BAULKHAM HILLS HIGH SCHOOL

YEAR 12 TASK 3 2022

Mathematics Extension 1

General Instructions
- Reading time – 5 minutes
- Working time – 1 hour
- Write using black pen
- Calculators approved by NESA may be used
- Show all necessary working in questions 5-7
- Marks may be deducted for careless or badly arranged work

Total marks – 35
Exam consists of 7 pages.

This paper consists of TWO sections.

Section I – (4 marks) Pages (2-3)
Questions 1-4
- Attempt Questions 1-4
- Answer on answer sheet provided
- Allow about 7 minutes for this section

Section II – (31 marks) Pages (4-7)
- Attempt Questions 5-7
- Allow about 53 minutes
- Answer each question on the appropriate page of your answer booklet

Section I - 4 marks
Use the multiple choice answer sheet for questions 1-4

1. A stone is thrown at an angle of θ to the horizontal. The position of the stone at time t seconds is given by $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$ where $g\,m/s^2$ is the acceleration due to gravity and $v\,m/s$ is the initial velocity of projection. What is the maximum height reached by the stone?

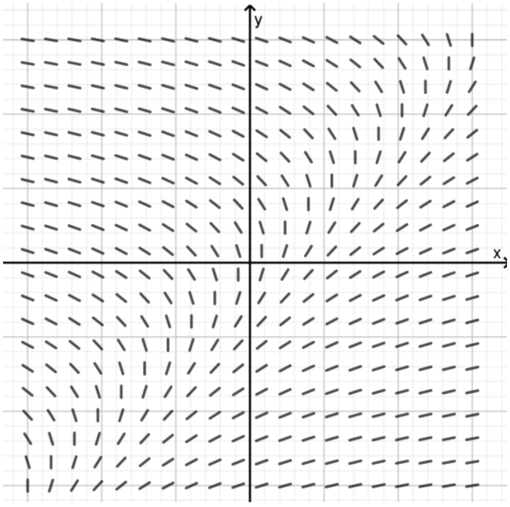
A) $\dfrac{V\sin\theta}{g}$

B) $\dfrac{g\sin\theta}{g}$

C) $\dfrac{V^2\sin^2\theta}{2g}$

D) $\dfrac{g\sin^2\theta}{2V^2}$

2.



Which of the following differential equations could produce the slope field shown above?

A) $\dfrac{dy}{dx} = x - y$

B) $\dfrac{dy}{dx} = \dfrac{1}{x-y}$

C) $\dfrac{dy}{dx} = y - x$

D) $\dfrac{dy}{dx} = \dfrac{1}{y-x}$

3.	Which of the following differential equations has $y = 5xe^{2x}$ as a solution? A) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ B) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$ C) $\frac{d^2y}{dx^2} - 4y = 20e^{2x}$ D) $\frac{d^2y}{dx^2} - 4y = e^{2x}$
4.	The logistic equation $\frac{dP}{dt} = a\left(1 - \frac{P}{k}\right)P$ Can be used to model the size of a population *P*, over time *t*. If a and *k* are both positive constants and $0 < P < k$, which of the following statements must be true? A) The population is increasing B) The population is constant C) The population is decreasing D) None of the above.

End of Section I

Section II

31 marks
Attempt Questions 5-7
Allow about 53 minutes for this section

Answer each question on the appropriate page of your answer booklet. Extra paper is available. Your responses should include relevant calculations and/or mathematical reasoning.

Question 5 (7 marks) **Marks**

a) Solve the following Differential equation given by

$\frac{dy}{dx} + 1 = \cos x$ **2**

b) Find the general solution of the differential equation given by **2**

$$\frac{dy}{dx} = e^{x-y}.$$

c) For the Differential equation $y' = \frac{6}{5x^2 + 4x - 1}$

i. Show that $y' = \frac{5}{5x-1} - \frac{1}{x+1}$ **1**

ii. Find the solution to the differential equation when $x = \frac{1}{2}$ and $y = 3$. **2**

Question 6 (13 marks) **Marks**

a) After time t years the number N of animals in a national park decreases according to the equation: $\frac{dN}{dt} = -0.09(N-100)$.

The initial number of animals in the national park is 500.

i. Verify that $N = 100 + Ae^{-0.09t}$ is a solution of the above equation, where A is a constant. **1**

ii. After one year the number of animals in the national park is 400. Find the time taken for the number of animals to reach 200. Answer correct to three significant figures. **2**

b) A heated metal ball is dropped into a liquid. As the ball cools, $T^\circ C$, t minutes after it enters the liquid, is given by $T = 400e^{-0.05t} + 25,\ t \geq 0$.

i. Find the temperature of the ball as it enters the liquid. **1**

ii. Find the value of t if T = 300. Answer correct to 3 significant figures. **1**

iii. Find the rate at which the temperature of the ball is decreasing at the instant when $t = 50$. **2**

iv. Using the equation for temperature T in terms of t given above, explain why the temperature of the ball can never fall to $20^\circ C$. **1**

Question 5 is continued on the next page

c) When the particle is fired in the open from a point at a speed of $40\,\mathrm{ms}^{-1}$ and at an angle of θ above the horizontal, where $0 < \theta < \frac{\pi}{2}$. You may assume without proof that the horizontal displacement (*x* metres) of the particle from O at time t seconds after firing, are given by $x = 40t\cos\theta$ and $y = 40t\sin\theta - 5t^2$.

The particle is fired with the same speed from a point O on the floor of a horizontal tunnel that has a height of 20 metres.

(You may assume the projectile has no size and can reach the maximum height of 20 metres without touching the roof of the tunnel)

i. Show that the time taken for the projectile to reach the maximum height is $t = 4\sin\theta$. **1**

ii. Find the angle of projection for this maximum height in the tunnel. **2**

iii. Show that the exact maximum horizontal range for the projectile in this tunnel is $80\sqrt{3}$. **2**

Question 7 (11 marks) **Marks**

a) A particle is projected upwards from the top of a building 12 metres high with an initial velocity of 40 *m/s*. The particle is launched at an angle of 30° with the horizontal. Air resistance is ignored and use $g = 9.8 m/s^2$ for the acceleration due to gravity.

i. If the origin is at the base of the building, show that the displacement vector of the particle is : **3**

$$\underset{\sim}{s}(t) = 20\sqrt{3}t\underset{\sim}{i} + \left(-4.9t^2 + 20t + 12\right)\underset{\sim}{j}.$$

ii. Show that the time of the flight is 4.61 seconds, correct to 2 decimal places. **2**

iii. Determine the speed and the angle (correct to the nearest degree) at which the particle hits the ground. **2**

b) The population *P(t)* 0f bacteria in a petri dish is modelled by the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{6}\left(1 - \frac{P}{8000}\right) \text{ where } P(0) = P_0 \text{ and } t \text{ is time in hours.}$$

i) If the initial population P_0 is 1000 bacteria, show that $P(t) = \dfrac{8000}{1 + 7e^{-\frac{t}{6}}}$. **3**

(You may use the fact that $\dfrac{Q}{P(Q-P)} = \dfrac{1}{P} + \dfrac{1}{Q-P}$)

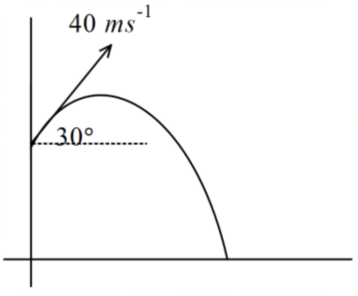
ii) If instead of the initial population P_0 were 12000 bacteria, describe what would have happened to the population as t increases. **1**

End of Exam

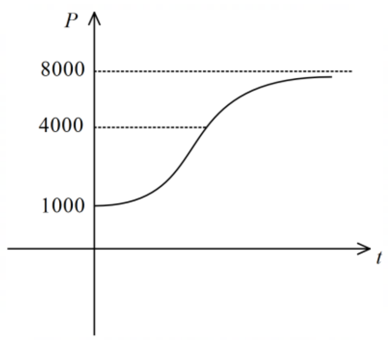
HSC Year12 Extension 1 Task3 2022(suggested marking guidline)

	Solutions	Mks	Comments
5a	$\frac{dy}{dx}=\cos x-1$ $\int dy=\int(\cos x-1)dx$ $y=\sin x-x+c$	**1**	**1 mark** • correct answer.
5b	$\frac{dy}{dx}=e^{x}e^{-y}$ $\frac{dy}{e^{-y}}=e^{x}dx$ $\int e^{y}dy=\int e^{x}dx$ $e^{y}=e^{x}+c$ $y=\ln(e^{x}+c)$	**2**	**2 mark** • correct solution. **1 mark** • separating the variables.
5c(i)	$\frac{5}{5x-1}-\frac{1}{x+1}=\frac{5x+5-5x+1}{5x^{2}+4x-1}$ $=\frac{6}{5x^{2}+4x-1}$	**1**	**1 mark** Adequate steps to show equality.
5c(ii)	$y=\int\frac{6}{5x^{2}+4x-1}dx=\int\left(\frac{5}{5x-1}-\frac{1}{x+1}\right)dx$ $=\ln(5x-1)-\ln(x+1)+C$ $=\ln\left(\frac{5x-1}{x+1}\right)+C$ $when\, x=\frac{1}{2},\ y=3$ $3=\ln\left(\frac{\frac{5}{2}-1}{\frac{1}{2}+1}\right)+C$ $3=\ln\left(\frac{\frac{3}{2}}{\frac{3}{2}}\right)+C$ $3=\ln(1)+C$ $3=0+C$ $C=3$ $y=\ln\left(\frac{5x-1}{x+1}\right)+3$	**2**	**2 mark** • Correct answer with valid working **1 mark** • finding the correct integration.

	Solutions	Mks	Comments
6b(i)	Ball enters liquid when $t = 0$ $T = 400e^{-0.05t} + 25$ $= 400e^{-0.05\times 0} + 25 = 425^{\circ}C$	**1**	**1 mark** • Correct answer.
6b(ii)	$300 = 400e^{-0.05t} + 25$ $e^{-0.05t} = \frac{275}{400}$ $-0.05t = \ln\left(\frac{11}{16}\right)$ $t = 7.4938 \approx 7.49\,\text{min}$	**1**	**1 mark** • Correct answer.
6b(iii)	$T = 400e^{-0.05t} + 25$ $\frac{dT}{dt} = -20^{-0.05t}$ $= -20^{-0.05\times 50}$ $\approx -1.64^{\circ}C$ per minute.	**2**	**2 mark** • correct solution. **1 mark** • Differentiates correctly to find the rate of change.
6b (iv)	When t approaches infinfinity then $e^{-0.05t} \to 0$ $\therefore T > 25$ and can never fall to 20°.	**1**	**1 mark** • correct answer.
6c(i)	$y = 40t\sin\theta - 5t^2$ $y' = 40\sin\theta - 10t$ For maximum height $y' = 0$ $40\sin\theta - 10t = 0$ $10t = 40\sin\theta$ $t = 4\sin\theta$	**1**	**1 marks** • correct solution.
6c(ii)	Maximum height $y = 20$ $y = 40t\sin\theta - 5t^2$ $20 = 40t(4\sin\theta)\sin\theta - 5(4\sin\theta)^2$ $20 = 160\sin^2\theta - 80\sin^2\theta$ $\sin^2\theta = \frac{1}{4}$ $\sin\theta = \pm\frac{1}{2}$ $\theta = \frac{\pi}{6}$	**2**	**2 mark** • correct solution. **1 mark** • significant progress towards the solution.

	Solutions	Mks	Comments
6c(iii)	Max range : $y=0$ and at $2\times$ max height$(t=8\sin\theta)$ $x=40t\cos\theta$ $x=40(8\sin\theta)\cos\theta$ $x=320\sin\theta\cos\theta$ $=160(2\sin\theta\cos\theta)$ $=160\sin 2\theta$ $=\sin 2\left(\frac{\pi}{6}\right)$ $=160\sin\left(\frac{\pi}{3}\right)$ $=80\sqrt{3}m$	**2**	**2 mark** • correct solution. **1 mark** • significant progress towards the solution.
7a(i)	 *When* $t=0$ $V(0)=40\cos 30\,\underset{\sim}{i}+40\sin 30\,\underset{\sim}{j}$ $=20\sqrt{3}\,\underset{\sim}{i}+20\,\underset{\sim}{j}$ *When* $t=0$ $20\sqrt{3}\,\underset{\sim}{i}+20\,\underset{\sim}{j}=C$ $\therefore V(t)=-9.8t\,\underset{\sim}{j}+20\sqrt{3}\,\underset{\sim}{i}+20\,\underset{\sim}{j}$ $s(t)=\int(20\sqrt{3}\,\underset{\sim}{i}+(-9.8t+20)\,\underset{\sim}{j}\,dt$	**3**	**3 marks** • correct solution. **2 mark** • finds the correct expression for *V(t)* **1 mark** • show some understanding

	Solutions	Mks	Comments
	$= 20\sqrt{3}t\,\underset{\sim}{i} + (\frac{-9.8t^2}{2} + 20t)\,\underset{\sim}{j} + C$ When $t = 0 \quad s(t) = 12\,\underset{\sim}{j}$ $C = 12\,\underset{\sim}{j}$ $s(t) = 20\sqrt{3}t\,\underset{\sim}{i} + (-4.9t^2 + 20t)\,\underset{\sim}{j} + 12\,\underset{\sim}{j}$ $s(t) = 20\sqrt{3}t\,\underset{\sim}{i} + (-4.9t^2 + 20t + 12)\,\underset{\sim}{j}$		
7a(ii)	The time of flight is when the y component of the displacement is 0. $i.e. \quad -4.9t^2 + 20t + 12 = 0$ $t = \frac{-20 \pm \sqrt{20^2 - 4(-4.9)12}}{2(-4.9)}$ $= -0.53 \text{ or } 4.61 \text{seconds}$ $but\, t > 0$ $\therefore t = 4.61 \text{seconds}$	**2**	**2 mark** • correct solution. **1 mark** • significant progress towards the solution.
7a(iii)	$V(t) = 20\sqrt{3}\,\underset{\sim}{i} + (-9.8t + 20)\,\underset{\sim}{j}$ $when\, t = 4.61$ $V(4.61) = 20\sqrt{3}\,\underset{\sim}{i} + (-9.8 \times 4.61 + 20)\,\underset{\sim}{j}$ $= 20\sqrt{3}\,\underset{\sim}{i} + -25.178\,\underset{\sim}{j}$ $\therefore velocity = \sqrt{\left(20\sqrt{3}\right)^2 + \left(-25.178\right)^2}$ $= 42.83\, m/s$ Angle of inclination : $\tan\theta = \frac{-25.178}{20\sqrt{3}}$ $\theta = -36^\circ \text{ } or \text{ } 36^\circ$ $\therefore$ the particle hits the ground with a velocity of $42.83\, m/s$ at an angle of 144° with the horizontal.	**2**	**2 mark** • correct solution. **1 mark** • significant progress towards the solution.

	Solutions	Mks	Comments
7b(i)	$\frac{dP}{dt}=\frac{P}{6}\left(1-\frac{P}{8000}\right)$ $\frac{dt}{dP}=\frac{48000}{P(8000-P)}$ $\int dt=6\int\frac{8000}{P(8000-P)}dP$ $\frac{1}{6}\int dt=\int\left(\frac{1}{P}+\frac{1}{8000-P}\right)dP$ using $\frac{Q}{P(Q-P)}=\frac{1}{P}+\frac{1}{Q-P}$ $\frac{1}{6}t+C=\ln\lvert P\rvert-\ln\lvert 8000-P\rvert$ $\frac{1}{6}t+C=\ln\left\lvert\frac{P}{8000-P}\right\rvert$ $e^{c}e^{\frac{t}{6}}=\left\lvert\frac{P}{8000-P}\right\rvert$ $\frac{P}{8000-P}=\pm e^{c}e^{\frac{t}{6}}$ $\frac{P}{8000-P}=Ae^{\frac{t}{6}}, where\ A=\pm e^{c}$ $When\, t=0, P=1000$ $\frac{1000}{8000-1000}=Ae^{0}$ $\therefore A=\frac{1}{7}$ $\therefore\frac{P}{8000-P}=\frac{1}{7}e^{\frac{t}{6}}$ $\therefore P=\frac{8000}{1+7e^{-\frac{t}{6}}}$	**3**	**3 mark** • correct solution. **2 mark** • Finding the expression $\frac{P}{8000-P}=\pm e^{c}e^{\frac{1}{6}}$ **1 mark** • Separating the variables.
7b(ii)	Trivial solutions 0 and 8000.  	**2**	**1 mark** • correct trivial solutions. **1 mark** • correct graph
7b(iii)	The carrying capacity for the population is 8000. If $P(0)>8000$, such as12000, the population will decay and approach the asymptotic population size of 8000 as time increases.	**1**	**1 mark** • correct explanation

	Solutions	Mks	Comments
	Alternatively: The equation for $\frac{dP}{dt}$ shows that when $P = 12000, \frac{dP}{dt} < 0$ (population will decrease). As *P* decreases and approaches 8000, $\frac{dP}{dt} \to 0$. Thus, the population will decrease and will approach 8000.		