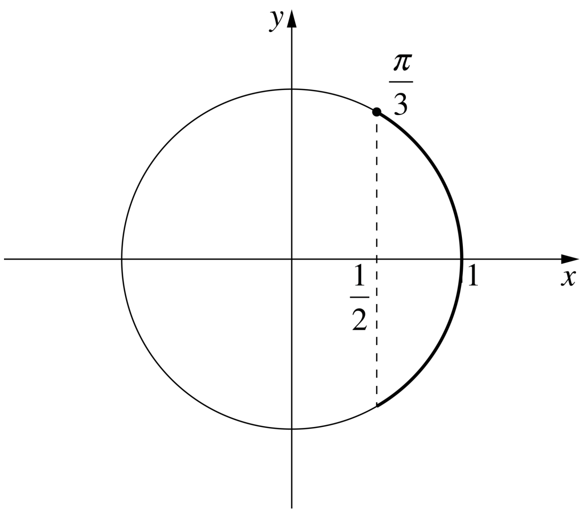
$$\begin{aligned}
f'(t) &= 2\cos 2t + 2\cos t \\
&= 2\left(2\cos^2 t + \cos t - 1\right) \\
&= 2(2\cos t - 1)(\cos t + 1)
\end{aligned}$$

For all t, $\cos t + 1 \geq 0$ therefore $f'(t)$ has the same sign as $2\cos t - 1$

$f(t)$ is odd so $|f(t)|$ is even and we only need to study $|f(t)|$ on $[0, \pi]$
where area of triangle $= \frac{1}{2}Rr|f(t)|$, so $f(t) \geq 0$ on $[0, \pi]$

$f'(t) \geq 0 \quad \Leftrightarrow \quad 2\cos t - 1 \geq 0 \quad \Leftrightarrow \quad \cos t \geq \dfrac{1}{2}$

which will happen when $-\dfrac{\pi}{3} \leq t \leq \dfrac{\pi}{3}$



t	0		$\dfrac{\pi}{3}$		π
$f'(t)$		$+$	0	$-$	
	0	$\nearrow$		$\searrow$	0

By symmetry, since $|f(t)|$ is even, we see that the area of the triangle is maximum when $t = -\dfrac{\pi}{3}$ or $t = \dfrac{\pi}{3}$

2023 HSC Mathematics Extension 1 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	ME-C1 Rates of Change	ME11-1
2	1	ME-S1 The Binomial Distribution	ME12-5
3	1	ME-C3 Applications of Calculus	ME12-1
4	1	ME-C3 Applications of Calculus	ME12-4
5	1	ME-T1 Inverse Trigonometric Functions	ME11-3
6	1	ME-V1 Introduction to Vectors	ME12-2
7	1	ME-T1 Inverse Trigonometric Functions	ME11-3
8	1	ME-F1 Further Work with Functions	ME11-1
9	1	ME-F1 Further Work with Functions	ME11-1
10	1	ME-A1 Working with Combinatorics	ME11-5

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	2	ME-F1 Further Work with Functions	ME11-2
11 (b)	2	ME-A1 Working with Combinatorics	ME11-5
11 (c)	3	ME-F2 Polynomials	ME11-2
11 (d)	2	ME-C2 Further Calculus Skills	ME12-1
11 (e)	3	ME-T3 Trigonometric Equations	ME12-3
11 (f) (i)	2	ME-S1 The Binomial Distribution	ME12-5
11 (f) (ii)	2	ME-S1 The Binomial Distribution	ME12-5
12 (a)	3	ME-C2 Further Calculus Skills	ME12-1
12 (b)	3	ME-P1 Proof by Mathematical Induction	ME12-1
12 (c) (i)	2	ME-S1 The Binomial Distribution	ME12-5
12 (c) (ii)	1	ME-S1 The Binomial Distribution	ME12-5
12 (d)	2	ME-A1 Working with Combinatorics	ME11-5
12 (e)	4	ME-C3 Applications of Calculus	ME12-4
13 (a) (i)	2	ME-C1 Rates of Change	ME11-4
13 (a) (ii)	2	ME-C3 Applications of Calculus	ME12-1, ME12-4
13 (a) (iii)	3	ME-C3 Applications of Calculus	ME12-1, ME12-7
13 (b) (i)	2	ME-V1 Introduction to Vectors	ME12-2
13 (b) (ii)	1	ME-V1 Introduction to Vectors	ME12-2
13 (b) (iii)	3	ME-V1 Introduction to Vectors	ME12-2
13 (b) (iv)	2	ME-V1 Introduction to Vectors	ME12-2

Question	Marks	Content	Syllabus outcomes
14 (a) (i)	1	ME-F1 Further Work with Functions	ME11-1
14 (a) (ii)	2	ME-C2 Further Calculus Skills	ME12-1
14 (b) (i)	1	ME-F2 Polynomials	ME11-6
14 (b) (ii)	3	ME-F2 Polynomials	ME11-2, ME11-6
14 (c) (i)	3	ME-V1 Introduction to Vectors	ME12-2
14 (c) (ii)	4	ME-V1 Introduction to Vectors ME-T3 Trigonometric Equations	ME12-3