2020 **ASSESSMENT TASK 1 DECEMBER**

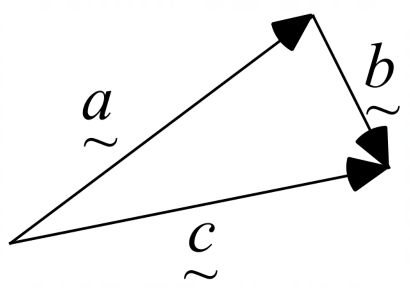
Mathematics Extension 1

General Instructions	• Reading time – 5 minutes • Working time – 1 hour • Write using black pen • Calculators approved by NESA may be used • In Questions 1 – 4, show relevant mathematical reasoning and/or calculations
Total marks: 38	Attempt all Questions 1 – 4 (pages 2-6)

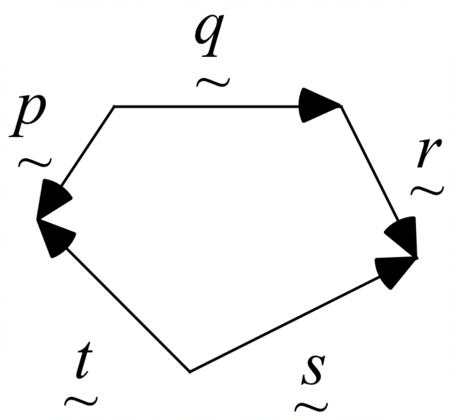
Question 1 **Start on the appropriate page of your answer booklet** **10 marks**

a) Simplify $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}$ **2**

b) A vector equation for the diagram below is $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$.



Write a possible vector equation for the following diagram. **1**



c) If $a = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, find

(i) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ **1**

(ii) $\underset{\sim}{a} - 2\underset{\sim}{b}$ **1**

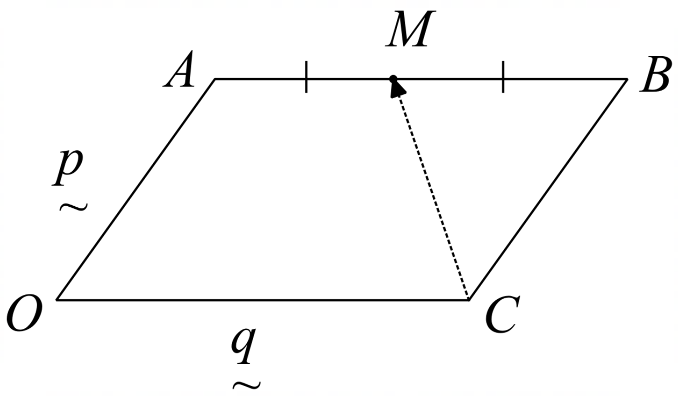
(iii) $\hat{\underset{\sim}{a}}$ **2**

d) Prove, by mathematical induction, that $3^{2n-1} + 2^{n+1}$ is divisible by 7 for all positive integers, $n \geq 1$. **3**

Question 2 **Start on the appropriate page of your answer booklet** **8 marks**

a) Find the vector projection of $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$ onto $\underset{\sim}{b} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$. **2**

b) $OABC$ is a parallelogram. Let $\underset{\sim}{p} = \overrightarrow{OA}$ and $\underset{\sim}{q} = \overrightarrow{OC}$. Let M be the midpoint of AB.



(i) Express $\overrightarrow{OB}$ and $\overrightarrow{OM}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$. **2**

(ii) If X is a point on CM and $\overrightarrow{CX} = \frac{2}{3}\overrightarrow{CM}$, express $\overrightarrow{CX}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$. **1**

(iii) Find $\overrightarrow{OX}$ and hence show that X lies on the line segment OB. **2**

(iv) Find the ratio in which the median CM divides the diagonal OB. **1**

Question 3 **Start on the appropriate page of your answer booklet** **10 marks**

a) If $|\underset{\sim}{a}| = 5$, $|\underset{\sim}{b}| = 6$ and the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $45°$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$ **1**

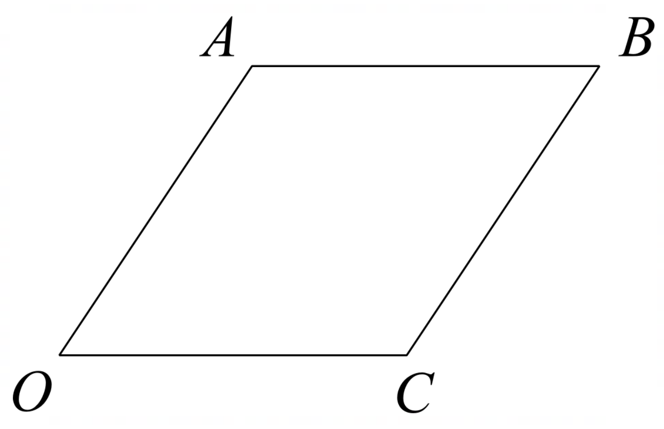
b) Let $S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}$, for all positive integer n.

(i) Prove, using mathematical induction, that $S_n = 1 - \frac{1}{(n+1)!}$ for all positive integers $n \geq 1$. **3**

(ii) Find the value of $\lim_{n \to \infty} S_n$. **1**

(iii) Find the smallest positive integer n such that $|S_n - 1| < 10^{-6}$. **1**

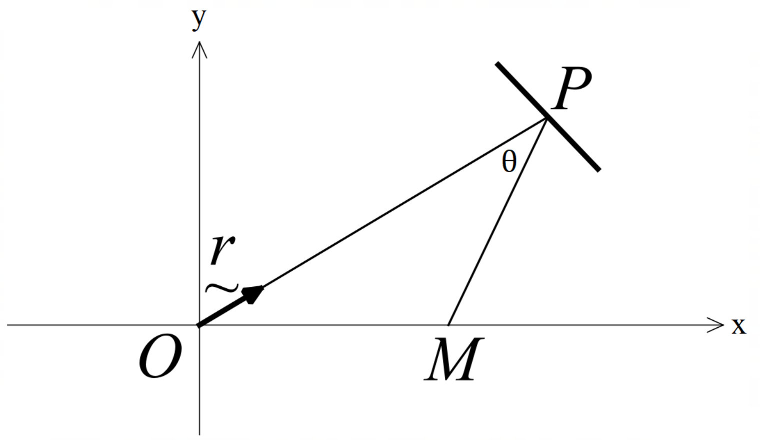
c) Consider a rhombus $OABC$ where $\overrightarrow{OA}$ is defined as $\underset{\sim}{m} + \underset{\sim}{n}$ and $\overrightarrow{OC}$ is defined as $\underset{\sim}{n}$.



Using these vector definitions, prove that the diagonals of $OABC$ are perpendicular. **4**

Question 4 **Start on the appropriate page of your answer booklet** **10 marks**

a) The diagram shows the path of a light beam from its source at O in the direction of the vector $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.



At point P, the beam is reflected by an adjustable mirror and meets the x-axis at M. The position of M varies, depending on the adjustment of the mirror at P.

(i) Given that $\overrightarrow{OP} = 4\underset{\sim}{r}$ find the coordinates of P. **1**

(ii) The point M has coordinates $(k,0)$. Find an expression, in terms of k, for vector $\overrightarrow{PM}$. **1**

(iii) Find the magnitude of the vectors $\overrightarrow{OP}$ and $\overrightarrow{PM}$. **2**

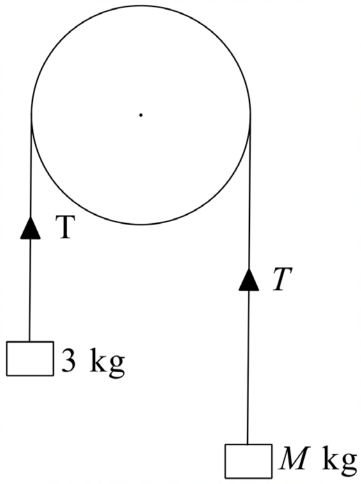
(iv) Hence, using vector techniques and the information in parts (i) to (iii), find the value of

(1) k if $\theta = 90°$. **1**

(2) θ if M has coordinates $(9,0)$. Answer to the nearest degree. **2**

Question 4 continues on the next page

b) Two blocks of masses 3 kg and M kg (where $M > 3$) are connected by a light inextensible string passing over a smooth fixed pulley. The two masses are at rest before being released.



Given that the tension in the string is 37.5 N, calculate the value of M. **3**

(Use $g = 9.8\ ms^{-2}$)

End of paper

EXTENSION 1 DECEMBER ASSESSMENT

1a) $\vec{AB} + \vec{BC} - \vec{DC}$

$= \vec{AC} + \vec{CD}$

$= \vec{AD}$

(2) correct solution
(1) simplifies two of the vectors OR adds $\vec{CD}$

b) $\underset{\sim}{q} + \underset{\sim}{r} - \underset{\sim}{s} + \underset{\sim}{t} = \underset{\sim}{p}$ (or equivalent)

(1) correct answer

c) i) $\begin{pmatrix} 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \end{pmatrix} = -3 - 10$

$= -13$

(1) correct answer

ii) $\underset{\sim}{a} - 2\underset{\sim}{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 3+2 \\ -2-10 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ -12 \end{pmatrix}$

(1) correct answer

iii) $\hat{\underset{\sim}{a}} = \dfrac{1}{\sqrt{3^2 + (-2)^2}} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$= \dfrac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(2) correct solution
(1) finds magnitude of $\underset{\sim}{a}$

d) Test $n = 1$

$3^{2-1} + 2^{1+1}$

$= 3 + 4$

$= 7 \times 1$

$\therefore$ True for $n = 1$

Assume true for $n = k$ (k is a positive integer)

$3^{2k-1} + 2^{k+1} = 7P$ (where P is an integer)

For $n = k+1$ we wish to prove

$3^{2(k+1)-1} + 2^{(k+1)+1} = 7Q$ (where Q is an integer)

ie $3^{2k+1} + 2^{k+2} = 7Q$

$3^{2k+1} + 2^{k+2} = 3^2 3^{2k-1} + 2^{k+2}$

$= 9(7P - 2^{k+1}) + 2 \times 2^{k+1}$ (using assumption)

(3) correct proof
(2) Successfully does 3 of 4 key parts below
(1) Successfully does 2 of 4 key parts below

4 key parts
(1) proves true for $n=1$
(2) states assumption and what must be proven
(3) uses assumption
(4) correctly proves required statement

$= 7\times 9P - 9\times 2^{k+1} + 2\times 2^{k+1}$

$= 7\times 9P - 7\times 2^{k+1}$

$= 7(9P - 2^{k+1})$

$= 7Q$ where $Q = 9P - 2^{k+1}$, which is an integer as P, k are integers

$\therefore$ If true for $n=k$, it is true for $n=k+1$. But it is true for $n=1$, therefore true for $n=1+1=2,3,4,\ldots$ and so on for all positive integers $n\geq 1$

Q2 a) $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\underset{\sim}{b}$

(2) correct solution
(1) progress towards

$= \dfrac{3-12}{25}(3\underset{\sim}{i} - 4\underset{\sim}{j})$

$$\left\{\begin{aligned} &= \frac{-9}{25}(3\underset{\sim}{i} - 4\underset{\sim}{j}) \\ &= \frac{-27}{25}\underset{\sim}{i} + \frac{36}{25}\underset{\sim}{j}\end{aligned}\right.$$

2 b) i) $\vec{OB} = \vec{OA} + \vec{AB}$

$= \vec{OA} + \vec{OC}$ (opposite sides of a parallelogram)

$= \underset{\sim}{p} + \underset{\sim}{q}$

(2) correct solution
(1) correct expression for one of the vectors

$\vec{OM} = \vec{OA} + \vec{AM}$

$= \underset{\sim}{p} + \dfrac{\underset{\sim}{q}}{2}$

ii) $\vec{OC} + \vec{CM} = \vec{OM}$

(1) correct answer

$\vec{CM} = \underset{\sim}{p} + \frac{1}{2}\underset{\sim}{q} - \underset{\sim}{q}$

$= \underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$

$$\left.\begin{aligned}\vec{CX} &= \frac{2}{3}\left(\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}\right) \\ &= \frac{1}{3}\left(2\underset{\sim}{p} - \underset{\sim}{q}\right)\end{aligned}\right\}$$

2b iii) $\vec{OX} = \vec{OC} + \vec{CX}$

$\vec{OX} = \frac{2}{3}\underset{\sim}{p} - \frac{1}{3}\underset{\sim}{q} + \underset{\sim}{q}$

$= \frac{2}{3}\underset{\sim}{p} + \frac{2}{3}\underset{\sim}{q}$

$= \frac{2}{3}(\underset{\sim}{p} + \underset{\sim}{q})$

$= \frac{2}{3}\vec{OB}$

$\therefore$ Since $\vec{OX} = \frac{2}{3}\vec{OB}$, $\vec{OX} \parallel \vec{OB}$ and lies on $\vec{OB}$ as it shares a common point O.

(2) correct solution
(1) finds $\vec{OX}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$

iv) X divides OB in the ratio 2:1

(1) correct answer

3a $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

$= 5 \times 6 \times \cos 45^\circ$

$= \frac{30}{\sqrt{2}}$

$= 15\sqrt{2}$

(1) correct solution

3b i) Test $n=1$

$S_1 = \frac{1}{(1+1)!} = \frac{1}{2}$

$RHS = 1 - \frac{1}{(1+1)!} = 1 - \frac{1}{2} = \frac{1}{2} = LHS$

$\therefore$ True for $n=1$

Assume true for $n=k$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$$

For $n = k+1$ we wish to prove

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

(3) Correct proof
(2) successfully does 3 of 4 key parts below
(1) successfully does 2 of 4 key parts below
(I) Proves true for $n=1$
(II) States assumption and what must be proven
(III) Uses assumption
(IV) Proves required result

36 (continued)

Now $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\dots+\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}$

$= 1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}$ (using assumption)

$= 1+\left(\frac{k+1}{(k+2)!}-\frac{k+2}{(k+2)(k+1)!}\right)$

$= 1+\frac{k+1\cancel{-k}-2}{(k+2)!}$

$= 1-\frac{1}{(k+2)!}$ as required.

Therefore, if true for $n=k$, it is true for $n=k+1$. Since true for $n=1$, by mathematical induction, it is true for $n=2,3,4$ and so on for all positive integers.

36 ii) $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(1-\frac{1}{(n+1)!}\right)$ ① Correct solution

$= 1-0$

$= 1$

36 iii) $|S_n-1|<10^{-6}$ ① Correct solution

$\left|\frac{-1}{(n+1)!}\right|<\frac{1}{10^6}$

$(n+1)!>10^6$

$9! = 362\,880$

$10! = 3628800$

$>10^6$

$(9+1)!>10^6$

$\therefore n=9$ is the smallest positive integer

3 c) $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= \underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{b}$

$= \underset{\sim}{a}$

$\overrightarrow{OB} \cdot \overrightarrow{CA} = (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{b}) \cdot \underset{\sim}{a}$

$= (\underset{\sim}{a} + 2\underset{\sim}{b}) \cdot \underset{\sim}{a}$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} + 2\underset{\sim}{a} \cdot \underset{\sim}{b}$ *

Since OABC is a rhombus

$|\overrightarrow{OA}| = |\overrightarrow{OC}|$

$|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{b}|$

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{b} \cdot \underset{\sim}{b}$

$\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{a} + \cancel{\underset{\sim}{b} \cdot \underset{\sim}{b}} = \cancel{\underset{\sim}{b} \cdot \underset{\sim}{b}}$

$\underset{\sim}{a} \cdot \underset{\sim}{a} + 2\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

From * $\overrightarrow{OB} \cdot \overrightarrow{CA} = 0$

$\therefore$ $OB \perp CA$, so the diagonals of OABC are perpendicular

④ Correct solution

③ uses $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{b}|$ and $\overrightarrow{OB} \cdot \overrightarrow{CA}$

② Expresses $\overrightarrow{OB} \cdot \overrightarrow{CA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$

① Expresses $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$

4a i) $\overrightarrow{OP} = 4\begin{pmatrix}3\\1\end{pmatrix}$

$= \begin{pmatrix}12\\4\end{pmatrix}$

$\therefore P = (12, 4)$

① Correct answer

ii) $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP}$

$= \begin{pmatrix}k\\0\end{pmatrix} - \begin{pmatrix}12\\4\end{pmatrix}$

$\overrightarrow{PM} = \begin{pmatrix}k-12\\-4\end{pmatrix}$

① Correct answer

iii) $|\overrightarrow{OP}| = \sqrt{12^2 + 4^2}$

$= \sqrt{160}$

$= 4\sqrt{10}$

$|\overrightarrow{PM}| = \sqrt{(k-12)^2 + (-4)^2}$

$= \sqrt{k^2 - 24k + 160}$

② Correct solution

① Finds one correct magnitude

4a iv) 1. If $\theta = 90°$ $\vec{PM} \cdot \vec{OP} = 0$

$$\begin{pmatrix} k-12 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 4 \end{pmatrix} = 0$$

$$12(k-12) - 4 \times 4 = 0$$

$$12k = 160$$

$$\therefore \; k = \frac{40}{3}$$

(1) Correct answer

2. If $k = 9$

$$\cos(180° - \theta) = \frac{\begin{pmatrix} -3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 4 \end{pmatrix}}{\sqrt{160} \times \sqrt{25}}$$

$$= \frac{-52}{20\sqrt{10}}$$

$$= -0.82219\ldots$$

$$\theta = 145.30\ldots$$

$$\theta = 35° \text{ (nearest degree)}$$

(2) Correct solution
(1) finds obtuse θ or significant progress

4 b) For the 3kg block

$$F = ma$$

$$3a = T - 3g$$

$$3a = 37.5 - 3g$$

$$a = 12.5 - 9.8$$

$$a = 2.7 \text{ ms}^{-2}$$

For the M kg block

$$F = ma$$

$$Ma = Mg - T$$

$$M(9.8 - 2.7) = 37.5$$

$$M = 5.28 \text{ kg (2 decimal places)}$$

(3) Correct solution
(2) Finds valid equations for both blocks and resolves into a value for m.
(1) Finds a valid equation for the 3kg block and obtains 8.1N or 2.7 ms^{-2}.