BAULKHAM HILLS HS PERSEVERE

2022	HSC Assessment Task 2

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 1 hour
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 6 - 8, show relevant mathematical reasoning and/or calculations

Total marks: 36

Section I – 5 marks (pages 2 – 3)

- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II – 31 marks (pages 4 - 6)

- Attempt Questions 6 - 8
- Allow about 52 minutes for this section

Section I

5 marks
Attempt Questions 1 – 5
Allow about 8 minutes for this section

Use the multiple-choice answer page in the writing booklet for Questions 1 – 5.

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1 Which of the following is the derivative of $\tan^{-1}(2x-1)$?

A. $\dfrac{1}{4x^2-4x+2}$

B. $\dfrac{1}{2x^2-2x+1}$

C. $\dfrac{2}{4x^2-2x+1}$

D. $\dfrac{2}{2x^2-2x+1}$

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2 Which of the following expressions is equal to $\int \cos^2 x\,dx$?

A. $\dfrac{\cos^3 8x}{16}+c$

B. $\dfrac{1}{24}(12x+\cos 12x)+c$

C. $\dfrac{1}{24}(12x+\sin 12x)+c$

D. $\dfrac{1}{24}(12x-\sin 12x)+c$

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3 Using the substitution $u=\tan x$, $\displaystyle\int \frac{\sec^2 x}{\sec^2 x-3\tan x+1}dx$ can be expressed as:

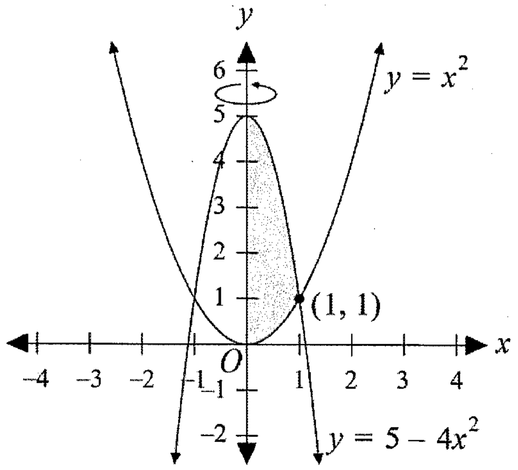
A. $\displaystyle\int \frac{du}{u^2-3u+2}$

B. $\displaystyle\int \frac{du}{u^2-3u}$

C. $\displaystyle-\int \frac{du}{u^2+3u-2}$

D. $\displaystyle-\int \frac{du}{u^2+3u}$

4 A solid of revolution is formed by rotating the shaded area about the y-axis.



Which of the following integrals would give the volume of this solid?

A. $V = \pi \int_0^1 \left[\left(5 - 4x^2\right)^2 - x^4 \right] dx$

B. $V = \pi \int_0^5 \left(\frac{5 - 5y}{4} \right) dy$

C. $V = \pi \left[\int_0^1 x^4 \, dx + \int_1^5 \left(5 - 4x^2\right)^2 dx \right]$

D. $V = \pi \left[\int_0^1 y \, dy + \frac{1}{4} \int_1^5 (5 - y) \, dy \right]$

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5 Which of the following equations is equivalent to $\cos x + \sin x = \sqrt{2}$?

A. $\cos\left(x + \frac{\pi}{4}\right) = 1$

B. $\cos\left(x + \frac{3\pi}{4}\right) = 1$

C. $\cos\left(x + \frac{5\pi}{4}\right) = 1$

D. $\cos\left(x + \frac{7\pi}{4}\right) = 1$

End of Section I

Section II

31 marks
Attempt Questions 6 – 8
Allow about 52 minutes for this section

Answer each question in the appropriate section of the writing booklet. Extra writing paper is available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 marks) Use the Question 6 section of the writing booklet.

(a) Given $f(x)$, find $f'(x)$ for each of the following:

(i) $f(x)=\left(\cos^{-1}x\right)^4$ **2**

(ii) $f(x)=x\sin^{-1}(2x)$ **2**

(b) Find $\int \frac{5}{2+x^2}\,dx$. **2**

(c) Evaluate $\int_0^{\frac{\pi}{3}} \cos^5 x \sin x\,dx$. **2**

(d) Use the substitution $u=1+\tan x$ to find $\int_0^{\frac{\pi}{4}} \frac{1}{(1+\tan x)^2\cos^2 x}\,dx$. **3**

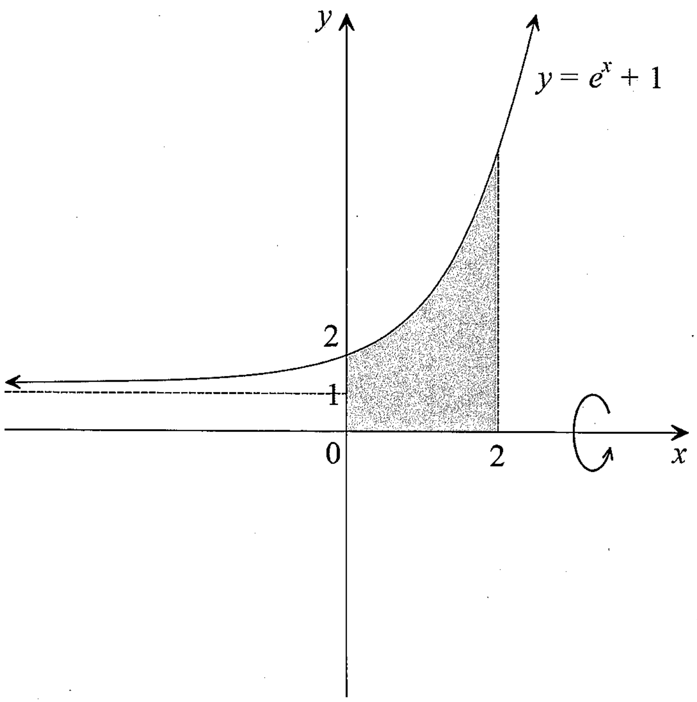
End of Question 6

Question 7 (8 marks) Use the Question 7 section of the writing booklet.

(a) Use the substitution $t = \tan\frac{x}{2}$ to solve the equation $\sin x - 7\cos x = 5$ for $-\pi < x \leq \pi$. **3**

Give your answers correct to three decimal places.

(b) The region bounded by the curve $y = e^x + 1$, the x-axis, the y-axis and the line $x = 2$ is rotated about the x-axis. **3**



Find the volume of the solid of revolution formed.

(c) Use the substitution $u = x^4$ to find $\int \frac{2x^3}{\sqrt{1-x^8}}\,dx$. **2**

End of Question 7

Question 8 (12 marks) Use the Question 8 section of the writing booklet.

(a) (i) Write $\sin\theta - \sqrt{3}\cos\theta$ in the form $R\sin(\theta-\alpha)$, where $R>0$ and $0<\alpha<\dfrac{\pi}{2}$. **2**

(ii) Hence solve the equation $\sin x - \sqrt{3}\cos x = 1$ for $0 \le x \le 2\pi$. **2**

(b) (i) Using a formula from the reference sheet, or otherwise, express $\sin 3x + \sin x$ in the form $2\sin A\cos B$. **1**

(ii) Hence or otherwise find all solutions of $\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$. **3**

(c) (i) Show that $\tan\left(\theta + \dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}\tan\theta + 1}{\sqrt{3} - \tan\theta}$. **1**

(ii) Hence or otherwise, solve for $0 \le \theta \le \pi$: **3**

$$\sqrt{3}\tan\theta + 1 = \left(\sqrt{3} - \tan\theta\right)\tan(\pi - \theta)$$

End of Examination

SECTION I

XI H2 2022 Solutions.

1. $\frac{d}{dx}\left(\tan^{-1}(2x-1)\right) = \frac{2}{1+(2x-1)^2}$

$= \frac{2}{4x^2-4x+2}$

$= \frac{1}{2x^2-2x+1}$ (B)

2. $\int \cos^2 6x \, dx = \int \frac{1}{2}(1+\cos 12x)\, dx$

$= \frac{1}{2}\left(x + \frac{1}{12}\sin 12x\right) + c$

$= \frac{1}{24}(12x + \sin 12x) + c$ (C)

3. $\int \frac{\sec^2 x}{1+\tan^2 x - 3\tan x + 1}\, dx$ $\quad u = \tan x$, $du = \sec^2 x \, dx$

$= \int \frac{du}{u^2 - 3u + 2}$ (A)

4. $V_1 = \pi \int_0^1 x^2 \, dy$ $\quad V_2 = \pi \int_1^5 x^2 \, dy$ $\quad \begin{cases} y = 5 - 4x^2 \\ x^2 = \frac{5-y}{4} \end{cases}$

$= \pi \int_0^1 y \, dy$ $\quad = \pi \int_1^5 \frac{5-y}{4}\, dy$

$\therefore$ Total $V = V_1 + V_2$ = (D)

5. $\cos x + \sin x = R\cos(x+\alpha)$

$= R\cos x \cos\alpha - R\sin x \sin\alpha$

$R\cos\alpha = 1$

$-R\sin\alpha = 1$

$R = \sqrt{2}$

$\tan\alpha = -1$

$\cos\alpha > 0$, $\sin\alpha < 0$

$\therefore$ 4th quad.

$= \sqrt{2}\cos\left(x + \frac{7\pi}{4}\right)$ $\quad = \sqrt{2}$

when $\cos\left(x + \frac{7\pi}{4}\right) = 1$ (D)

Part II

Q6 a) (i) $f(x) = (\cos^{-1} x)^4$

$f'(x) = 4(\cos^{-1}x)^3 \cdot \frac{-1}{\sqrt{1-x^2}}$

$= \frac{-4(\cos^{-1}x)^3}{\sqrt{1-x^2}}$

(2) correct solution
(1) Significant progress using chain rule

(ii) $f(x) = x \sin^{-1} 2x$

$f'(x) = x \cdot \frac{2}{\sqrt{1-4x^2}} + \sin^{-1}2x \cdot 1$

$= \frac{2x}{\sqrt{1-4x^2}} + \sin^{-1} 2x$

(2) Correct soln
(1) Significant progress using product rule

b) $\int \frac{5}{2+x^2}\, dx = 5\int \frac{1}{2+x^2}\, dx$

$= \frac{5}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + c$

(2) Corect answer
(1) Uses inv tan

c) $\int_0^{\pi/3} \cos^5 x \cdot \sin x \cdot dx$

$= \frac{-1}{6}\left[\cos^6 x\right]_0^{\pi/3}$

(2) Correct soln
(1) Obtains cos^6(x) in integrand

$= -\frac{1}{6}\left(\left(\cos\frac{\pi}{3}\right)^6 - (\cos 0)^6\right)$

$= -\frac{1}{6}\left(\frac{1}{64} - 1\right)$

$= \frac{63}{384} = \frac{21}{128}$

d) $u = 1+\tan x$
$du = \sec^2 x \, dx$
$x = 0 : u = 1$
$x = \frac{\pi}{4} : u = 1+1 = 2$

$$\int_0^{\frac{\pi}{4}} \frac{1}{(1+\tan x)^2} \cdot \sec^2 x \cdot dx$$

$$= \int_1^2 \frac{1}{u^2} \, du$$

$$= \left[-\frac{1}{u} \right]_1^2$$

(3) Correct soln
(2) Correct primitive
(1) Correctly substitutes u and du

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}$$

Q7

a) $\sin x - 7\cos x = 5 \quad (-\pi \leq \theta \leq \pi)$

$$\frac{2t}{1+t^2} - \frac{7(1-t^2)}{1+t^2} = 5$$

(3) Correct solution
(2) Solves quad eqn
(1) Obtains quadratic eqn

$$\cancel{2t - 7 + 7t^2 = 5 + 5t^2}$$

$$2t^2 + 2t - 12 = 0$$

$$t^2 + t - 6 = 0$$

$$(t+3)(t-2) = 0$$

$$\tan\frac{x}{2} = -3 \quad \text{or} \quad \tan\frac{x}{2} = 2.$$

$$\frac{x}{2} = -1.249 \qquad \frac{x}{2} = 1.107$$

$$x = -2.498 \qquad x = 2.214$$

Now test if $x = \pi$:

$$\text{LHS} = \sin\pi - 7\cos\pi$$

$$= 0 - 7(-1)$$

$$\neq 5 \quad \therefore \text{Not a soln.}$$

$$\therefore x = -2.498, 2.214$$

b) $V = \pi \int_0^2 y^2 \, dx$

$y = e^x + 1$
$y^2 = e^{2x} + 2e^x + 1$

$$= \pi \int_0^2 e^{2x} + 2e^x + 1 \, dx$$

(3) Correct soln
(2) Correct primitive
(1) Correctly applies volume formula and expands y^2

$$= \pi \left(\frac{1}{2}e^{2x} + 2e^x + x \right)_0^2$$

$$= \pi \left(\left(\frac{1}{2}e^4 + 2e^2 + 2 \right) - \left(\frac{1}{2} + 2 + 0 \right) \right)$$

$$= \pi \left(\frac{1}{2}e^4 + 2e^2 - \frac{1}{2} \right) \cdot \text{units}^3$$

7c) $\int \frac{2x^3}{\sqrt{1-x^8}} \, dx$

$u = x^4$
$du = 4x^3 \, dx$

(2) Correct soln
(1) Correctly substitutes u and du

$$= \frac{1}{2} \int \frac{4x^3}{\sqrt{1-(x^4)^2}} \, dx$$

$$= \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2} \sin^{-1} u + c$$

$$= \frac{1}{2} \sin^{-1}(x^4) + c$$

Q8

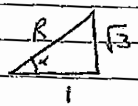
a)(i) $\sin\theta - \sqrt{3}\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

(2) Finds R and alpha
(1) Finds R or alpha

$\cancel{R}\cos\alpha = 1/R$

$\cancel{R}\sin\alpha = \sqrt{3}/R$



$R = 2$

$\alpha = \frac{\pi}{3}$

$\therefore \sin\theta - \sqrt{3}\cos\theta = 2\sin(\theta - \frac{\pi}{3})$

(ii) $\sin x - \sqrt{3}\cos x = 1$

$\cancel{2}\sin(x - \frac{\pi}{3}) = \frac{1}{2}$

$\left(\begin{matrix} -\pi < x < \pi \\ -\frac{4\pi}{3} < x - \frac{\pi}{3} < \frac{2\pi}{3} \end{matrix}\right)$

$x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}$

(2) Correct solns
(1) Significant progress

$x = \frac{\pi}{2}, \frac{7\pi}{6}$

b)

(i) From ref. sheet: $\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$

Let $A = 2x$, $B = x$

(1) Correct expression

$2\sin 2x\cos x = \sin 3x + \sin x$

(ii) $\sin x + \sin 2x + \sin 3x = 0$

$2\sin 2x\cos x + \sin 2x = 0$

$\sin 2x(2\cos x + 1) = 0$

$\sin 2x = 0$ $\qquad$ $\cos x = -\frac{1}{2}$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$ $\qquad$ $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(7 solns)

(3) All correct solns
(2) Solving sin2x=0 and cosx=-1/2 (but some solns be missing)
(1) Writing sin2x as a common factor
(1) Solve cosx=-1/2

8. c)(i) $\tan(\theta + \frac{\pi}{6}) = \dfrac{\tan\theta + \tan\frac{\pi}{6}}{1 - \tan\theta\tan\frac{\pi}{6}}$

$= \dfrac{\tan\theta + \frac{1}{\sqrt{3}}}{1 - \tan\theta\cdot\frac{1}{\sqrt{3}}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{\sqrt{3}\tan\theta + 1}{\sqrt{3} - \tan\theta}$

(1) correct soln.

(ii) $\sqrt{3}\tan\theta + 1 = (\sqrt{3} - \tan\theta)\tan(\pi - \theta)$

$\dfrac{\sqrt{3}\tan\theta + 1}{\sqrt{3} - \tan\theta} = \tan(\pi - \theta)$

$\tan(\theta + \frac{\pi}{6}) = \tan(-\theta)$

$\theta + \frac{\pi}{6} = -\theta$ $\qquad$ $2\theta = -\frac{\pi}{6}$ $\qquad$ $\theta = -\frac{\pi}{12}$ (this is not in correct domain)

OR

$\theta + \frac{\pi}{6} = -\theta + \pi$ $\qquad$ $2\theta = \frac{5\pi}{6}$ $\qquad$ $\theta = \frac{5\pi}{12}$

OR

$\theta + \frac{\pi}{6} = -\theta + 2\pi$ $\qquad$ $2\theta = \frac{11\pi}{6}$ $\qquad$ $\theta = \frac{11\pi}{12}$

OR

$\theta + \frac{\pi}{6} = -\theta + 3\pi$ $\qquad$ $2\theta = \frac{17\pi}{6}$ $\qquad$ $\theta = \frac{17\pi}{12}$ (also not in correct domain)

Solution $\theta = \frac{5\pi}{12}, \frac{11\pi}{12}$

(3) correct solutions.
(2) Obtains a correct solution (regardless of domain).
(1) Uses result from (i)