
Student Number

Barker College

2021 YEAR 12 ASSESSMENT BLOCK

Mathematics Extension 1

Staff Involved:
- ARM • ARP* • BHC • DZP
- PDJ • PJR

Wednesday March 31

95 copies

TOPICS COVERED:		
Combinatorics	Binomial Theorem	Further Rates
Trigonometry	Further Graphs	Polynomials
Induction		

General Instructions:
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A separate reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and / or calculations
- Marks may not be awarded for careless or poorly arranged working

Total marks: 80

Section I – 8 marks (pages 2 - 5)
- Attempt Multiple Choice Questions 1 - 8
- Allow about 15 minutes for this section

Section II – 72 marks (pages 7 - 12)
- Attempt Questions 9 - 14
- Show all necessary working
- Allow about 1 hour and 45 minutes for this section

Section I

8 Marks
Attempt Questions 1-8
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-8.

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1 A polynomial equation has roots α, β and γ where

$\alpha+\beta+\gamma=-2,\ \alpha\beta+\alpha\gamma+\beta\gamma=3$ and $\alpha\beta\gamma=1$.

Which polynomial equation has roots α, β and γ?

(A) $x^3+2x^2+3x+1=0$

(B) $x^3+2x^2+3x-1=0$

(C) $x^3-2x^2+3x+1=0$

(D) $x^3-2x^2+3x-1=0$

2 A family of 8, comprising 2 parents and 6 children, are seated randomly around a circular table.

What is the probability that the 2 parents sit together?

(A) $\frac{1}{4}$

(B) $\frac{2}{7}$

(C) $\frac{1}{14}$

(D) $\frac{1}{28}$

3 A circle with centre $(3, -4)$ passes through the origin.

Which of these is the equation of the circle?

(A) $x = 3 - 5\cos\theta$ $\quad y = 4 + \sin\theta$

(B) $x = 3 - 5\cos\theta$ $\quad y = 4 - \sin\theta$

(C) $x = 5\cos\theta - 3$ $\quad y = 5\sin\theta + 4$

(D) $x = 5\cos\theta + 3$ $\quad y = 5\sin\theta - 4$

4 What is the smallest number of people that can be assembled such that it is guaranteed that at least two people share the same initials of their first and last name? (eg Jessie Cummings and Joshua Cavill)

(A) 27

(B) 53

(C) 677

(D) 1353

5 The angle θ satisfies $\sin\theta = \frac{5}{13}$ and $\frac{\pi}{2} < \theta < \pi$.
What is the value of $\sin 2\theta$?

(A) $\frac{10}{13}$

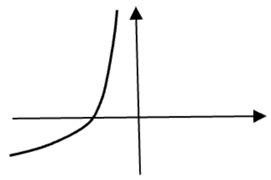
(B) $-\frac{10}{13}$

(C) $\frac{120}{169}$

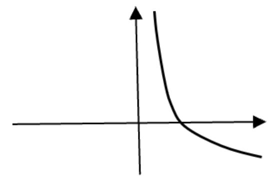
(D) $-\frac{120}{169}$

6 Which of the following graphs best represents the inverse of $y = 2^{-x}$?

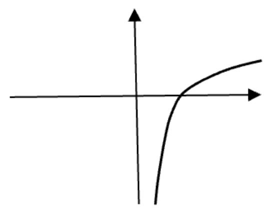
(A)



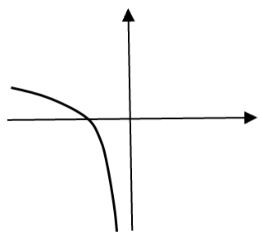
(B)



(C)



(D)



7 When expanded, which expression has a non-zero constant term?

(A) $\left(x+\dfrac{1}{x^2}\right)^7$

(B) $\left(x^2+\dfrac{1}{x^3}\right)^7$

(C) $\left(x^3+\dfrac{1}{x^4}\right)^7$

(D) $\left(x^4+\dfrac{1}{x^5}\right)^7$

8 A tanker has suffered an oil spill and a **circular** oil slick is floating on the surface of the water.

The area of the oil slick is increasing by $0.01\,\text{m}^2\,/\,\text{minute}$.

At what rate is the radius increasing when the area of the oil slick is $0.03\,\text{m}^2$?

(A) $0.006\,\text{m}\,/\,\text{minute}$

(B) $0.03\,\text{m}\,/\,\text{minute}$

(C) $0.0163\,\text{m}\,/\,\text{minute}$

(D) $0.0017\,\text{m}\,/\,\text{minute}$

Section II

72 Marks
Attempt Questions 9-14
Allow about 1 hour and 45 minutes for this section

Answer each question on the appropriate writing booklet. Extra writing booklets are available.

In Questions 9-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 9 (12 Marks) Use the Question 9 Writing Booklet

(a) Solve the inequality $\frac{4x}{x+3} \geq 1$. **3**

(b) The roots of $x^3 - x^2 - 5x + 2 = 0$ are α, β and γ.

(i) Show that $\beta + \gamma = 1 - \alpha$. **1**

(ii) Using part (i), and similar results, evaluate **3**

$$\frac{\beta+\gamma}{\alpha} + \frac{\gamma+\alpha}{\beta} + \frac{\alpha+\beta}{\gamma}.$$

(c) The polynomial $P(x) = ax^3 - 4x^2 + 3bx + 2$ has a factor of $(x - 1)$ and leaves a remainder of 24 when divided by $(x + 2)$. **2**

Find the values of a and b.

(d) Use mathematical induction to prove that $15^n + 2^{3n} - 2$ is a multiple of 7 for integers $n \geq 1$. **3**

Question 10 (12 Marks) Use the Question 10 Writing Booklet

(a) Showing all working, find the exact values of:

(α) $\tan\left(\cos^{-1}\frac{3}{7}\right)$. **1**

(β) $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$. **2**

(γ) $\cos 15^\circ$. **2**

(b) Consider the function $f(x) = \frac{1}{2}\cos^{-1}(2x)$.

(i) State the domain and range of $f(x)$. **2**

(ii) Sketch the function $f(x)$, clearly indicating all intercepts. **2**

(c) Use the double angle result to prove the following: **3**

$$\cos(4x) = 8\cos^4 x - 8\cos^2 x + 1.$$

Question 11 (12 Marks) Use the Question 11 Writing Booklet

(a) Given that $\left(\sqrt{3}-1\right)^4 = a + b\sqrt{3}$, evaluate a and b. **3**

(b) Consider the expansion of $\left(2x - \dfrac{3}{x}\right)^7$.

(i) Show that there is no term in x^4. **2**

(ii) By considering the co-efficients of x^3 and x^5 in the above expansion, find the co-efficient of x^5 in the expansion of **3**

$$(x^2+1)\left(2x - \frac{3}{x}\right)^7.$$

(c) A standard pack of 52 cards consists of 13 cards of each of the four suits: spades, hearts, clubs and diamonds.

(i) In how many ways can six cards be selected so that exactly two are hearts and four are diamonds? **1**

(ii) In how many ways can six cards be selected if at least five must be of the same suit? **3**

Question 12 (12 Marks) Use the Question 12 Writing Booklet

(a) Consider the function $f(x) = \dfrac{x+4}{x+2}$.

(i) State the equations of the horizontal and vertical asymptotes for $f(x)$. **1**

(ii) Find the inverse function $y = f^{-1}(x)$. **2**

(iii) State the domain of $y = f^{-1}(x)$. **1**

(iv) Sketch the graph of $y = f^{-1}(x)$ indicating all asymptotes and intercepts. **2**

Answer Question 12 (b) and (c) on the separate answer page supplied.
Hand it in with your booklet for Question 12.

Question 13 (12 Marks) Use the Question 13 Writing Booklet

(a) Dr Watson wishes to find the temperature of a very hot substance using his thermometer, which only measures up to 100°C. Dr Watson takes a sample of the substance and places it in a room with a surrounding air temperature of 20°C, and allows it to cool.

After 6 minutes the temperature of the substance is 80°C, and after a further 2 minutes it is 50°C. If T is the temperature of the substance after t minutes, then Newton's law of cooling states that T satisfies the equation

$$\frac{dT}{dt} = k(T - A).$$

where k is a negative constant and A the surrounding air temperature.

(i) Verify that $T = A + Be^{kt}$ satisfies the above equation. **1**

(ii) Show that $k = -\frac{\ln 2}{2}$, and hence find the value of B. **3**

(iii) Hence find the initial temperature of the substance. **1**

(b) Solve $(n+1)! = 240(n-1)!$ **2**

(c) Consider the set of integers from $1-15$ inclusive. How many integers must be selected to guarantee at least 2 pairs of integers have a sum of 19? **2**

(d) A 4 metre ladder rests on level ground against a vertical wall. The foot of the ladder is slipping away from the wall at the constant rate of 0.6m/min. Show that the top of the ladder is slipping down the wall at $\frac{\sqrt{3}}{5}$ m / min at the instant when the foot of the ladder is 2m away from the wall. **3**

Question 14 (12 Marks) Use the Question 14 Writing Booklet

(a) By considering the expansion of $(1+x)^{10} + (1-x)^{10}$ and using an appropriate substitution prove the following: **3**

$$1 + {}^{10}C_2(3)^2 + {}^{10}C_4(3)^4 + {}^{10}C_6(3)^6 + {}^{10}C_8(3)^8 + 3^{10} = 2^9(2^{10}+1).$$

(b) Use the principle of mathematical induction to show that for all positive integers n **4**

$$\frac{5}{6} + \frac{1}{4} + ... + \frac{n+4}{n(n+1)(n+2)} = \frac{3}{2} - \frac{n+3}{(n+1)(n+2)}.$$

(c) In a colony of 200 birds the number N infected with a disease at time t is given by

$$N = \frac{200}{1 + ke^{-200t}}$$

where k is a constant and t is in years.

(i) Show that all birds will eventually become infected. **1**

(ii) Show that $\frac{dN}{dt} = N(200 - N)$. **3**

(iii) By considering the graph of $\frac{dN}{dt}$, or otherwise, determine the infected population when the rate of infection is greatest. **1**

END OF PAPER

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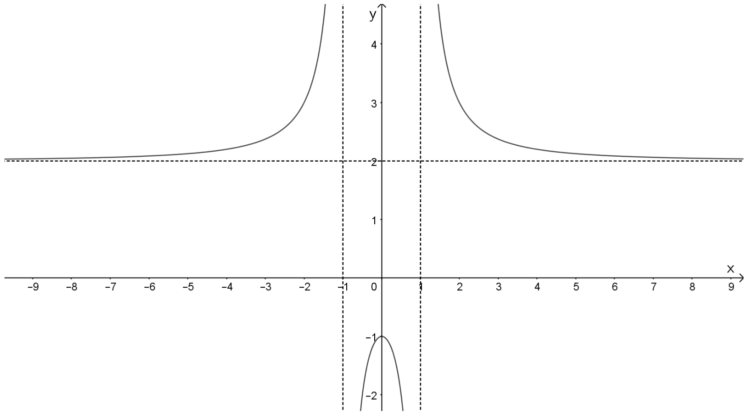
Additional Answer page for Question 12 (b) and (c)

Please place inside your Question 12 booklet.

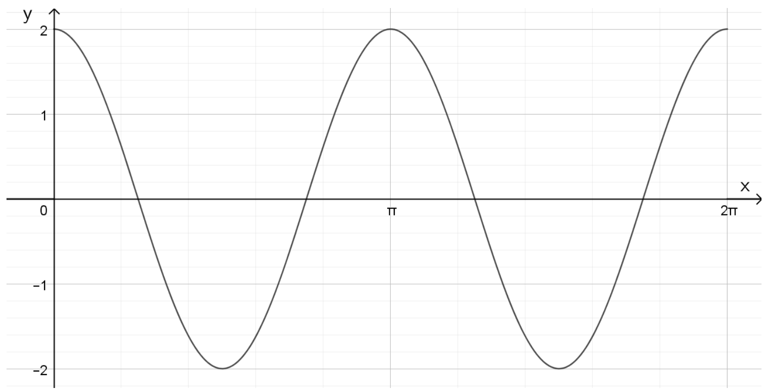
(b) The diagram below shows the graph of $y = f(x)$. **3**

Sketch on this answer page $y = \dfrac{1}{f(x)}$ showing all asymptotes and intercepts.



(c) Below is a graph of $f(x) = 2\cos 2x$ for $0 \le x \le 2\pi$. **3**

Sketch on this answer page $y^2 = f(x)$ for $0 \le x \le 2\pi$ indicating all intercepts.



Solutions

1.

$-\frac{b}{a} = -2$

but $a=1$ $\therefore b=2$

$\frac{c}{a} = 3$

but $a=1$ $\therefore c=3$

$-\frac{d}{a} = 1$

but $a=1$ $\therefore d=-1$

$\therefore x^3+2x^2+3x-1=0$

(B)

2.

Number of arrangements without restriction $= (8-1)!$

$= 7!$

Number of ways with restriction:

Group 2 parents to be 1 object.

$(7-1)! \times 2 = 6! \times 2$

$\therefore$ Prob $= \frac{6! \times 2}{7!} = \frac{2}{7}$

So (B)

3.

Cartesian Equation of circle:

$(x-3)^2+(y+4)^2=r^2$

Passes through (0,0)

$(0-3)^2+(0+4)^2=r^2$

$25 = r^2$

$\therefore r = 5$

$(x-3)^2+(y+4)^2=25$

$(x-3)^2+(y+4)^2=25(\sin^2\theta+\cos^2\theta)$

Parametric equations (D) satisfy this.

4. The number of possible initial combinations

$= 26 \times 26$

$= 676$

To guarantee at least 2 people have the same combination of letters, need $676+1 = 677$ people.

So (C)

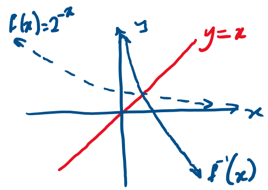
5. θ is in 2^{nd} quadrant

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right)$

$= \frac{-120}{169}$ So (D)

6.



So (B)

7. $\left(x+\frac{1}{x^3}\right)^7 \rightarrow$ general term is $\binom{7}{r}x^{7-r}\left(\frac{1}{x^3}\right)^r$

$= \binom{7}{r}x^{7-3r}$

For integer r, $7-3r \neq 0$

$\left(x^2+\frac{1}{x^3}\right)^7 \rightarrow$ general term is $\binom{7}{r}(x^2)^{7-r}\left(\frac{1}{x^3}\right)^r$

$= \binom{7}{r}x^{14-5r}$

For integer r, $14-5r \neq 0$

$\left(x^3+\frac{1}{x^4}\right)^7 \rightarrow$ general term $\binom{7}{r}(x^3)^{7-r}\left(\frac{1}{x^4}\right)^r$

$= \binom{7}{r}x^{21-7r}$

$21-7r=0$ when $r=3$

So (C) has a term independent of x.

8. want $\frac{dr}{dt}$ when $A=0.03$

$$\frac{dr}{dt}=\frac{dr}{dA}\times\frac{dA}{dt}$$

$\frac{dA}{dt}=0.01 \qquad A=\pi r^2$

$\frac{dA}{dr}=2\pi r$

$\frac{dr}{dA}=\frac{1}{2\pi r}$

when $0.03=\pi r^2$

$r^2=\frac{0.03}{\pi}$, $r=\sqrt{\frac{0.03}{\pi}}$

$\therefore \frac{dr}{dt}=\frac{1}{2\pi\sqrt{\frac{0.03}{\pi}}}\times 0.01 \approx 0.0163$

$\therefore$ (C)

9. a)

$\frac{4x}{(x+3)}\times(x+3)^2 \geq (x+3)^2$

$4x(x+3)\geq(x+3)^2$

$4x(x+3)-(x+3)^2\geq 0$

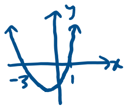
$(x+3)[4x-(x+3)]\geq 0$

$(x+3)(3x-3)\geq 0$

$x\geq 1$, $x\leq -3$

but $x\neq -3$

$\therefore x\geq 1, x<-3$



9b i)

i) Sum of roots: $\alpha+\beta+\gamma=-\frac{b}{a}$

$=1$

$\therefore \beta+\gamma=1-\alpha$

9b ii)

ii) $\frac{\beta+\gamma}{\alpha}+\frac{\gamma+\alpha}{\beta}+\frac{\alpha+\beta}{\gamma}$

$=\frac{\gamma\beta(\beta+\gamma)+\alpha\gamma(\gamma+\alpha)+\alpha\beta(\alpha+\beta)}{\alpha\beta\gamma}$

From part i), $\gamma+\beta=1-\alpha$, $\gamma+\alpha=1-\beta$, $\alpha+\beta=1-\gamma$

$=\frac{\gamma\beta(1-\alpha)+\alpha\gamma(1-\beta)+\alpha\beta(1-\gamma)}{\alpha\beta\gamma}$

$=\frac{\gamma\beta-\gamma\beta\alpha+\alpha\gamma-\alpha\gamma\beta+\alpha\beta-\alpha\beta\gamma}{\alpha\beta\gamma}$

$=\frac{\gamma\beta+\alpha\gamma+\alpha\beta-3\alpha\beta\gamma}{\alpha\beta\gamma}$

$=\frac{-5-3(-2)}{-2}$

$=-\frac{1}{2}$

9c)

$x-1$ is a factor $\Rightarrow P(1)=0$

$a-4+3b+2=0$

$a+3b=2$ —(1)

$x+2$ leaves remainder of 24 $\Rightarrow P(-2)=24$

$-8a-16-6b+2=24$

$-8a-6b=38$

$4a+3b=-19$ —(2)

From (1): $a=2-3b$ —(3)

Substitute (3) into (2)

$4(2-3b)+3b=-19$

$8-12b+3b=-19$

$8-9b=-19$

$-9b=-27$

$b=3$

$\therefore a=2-3(3)$

$=-7$

$a=-7, b=3$

9d)

Test $n=1$

$15^1+2^3-2=21$

$=7\times 3$ $\quad\therefore$ true for $n=1$

Assume true for $n=k$,

ie $\boxed{15^k+2^{3k}-2=7Q}$ for some integer Q

Test $n=k+1$

$15^{k+1}+2^{3(k+1)}-2=15\times 15^k+2^{3k+3}-2$

from assumption, $15^k=7Q+2-2^{3k}$

$=15(7Q+2-2^{3k})+2^{3k+3}-2$

$=105Q+30-15(2^{3k})+8(2^{3k})-2$

$=105Q+28-7(2^{3k})$

$=7[15Q+4-2^{3k}]$

$\therefore$ true for $n=k+1$

Since true for $n=1$, and from inductive step true for $n=2,3,4\ldots$, true for all integers $n\geq 1$.

10a) α)

let $\theta=\cos^{-1}\left(\frac{3}{7}\right)$

θ is in 1^{st} quad.

$\tan\theta=\frac{\sqrt{40}}{3}$

$=\frac{2\sqrt{10}}{3}$

β)

$\frac{5\pi}{4}$ is not in the restricted domain of $\sin x$.

$\frac{5\pi}{4}=\pi+\frac{\pi}{4}$, which is in 3^{rd} quadrant, ie $\sin\frac{5\pi}{4}<0$

Equivalent angle in restricted domain of $\sin x$ is $-\frac{\pi}{4}$.

$\therefore \sin^{-1}\left(\sin\frac{5\pi}{4}\right)$

$=\sin^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right)$

$=-\frac{\pi}{4}$

γ)

$\cos 15^\circ=\cos(45^\circ-30^\circ)$

$=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ$

$=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)$

$=\frac{\sqrt{3}+1}{2\sqrt{2}}$

10b)

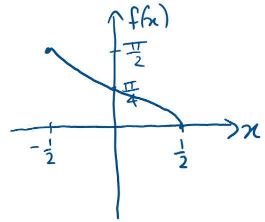
i) Domain: $-1\leq 2x\leq 1$

$-\frac{1}{2}\leq x\leq\frac{1}{2}$

Range: $0\leq 2f(x)\leq\pi$

$0\leq f(x)\leq\frac{\pi}{2}$

ii)



10c) $LHS=\cos(4x)$

$=2\cos^2(2x)-1$

$=2\left[2\cos^2x-1\right]^2-1$

$=2(4\cos^4x-4\cos^2x+1)-1$

$=8\cos^4x-8\cos^2x+2-1$

$=8\cos^4x-8\cos^2x+1$

$=RHS$

12

11a)

Applying the binomial expansion of $(x+y)^4$:

$$(\sqrt{3}-1)^4 = (\sqrt{3})^4 + \binom{4}{1}(\sqrt{3})^3(-1) + \binom{4}{2}(\sqrt{3})^2(-1)^2 + \binom{4}{3}(\sqrt{3})(-1)^3 + \binom{4}{4}(-1)^4$$
$$= 9 - 4(3\sqrt{3}) + 6(3) - 4\sqrt{3} + 1$$
$$= 28 - 16\sqrt{3}$$

$\therefore a = 28 \quad b = -16$

11b)

i) $\left(2x - \frac{3}{x}\right)^7 \rightarrow$ a general term is of the form:

$\binom{7}{r}(2x)^{7-r}\left(-\frac{3}{x}\right)^r$ where r is an integer from 0 to 7

$= \binom{7}{r}2^{7-r}(-3)^r x^{7-2r}$

For a term in x^4: $7-2r=4$
$2r=3$
$r=1.5$, not possible

$\therefore$ There is no term in x^4

ii) $(x^2+1)\left(2x-\frac{3}{x}\right)^7$

Form terms in x^5: x^2 must be multiplied by term in x^3
1 must be multiplied by term in x^5

In expansion of $\left(2x-\frac{3}{x}\right)^7$, general term is $\binom{7}{r}(2x)^{7-r}\left(-\frac{3}{x}\right)^r$
$= \binom{7}{r}2^{7-r}(-3)^r x^{7-2r}$

For term in x^3: $7-2r=3$
$2r=4$
$r=2$

Term in $x^3 = \binom{7}{2}2^5(-3)^2x^3$
$= 6048x^3$

For term in x^5: $7-2r=5$
$2r=2$
$r=1$

Term in $x^5 = \binom{7}{1}2^6(-3)^1x^5$
$= -1344x^5$

$\therefore$ Overall term in $x^5 = x^2(6048x^3) + 1(-1344x^5)$
$= 4704x^5$

$\therefore$ coefficient of x^5 is 4704.

11c)

i) 13H, 13D
number of ways $= {}^{13}C_2 \times {}^{13}C_4$
$= 55770$

ii) number of ways = 5 of same suit, 1 other
+
6 of same suit

number of ways 5 of same suit and one other can be selected $= {}^{13}C_5 \times {}^{39}C_1 \times 4$
$= 200\ 772$

number of ways 6 of same suit can be selected
$= {}^{13}C_6 \times 4$
$= 6864$

$\therefore$ total number of ways $= 200\ 772 + 6864$
$= 207\ 636$

12a)

i) Hor. asymptote: $\lim_{x\to\infty}\left(\frac{x+4}{x+2}\right) = \lim_{x\to\infty}\left(\frac{1+\frac{4}{x}}{1+\frac{2}{x}}\right)$
$= \frac{1+0}{1+0}$
$= 1$

Vert. asymptote: $x=-2$

ii) For inverse: $x = \frac{y+4}{y+2}$
$x(y+2) = y+4$
$xy + 2x = y+4$
$xy - y = 4-2x$
$y(x-1) = 4-2x$
$y = \frac{4-2x}{x-1}$
$f^{-1}(x) = \frac{4-2x}{x-1}$

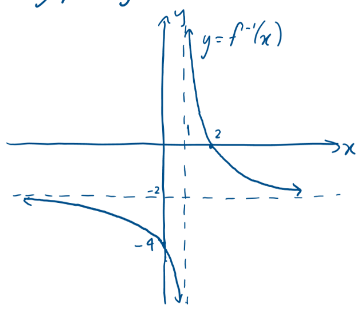
iii) Domain of $f^{-1}(x)$: $x \neq 1$

12a)

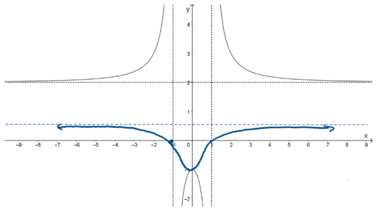
iv) $f^{-1}(x) = \frac{4-2x}{x-1}$

Intercepts:
When $x=0$ $y=-4$
When $y=0$, $4-2x=0$
$x=2$

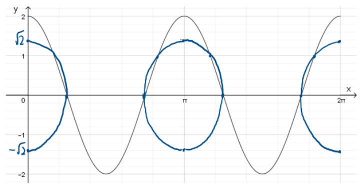
Vertical asymptote: $x=1$
Hor. asymptote: $y=-2$



12b)



12c)



13a)

i) $T = A + Be^{kt}$

$\frac{dT}{dt} = Bke^{kt}$

but $Be^{kt} = T - A$

$\therefore \frac{dT}{dt} = k(T-A)$

ii) $A = 20$

$T = 20 + Be^{kt}$

when $t = 6$ $T = 80$

$80 = 20 + Be^{6k}$

$60 = Be^{6k}$ — ①

when $t = 8$ $T = 50$

$50 = 20 + Be^{8k}$

$30 = Be^{8k}$ — ②

① ÷ ②

$\frac{60}{30} = \frac{Be^{6k}}{Be^{8k}}$

$2 = e^{-2k}$

$-2k = \ln 2$

$k = -\frac{1}{2}\ln 2$

ii) (continued)

Sub $k = -\frac{1}{2}\ln 2$ into ①

$60 = Be^{6(-0.5\ln 2)}$

$60 = Be^{-3\ln 2}$

$60 = Be^{\ln(\frac{1}{8})}$

$60 = B \times \frac{1}{8}$

$\therefore B = 480$

iii) $T = 20 + 480e^{kt}$

when $t = 0$

$T = 20 + 480e^{0}$

$= 500°C$

13b)

$(n+1)(n)(n-1)! = 240(n-1)!$

$n(n+1) = 240$

$n^2 + n - 240 = 0$

$(n+16)(n-15) = 0$

$n = -16 \quad n = 15$

reject since $n \geq 1$

13c)

Pairs that result in sum of 19: (4,15) (5,14) (6,13) (,12) (8,11) (9,10)

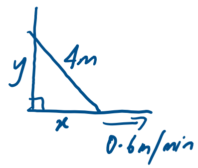
there are 6 pairs, and 3 numbers that don't app in any pair.

Worst case scenario: First 9 selections result i 1 from each pair and the 3 that do appear in any pair.

Next 2 selections will en e 2 pairs are completed.

∴ Need to select 9+2=11 integers to ensure at least 2 pairs.

13d)



Let x be the horizontal distance of the foot from the wall.

Let y be the vertical distance of the top from the floor.

Want $\frac{dy}{dt}$ when $x = 2$

$\frac{dx}{dt} = 0.6$

$y = \sqrt{16 - x^2}$

$\frac{dy}{dx} = \frac{1}{2}(16-x^2)^{-\frac{1}{2}}(-2x)$

$= \frac{-x}{\sqrt{16-x^2}}$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= \frac{-x}{\sqrt{16-x^2}} \times 0.6$

When $x = 2$

$\frac{dy}{dt} = \frac{-2}{\sqrt{16-4}} \times 0.6$

$= \frac{-1.2}{\sqrt{12}}$

$= \frac{-1.2}{2\sqrt{3}}$

$= \frac{-1.2\sqrt{3}}{6}$

$= -\frac{\sqrt{3}}{5}$ m/min

∴ Ladder is slipping down at $\sqrt{3}/5$ m/min

14a)

$$(1+x)^{10}+(1-x)^{10} = \binom{10}{0}+\binom{10}{1}x+\binom{10}{2}x^2+\cdots+\binom{10}{9}x^9+\binom{10}{10}x^{10}$$

$$+$$

$$\binom{10}{0}-\binom{10}{1}x+\binom{10}{2}x^2-\binom{10}{3}x^3+\cdots-\binom{10}{9}x^9+\binom{10}{10}x^{10}$$

$$=2\left[\binom{10}{0}+\binom{10}{2}x^2+\binom{10}{4}x^4+\binom{10}{6}x^6+\binom{10}{8}x^8+\binom{10}{10}x^{10}\right]$$

Sub $x=3$ into identity:

$$4^{10}+(-2)^{10}=2\left[1+\binom{10}{2}3^2+\binom{10}{4}3^4+\binom{10}{6}3^6+\binom{10}{8}3^8+3^{10}\right]$$

$$\therefore 1+\binom{10}{2}3^2+\binom{10}{4}3^4+\cdots+3^{10}=\frac{4^{10}+(-2)^{10}}{2}$$

$$=\frac{(2^2)^{10}+2^{10}}{2}$$

$$=\frac{2^{20}+2^{10}}{2}$$

$$=2^{19}+2^9$$

$$=2^9(2^{10}+1) \quad //$$

14b)

Test $n=1$

$LHS=\frac{1+4}{1(2)(3)}$ $\quad$ $RHS=\frac{3}{2}-\frac{4}{2\times 3}$

$=\frac{5}{6}$ $\quad$ $=\frac{3}{2}-\frac{2}{3}$

$=\frac{5}{6}$

$=LHS$ $\quad$ $\therefore$ true for $n=1$

Assume true for $n=k$

ie $\boxed{\frac{5}{6}+\cdots+\frac{k+4}{k(k+1)(k+2)}=\frac{3}{2}-\frac{(k+3)}{(k+1)(k+2)}}$

Test $n=k+1$

$$LHS=\underbrace{\frac{5}{6}+\cdots+\frac{k+4}{k(k+1)(k+2)}}_{\text{substituting in assumption}}+\frac{(k+1)+4}{(k+1)(k+2)(k+3)}$$

$$=\frac{3}{2}-\frac{(k+3)}{(k+1)(k+2)}+\frac{(k+5)}{(k+1)(k+2)(k+3)}$$

$$=\frac{3}{2}+\frac{(k+5)}{(k+1)(k+2)(k+3)}-\frac{(k+3)^2}{(k+1)(k+2)(k+3)}$$

$$=\frac{3}{2}+\frac{k+5-(k+3)^2}{(k+1)(k+2)(k+3)}$$

$$=\frac{3}{2}+\frac{k+5-k^2-6k-9}{(k+1)(k+2)(k+3)}$$

$$=\frac{3}{2}+\frac{(-k^2-5k-4)}{(k+1)(k+2)(k+3)}$$

$$=\frac{3}{2}+\frac{-(k^2+5k+4)}{(k+1)(k+2)(k+3)}$$

$$=\frac{3}{2}-\frac{(k+4)(k+1)}{(k+1)(k+2)(k+3)}$$

$$=\frac{3}{2}-\frac{(k+4)}{(k+2)(k+3)}$$

$=RHS$ for $n=k+1$

By process of induction, true for all positive integers.

14c)

i) $N=\frac{200}{1+ke^{-200t}}$

$$\lim_{t\to\infty}\left(\frac{200}{1+ke^{-200t}}\right)=\lim_{t\to\infty}\left(\frac{200}{1+\frac{k}{e^{200t}}}\right)$$

$$=\frac{200}{1+0}$$

$$=200$$

$\therefore$ as $t\to\infty$, the number infected N approaches 200.

ii) $N=200(1+ke^{-200t})^{-1}$

$$\frac{dN}{dt}=-200(1+ke^{-200t})^{-2}\times(-200ke^{-200t})$$

$$=\frac{200^2ke^{-200t}}{(1+ke^{-200t})^2}$$

$$=\left[\frac{200}{(1+ke^{-200t})}\right]^2\times ke^{-200t}$$

but $\frac{200}{1+ke^{-200t}}=N$, and $\frac{200}{N}=1+ke^{-200t}$

$\frac{200}{N}-1=ke^{-200t}$

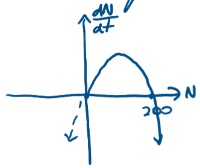
$$\therefore \frac{dN}{dt}=N^2\left(\frac{200}{N}-1\right)$$

$$=N(200-N) \quad //$$

iii) Want N when $\frac{dN}{dt}$ is greatest, ie a maximum.

Consider $\frac{dN}{dt}=N(200-N)$

This is a quadratic in N.



$\frac{dN}{dt}$ is greatest at the vertex,

when $N=100$