Student Number

Barker College

HONOR NON HONORES

Overall Median
47/77

2024 YEAR 12 ASSESSMENT BLOCK
SEMESTER 1

Mathematics Extension 1

Staff Involved:
- JZT - (J. Thomas)
- JAI - (J. Illes)
- GPF - (G. Fitzmaurice)
- ARM - (A. Mallam)
- AXD - (A. Davis)
- DZP* - (D. Peattie)

Thursday 4th April
8:30 am

95 copies

TOPICS COVERED:		
Combinatorics	Binomial Theorem	Further Rates
Further Trigonometry	Further Graphs	Polynomials
Trigonometric Equations	Induction	

General Instructions:
- Write your Student Number at the top of this page
- Reading time - 5 minutes
- Working time - 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A separate reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and / or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale

Total marks: 80

Section I - 10 marks (pages 2 - 4)
- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II - 70 marks (pages 5 - 9)
- Attempt Questions 11 - 15
- Allow about 1 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1 – 10.

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1 What is the derivative of $y = x^2 e^{2x}$?

A. $y' = 4xe^{2x}$

B. $y' = 2e^{2x}(1+x)$

C. $y' = 2xe^{2x}(1+x)$

D. $y' = 2xe^2$

2 Which of the following is equal to $\cos\theta$?

A. $\sin^2\dfrac{\theta}{2} - \cos^2\dfrac{\theta}{2}$

B. $2\cos^2\dfrac{\theta}{2} + 1$

C. $1 - 2\cos^2\theta$

D. $\dfrac{\sin\theta}{\tan\theta}$

3 How many arrangements of 3 digits can be formed using the digits 1 to 9 if each digit can only be used once?

A. $\dfrac{9!}{3!}$

B. $\dfrac{9!}{(9-3)!}$

C. $\dfrac{9!}{3!(9-3)!}$

D. $\dfrac{9!3!}{(9-3)!}$

4 Which of the following is the range of the function $y = \frac{1}{x^2+1}$?

A. $(-\infty, \infty)$

B. $(-\infty, 1]$

C. $(0, 1]$

D. $[0, 1]$

5 A curve is defined parametrically by the equation $x = \sin\theta, \; y = 2\cos\theta$. What is the Cartesian Equation of the curve?

A. $4x^2 + y^2 = 4$

B. $4x^2 + y^2 = 1$

C. $x^2 + y^2 = 1$

D. $x^2 + y^2 = 4$

6 If $t = \tan\frac{\theta}{2}$ which of the following expressions is equivalent to $4\sin\theta + 3\cos\theta + 5$?

A. $\frac{2(t+2)^2}{1-t^2}$

B. $\frac{(t+4)^2}{1-t^2}$

C. $\frac{2(t+2)^2}{1+t^2}$

D. $\frac{(t+4)^2}{1+t^2}$

7 Which group of three numbers could be the roots of the polynomial equation $x^3 + ax^2 - 41x + 42 = 0$?

A. 2, 3, 7

B. 1, –6, 7

C. –1, –2, 21

D. –1, –3, –14

8 How many solutions does the equation $(\cot x - 1)(\cos x + 1) = 0$ have in the domain $0 \le x \le 2\pi$?

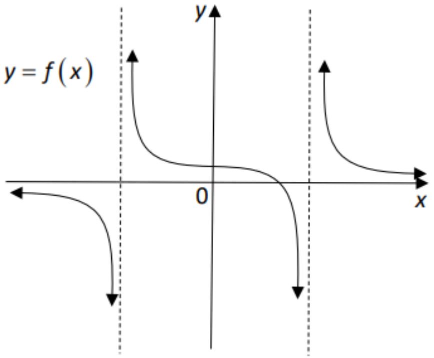
A. 1

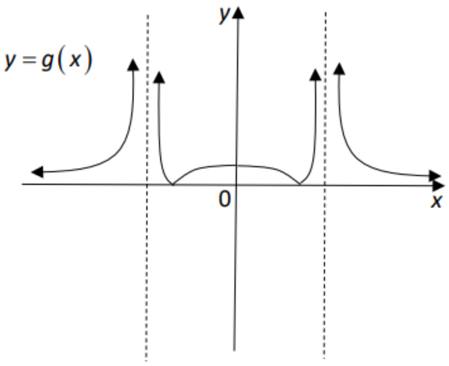
B. 2

C. 3

D. 4

9 Consider the following graphs $y = f(x)$ and $y = g(x)$.





What could be the possible relationship between the two graphs?

A. $g(x) = |f(x)|$

B. $g(x) = |f(|x|)|$

C. $g(x) = \sqrt{f(x)}$

D. $g(x) = \dfrac{1}{|f(x)|}$

10 A function is defined by $f(x) = -\sqrt{4 - x^2}$ for $0 \le x \le 2$. Which of the following correctly represents the inverse function of $f(x)$?

A. $f^{-1}(x) = -\sqrt{4 - x^2}$ for $0 \le x \le 2$

B. $f^{-1}(x) = \sqrt{4 - x^2}$ for $0 \le x \le 2$

C. $f^{-1}(x) = -\sqrt{4 - x^2}$ for $-2 \le x \le 0$

D. $f^{-1}(x) = \sqrt{4 - x^2}$ for $-2 \le x \le 0$

End of Section I

Section II

70 marks
Attempt Questions 11 – 15
Allow about 1 hour 45 minutes for this section

- Answer each question in a separate writing booklet. Extra writing booklets are available.
- Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) **Start the Question 11 Writing Booklet**

(a) Find $\dfrac{d}{dx}\dfrac{\sin^2 x}{x}$. **1**

(b) Evaluate $\int_0^{\frac{\pi}{6}} \cos x \sin^3 x \, dx$. **2**

(c) For what values of m is the polynomial $x^2-(m+2)x+2(m+2)$ positive for all values of x ? **3**

(d) Consider the expansion $(2x-p)^9$. The coefficient of x^6 is –344 064. Find the value of p. **3**

(e) A mixed volleyball team of eight players is selected from ten males and nine females. In how many ways can this be done if the team must have at least two female players? **2**

(f) Calculate the area enclosed between the curve $y=\sqrt{x+1}$, the y-axis and the lines $y=0$ and $y=3$. **3**

Question 12 (14 marks) **Start the Question 12 Writing Booklet**

(a) Determine whether the function $f(x) = \tan(\sin x)$ is odd, even or neither. **2**

(b) In a playground 11 girls are playing together, of which 3 are sisters. In a particular game they are required to sit in a circle. In how many ways can all the girls be seated in the circle if all 3 sisters are **NOT** to be seated together as a group of 3? **2**

(c)

(i) Express $\sqrt{3}\cos x + \sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and α is acute. **2**

(ii) Hence solve $\sqrt{3}\cos x + \sin x = 1$ for $0 \le x \le 2\pi$. **2**

(d) Given that $\sin\theta = \dfrac{3}{4}$ and $\dfrac{\pi}{2} \le \theta \le \pi$ determine the exact value of $\tan 2\theta$. **3**

(e) The polynomial $P(x) = (x - p)^3 + q$ has a zero at $x = 1$ and when divided by x the remainder is –7. Find the possible values of p. **3**

Question 13 (14 marks) **Start the Question 13 Writing Booklet**

(a) Solve $\dfrac{2x}{x-1} \le 1$. **3**

(b)

(i) Show that $\tan 2x + \tan x = \dfrac{\sin 3x}{\cos 2x \cos x}$. **2**

(ii) Hence solve the equation $\tan 2x + \tan x = 0$ for $0 < x < \dfrac{\pi}{2}$. **1**

(c) Let $f(x) = x - \dfrac{1}{2}x^2$ for $x \le 1$.

This function has an inverse function, $f^{-1}(x)$.

(i) Sketch, a half page diagram, of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. **2**

(ii) Find an expression for $f^{-1}(x)$. **2**

(iii) Evaluate $f^{-1}\left(\dfrac{3}{8}\right)$. **1**

(d) Using mathematical induction, prove that, for all positive integers n, **3**

$$\frac{5}{6} + \frac{1}{4} + \frac{7}{60} + \ldots + \frac{n+4}{n(n+1)(n+2)} = \frac{3}{2} - \frac{n+3}{(n+1)(n+2)}$$

Question 14 (14 marks) **Start the Question 14 Writing Booklet**

(a)

(i) State the domain and range of $y = 3\sin^{-1}(2x)$. **2**

(ii) Hence, sketch $y = 3\sin^{-1}(2x)$. **1**

(b) At a school activity day, each student is required to choose four different sports out of a list of ten sports.
Each student is then placed in a group with the other students that chose the same four sports.
If there are 1500 students at the school, what is the minimum possible number of students that would have been placed in the most popular four-sport group? **2**

(c) A refrigerator has a constant temperature of 3°C. A can of drink with temperature 30°C is placed in the refrigerator.
After being in the refrigerator for 15 minutes, the temperature of the can of drink is 20°C.
The change in the temperature of the can of drink can be modelled by $\frac{dT}{dt} = k(T-3)$,
where T is the temperature of the can of drink in degrees celsius, *t* is the time in minutes after the can is placed in the refrigerator and *k* is a negative constant.

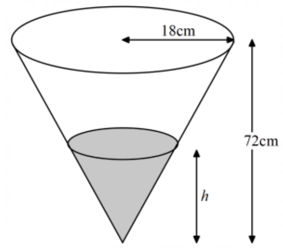
(i) Show that $T = 3 + Ae^{kt}$, where A is a constant, satisfies $\frac{dT}{dt} = k(T-3)$. **1**

(ii) After 60 minutes, at what rate is the temperature of the can of drink changing (2 d.p.)? **3**

(d) **Answer Question 14 (d) on the separate answer page supplied, which can be detached from the end of this question booklet.** **5**
Hand it in with your booklet for Question 14.

Question 15 (14 marks) **Start the Question 15 Writing Booklet**

(a) Flowers at a florist are stored in vases which are in the shape of hollow inverted right circular cones with height 72cm and radius 18cm.
One such vase is initially empty and placed, with its axis vertical, under a tap where the water is flowing into the vase at the constant rate of $6\pi \text{cm}^3\text{s}^{-1}$.



(i) Show that the volume, $V\ cm^3$, of the water in the vase is given by $V = \dfrac{1}{48}\pi h^3$, where, h cm, is the height of the water in the vase. **1**

(ii) Find the rate at which h is rising when h = 4cm. **1**

(iii) Determine the rate at which h is rising 12.5 minutes after the vase was placed under the tap. **2**

(b) Solve $\sin 5x = \sin 3x$, where $0 \le x \le \dfrac{\pi}{2}$. **3**

(c) By considering the expansion: $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \ldots + {}^{2n}C_{2n} x^{2n}$,

prove that:

$$2(2-1)\,{}^{2n}C_2 + 3(3-1)\,{}^{2n}C_3 + 4(4-1)\,{}^{2n}C_4 + \ldots + 2n(2n-1)\,{}^{2n}C_{2n} = 2n(2n-1)\,4^{n-1}.$$ **3**

(d) The polynomial equation $x^3 + bx^2 + cx + d = 0$ has roots α, α^2 and α^3 for some real number, $\alpha \ne 0$.

(i) Simplify $\dfrac{1}{\alpha} + \dfrac{1}{\alpha^2} + \dfrac{1}{\alpha^3}$, leaving your answer without α. **2**

(ii) Show that $b^3 d - c^3 = 0$. **2**

End of Paper

Additional Answer page for Question 14 (d)
Detach and hand in with your Question 14 booklet.

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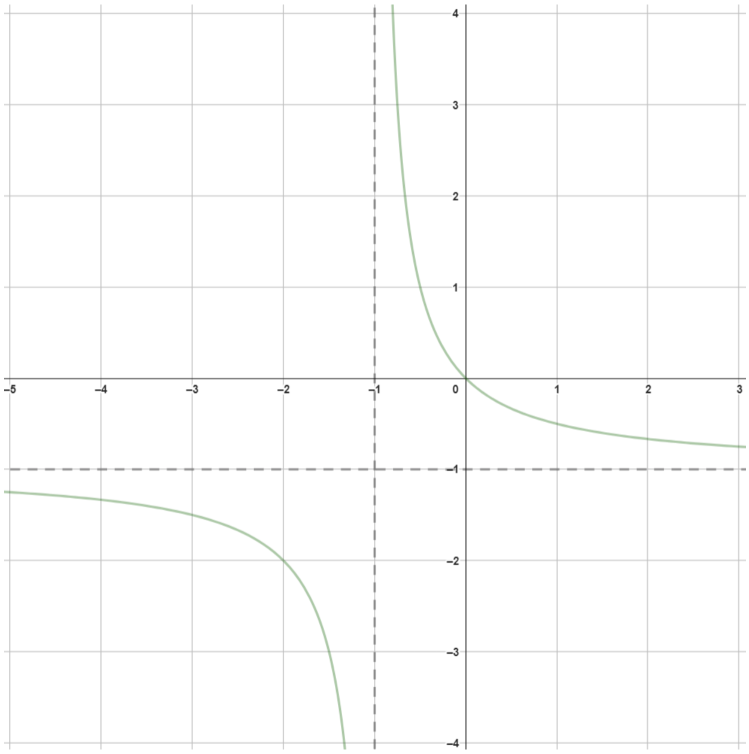
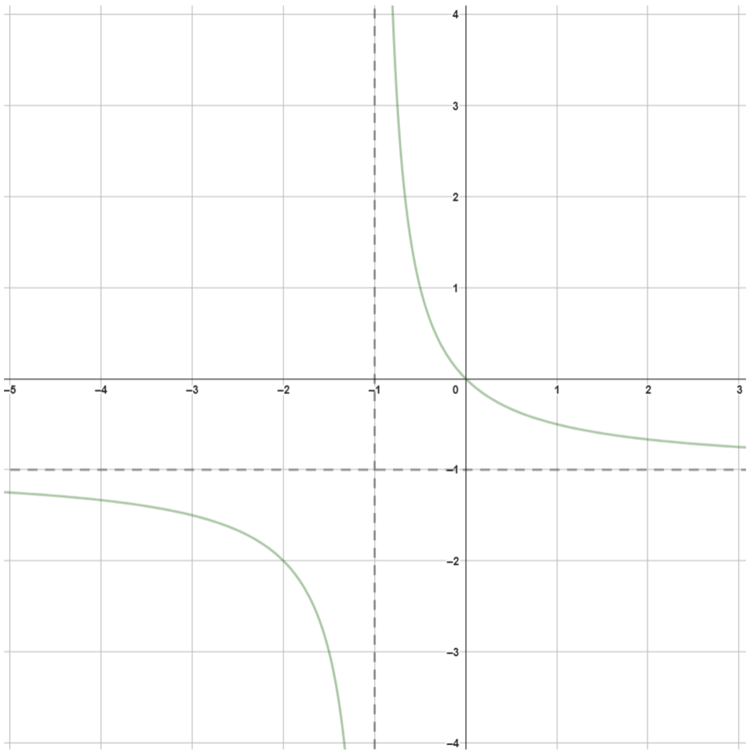
(d) The diagram below shows the graph of $y = f(x)$.

(i) On the axes below, sketch $\dfrac{1}{f(x)}$ showing **all** asymptotes and important features. **2**



(ii) On the axes below, sketch $y^2 = f(x)$ showing **all** asymptotes and important features. **3**



MC

Tuesday, 2 April 2024 11:01 AM

1. C $2xe^{2x}(1+x)$

2. D $\frac{\sin\theta}{\tan\theta} = \cancel{\sin\theta} \times \frac{\cos\theta}{\cancel{\sin\theta}}$

$= \cos\theta$

3. B ${}^9P_3 = \frac{9!}{6!}$

4. C $0 < y \le 1$

5. A $x^2 + \left(\frac{y}{2}\right)^2 = 1$

$x^2 + \frac{y^2}{4} = 1$

$4x^2 + y^2 = 4$

6. C $\frac{8t}{1+t^2} + \frac{3-3t^2}{1+t^2} + 5$

$= \frac{2(t+2)^2}{1+t^2}$

7. B $\alpha+\beta+\gamma = -a$

$\alpha\beta+\alpha\gamma+\beta\gamma = -41$

$\alpha\beta\gamma = -42$

8. C $\cot x = 1$ $\cos x = -1$

$\tan x = 1$ $x = \pi$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

9. B

10. D $f(x)$ $0 \le x \le 2$

$-2 \le y \le 0$

$\therefore f^{-1}(x)$ $-2 \le x \le 0$

$0 \le y \le 2$

11. a) $u = \sin^2 x$

$u' = 2\sin x\cos x$

$v = x,\ v' = 1$

$\therefore y' = \dfrac{2x\sin x\cos x - \sin^2 x}{x^2}$

b) $\displaystyle\int_0^{\frac{\pi}{6}} \cos x \sin^3 x\, dx$

$= \left.\dfrac{\sin^4 x}{4}\right]_0^{\frac{\pi}{6}}$

$= \dfrac{1/16}{4} - 0$

$= \dfrac{1}{64}$

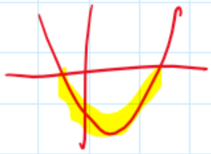
c) $\Delta < 0$

$\Delta = (m+2)^2 - 8(m+2)$

$= m^2 + 4m + 4 - 8m - 16$

$= m^2 - 4m - 12 < 0$

$(m-6)(m+2) < 0$



$\therefore -2 < m < 6$

d) $T_{r+1} = \binom{9}{r} 2x^{9-r} (-p)^r$

$r = 3$

$-344064 = \binom{9}{3} 2^6 - p^3$

$\therefore p^3 = \dfrac{344064}{5376}$

$= 64$

$\therefore p = 4$

e) Total $= \binom{19}{8}$

0 Female $= \binom{10}{8}$

1 Female $= \binom{10}{7} \times \binom{9}{1}$

$\therefore$ At least 2F

$= \binom{19}{8} - \binom{10}{7} + \binom{9}{1} = 74457$

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f) 

$y = \sqrt{x+1}$

$y^2 = x+1$

$x = y^2 - 1$

Area $= \int x\, dy$

$= \left|\int_0^1 y^2 - 1\, dy\right| + \int_1^3 y^2 - 1\, dy$

$= \left|\left.\dfrac{y^3}{3} - y\right]_0^1\right| + \left.\dfrac{y^3}{3} - y\right]_1^3$

$= 1 - \dfrac{1}{3} + \dfrac{27}{3} - 3 - \left(\dfrac{1}{3} - 1\right)$

$= \dfrac{22}{3}$ or $7\frac{1}{3}\ u^2$

Q12.

a) $\sin x$ is odd

$\therefore \sin(-x) = -\sin(x)$

$\tan x$ is odd

$\therefore \tan(-x) = -\tan(x)$

$f(-x) = \tan(\sin(-x))$

$= -\tan \sin(x)$

$= -f(x)$

$\therefore$ odd

b) Total $= 10!$

3 together $= 3! \times 8!$

$\therefore$ Not 3 together $= 10! - 3!\,8!$

$= 3386880$

c)(i) $R = \sqrt{3+1}$

$= 2$

$\theta = \tan^{-1} \frac{1}{\sqrt{3}}$

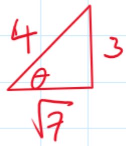
$= \frac{\pi}{6}$

$\therefore \sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right)$

(ii) $\cos\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

$x - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{2}, \frac{11\pi}{6}$

d) 

$\sin\theta = \frac{3}{4}$

$\cos\theta = -\frac{\sqrt{7}}{4}$

$\tan\theta = -\frac{3}{\sqrt{7}}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{-6/\sqrt{7}}{1 - \frac{9}{7}}$

$= \frac{21}{\sqrt{7}} = 3\sqrt{7}$

e) $P(1) = (1-p)^3 + q = 0$

$\therefore q = (p-1)^3$

$P(0) = -p^3 + q = -7$

$q = -7 + p^3$

$\therefore p^3 - 3p^2 + 3p - 1 = -7 + p^3$

$3p^2 - 3p - 6 = 0$

$(p-2)(p+1) = 0$

$p = 2, -1$

Q13 1 of 2

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13. $x \neq 1$

a) $2x(x-1) \le (x-1)^2$

$2x^2 - 2x \le x^2 - 2x + 1$

$x^2 - 1 \le 0$

$(x+1)(x-1) \le 0$

$-1 \le x < 1$

b) (i) $LHS = \dfrac{\sin 2x}{\cos 2x} + \dfrac{\sin x}{\cos x}$

$= \dfrac{\sin 2x \cos x + \sin x \cos 2x}{\cos 2x \cos x}$

$= \dfrac{\sin(2x+x)}{\cos 2x \cos x}$

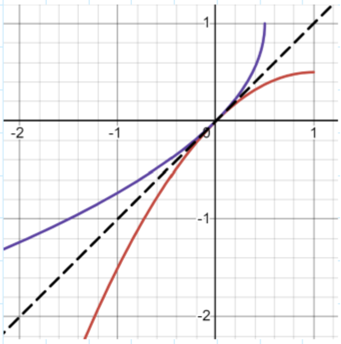
$= \dfrac{\sin 3x}{\cos 2x \cos x}$

(ii) $\sin 3x = 0 \quad 0 < 3x < \frac{3\pi}{2}$

$3x = \pi$

$x = \frac{\pi}{3}$

c) (i)



(ii) $x = y - \frac{y^2}{2}$

$y^2 - 2y + 2x = 0$

$y = \dfrac{2 \pm \sqrt{4-8x}}{2}$

$= 1 \pm \sqrt{1-2x}$

but $y \le 1$

$\therefore f^{-1}(x) = 1 - \sqrt{1-2x}$

(iii) $f^{-1}\left(\frac{3}{8}\right) = 1 - \frac{1}{2}$

$= \frac{1}{2}$

d) Step 1

let $n=1$ $LHS = \frac{5}{6}$

$RHS = \frac{3}{2} - \frac{4}{6}$

$= \frac{5}{6}$

Step 2

assume $n=k$

$\therefore \frac{5}{6} + \frac{1}{4} + \frac{7}{60} + \dots \frac{k+4}{k(k+1)(k+2)} = \frac{3}{2} - \frac{k+3}{(k+1)(k+2)}$

Step 3

let $n = k+1$

RTP $\frac{5}{6} + \frac{1}{4} + \frac{7}{60} + \dots \frac{k+4}{k(k+1)(k+2)} + \frac{k+5}{(k+1)(k+2)(k+3)} = \frac{3}{2} - \frac{k+4}{(k+2)(k+3)}$

$LHS = \frac{3}{2} - \frac{k+3}{(k+1)(k+2)} + \frac{k+5}{(k+1)(k+2)(k+3)}$ from assumption

$= \frac{3}{2} - \frac{(k+3)^2 - k - 5}{(k+1)(k+2)(k+3)}$

$= \frac{3}{2} - \frac{k^2+5k+4}{(k+1)(k+2)(k+3)}$

$= \frac{3}{2} - \frac{\cancel{(k+1)}(k+4)}{\cancel{(k+1)}(k+2)(k+3)} = \frac{3}{2} - \frac{k+4}{(k+2)(k+3)} = RHS$

$\therefore$ True by PMI

Q14 1 of 2

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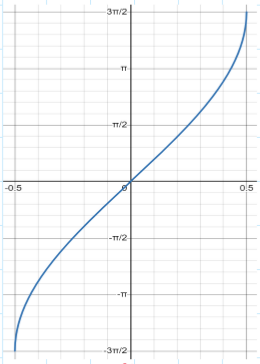
14.

a)(i) D $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

R $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

(ii)



b) Combos $= \binom{10}{4}$

$= 210$

$\frac{1500}{210} = 7.14$

∴ Min possible = 8

can justify 9 if 1 most popular for sport group but need explanation

c)(i) $T = 3 + Ae^{kt}$

$\frac{dT}{dt} = kAe^{kt}$

$= k(T-3)$

∴ Yes satisfies

(ii) $t=0$ $T=30$

∴ $A = 27$

$T = 3 + 27e^{kt}$

$t=15$ $T=20$

∴ $20 = 3 + 27e^{15k}$

$e^{15k} = \frac{17}{27}$

$k = \frac{\ln\frac{17}{27}}{15} = -0.0308$

$t = 60$ $T = 3 + 27e^{60k}$

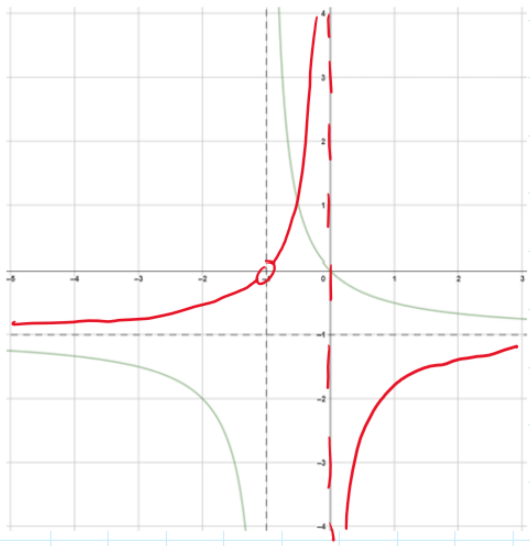
$= 7.24$

∴ $\frac{dT}{dt} = k(7.24 - 3)$

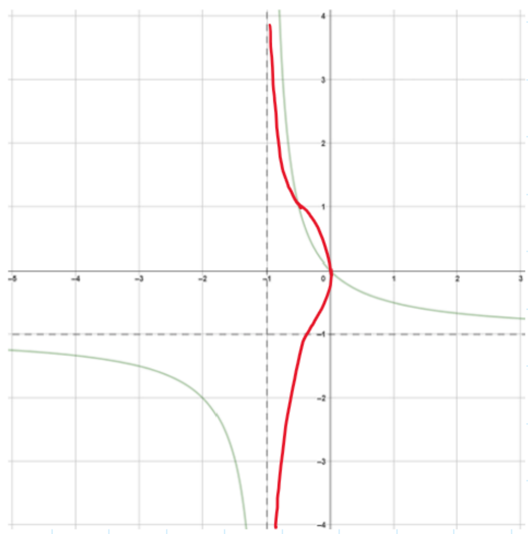
$= -0.13$

∴ reducing at 0.13°/min

(i)



(ii)



15.

a)(i) $V = \frac{1}{3}\pi r^2 h$

$r = \frac{h}{4}$

$\therefore V = \frac{1}{3}\pi \times \frac{h^2}{16} h$

$= \frac{1}{48}\pi h^3$

(ii) $\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$

$\frac{dv}{dh} = \frac{\pi h^2}{16}$

$\frac{dv}{dt} = 6\pi$

$\therefore \frac{dh}{dt} = \frac{16}{\pi h^2} \times 6\pi$

$= \frac{96}{h^2}$ at $h = 4$

$= 6$ cm/sec

(iii) $V = 12.5 \times 60 \times 6\pi$

$= 4500\pi \text{ cm}^3$

$\therefore 4500\pi = \frac{1}{48}\pi h^3$

$h = 60$ cm

$\frac{dh}{dt} = \frac{96}{h^2}$

$= \frac{2}{75}$ cm/sec

$= 0.02\dot{6}$ cm/sec

b) let $5x = 4x + x$, $3x = 4x - x$

$\therefore \sin(4x+x) - \sin(4x-x) = 0$

$\therefore 2\cos 4x \sin x = 0$

$\sin x = 0$ $\quad \cos 4x = 0$

$x = 0$ $\quad 4x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{8}, \frac{3\pi}{8}$

c) Diff both sides

$2n(1+x)^{2n-1} = \binom{2n}{1} + 2\binom{2n}{2}x + \dots 2n\binom{2n}{2n}x^{2n-1}$

Diff BS again

$2n(2n-1)(1+x)^{2n-2} = 2\binom{2n}{2} + 3\times 2\binom{2n}{3} + \dots 2n(2n-1)\binom{2n}{2n}x^{2n-2}$

let $x = 1$

$LHS = 2n(2n-1)2^{2n-2}$

$= 2n(2n-1)4^{n-1}$

$RHS = 2\binom{2n}{2} + 3\times 2\binom{2n}{3} + 4\times 3\binom{2n}{4} + \dots 2n(2n-1)\binom{2n}{2n}$

$= 2(2-1)\binom{2n}{2} + 3(3-1)\binom{2n}{3} + 4(4-1)\binom{2n}{4} + \dots 2n(2n-1)\binom{2n}{2n}$

$\therefore$ Proven

d) (i) $\alpha+\alpha^2+\alpha^3=-b$ — (1)

$\alpha^3+\alpha^4+\alpha^5=c$ — (2)

$\alpha^6=-d$ — (3)

$$\frac{1}{\alpha}+\frac{1}{\alpha^2}+\frac{1}{\alpha^3}=\frac{\alpha^5+\alpha^4+\alpha^3}{\alpha^6}$$

$$=-\frac{c}{d}$$

(ii) From (2) $\alpha^2(\alpha+\alpha^2+\alpha^3)=c$

$$\therefore -b\alpha^2=c$$

$$\alpha^2=-\frac{c}{b}$$

$$\therefore \alpha^6=\frac{-c^3}{b^3}$$

$$\therefore -d=\frac{-c^3}{b^3}$$

$$\therefore b^3d=c^3$$

$b^3d-c^3=0$ as required