BAULKHAM HILLS
HIGH
SCHOOL

2024

YEAR 12
H3: ASSESSMENT TASK 3

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 6 – 8, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for careless or badly arranged work

Total marks: 39

Section I – 5 marks (pages 2 – 3)
- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II – 34 marks (pages 4 – 7)
- Attempt Questions 6 – 8
  Allow about 82 minutes for this section

Section I

5 marks
Attempt Questions 1 – 5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 5

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1 Which of the following is **NOT** a first order linear differential equation?

(A) $\frac{dy}{dx} = x^2 + x + 1$

(B) $2x\frac{dy}{dx} - y^3 = 0$

(C) $e^{2x}\frac{dy}{dx} - 4e^{3x}y = \cos x$

(D) $2(y - 4x^2)dx + xdy = 0$

2 If $\frac{dy}{dt} = ky$ and k is a non-zero constant, which of the following could be an equation for y?

(A) $kty + 2$

(B) $\frac{1}{2}ky^2 + \frac{1}{2}$

(C) $e^{kt} + 2$

(D) $2e^{kt}$

3 A ball is thrown into the air from a point on the ground, at an angle of 30° to the ground and with an initial velocity of 30 m/s.

If air resistance is neglected and the acceleration due to gravity is taken as 10 m/s^2, how long will the ball take to reach its greatest height?

(A) 1.5 seconds

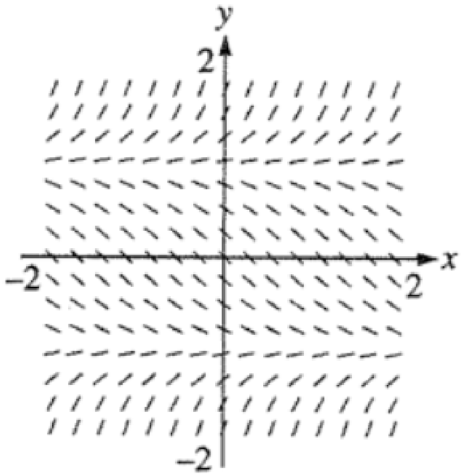
(B) 2.6 seconds

(C) 15 seconds

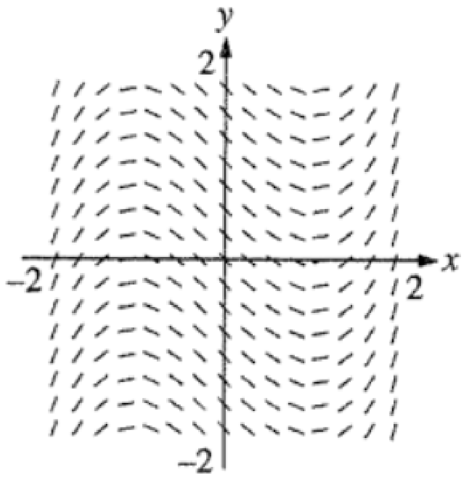
(D) 26 seconds

4 Which of the following could be the slope field for the differential equation $\frac{dy}{dx} = y^2 - 1$?

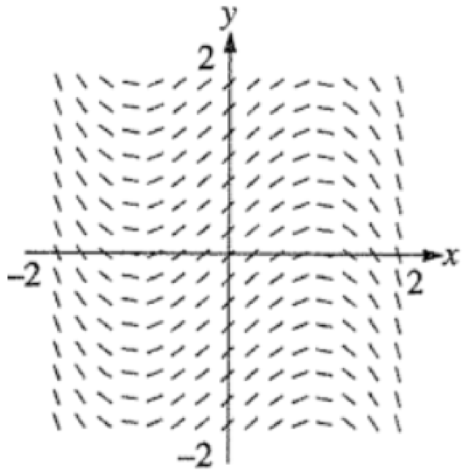
(A)



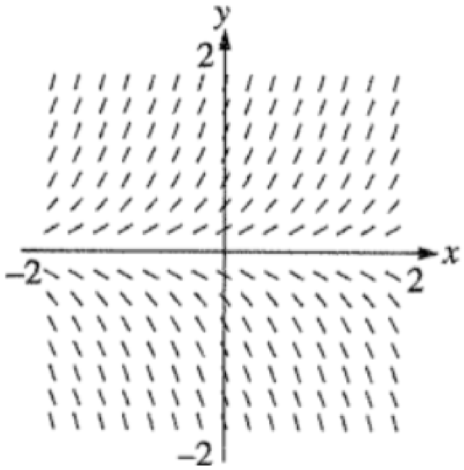
(B)



(C)



(D)



5 Two balls, A and B, are rolled horizontally off a 10 metre cliff at 10 m/s and 20 m/s respectively.

Which of the following statements is **FALSE**?

(A) A and B are in the air for the same amount of time.

(B) Whilst in the air, A and B travel with the same vertical speed.

(C) B's impact speed with the ground is twice A's impact speed.

(D) B lands twice as far from the base of the cliff than A.

END OF SECTION I

Section II

34 marks
Attempt Questions 6 – 8
Allow about 82 minutes for this section

Answer each question on the appropriate answer sheet. Extra paper is available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 marks) Use the pages labelled Question 6 in the answer booklet

(a) Given that $y = e^{2x}$ satisfies the second order differential equation *2*

$$y'' - y' - ky = 0,$$

find the value of k.

(b) The rate at which the water level in a barrel goes down is modelled by the equation *2*

$$\frac{dh}{dt} = -\sqrt{h}$$

where h is the height of the water level above the tap, in metres, and t is the time in minutes.

If there is initially 1 metre of water above the tap, how long does it take for the water flow to stop?

(c) Find the curve that satisfies the differential equation *3*

$$\frac{dy}{dx} = x^2 e^{-y}$$

and passes through the point (0, 1). Express your answer with y as the subject.

(d) A large wasp nest initially had a population of 1000 wasps. The population growth of the nest is given by the logistic equation *4*

$$\frac{dP}{dt} = \frac{P}{5}\left(\frac{C-P}{C}\right)$$

where t is the time in months, P is the population of wasps and C is the carrying capacity of the nest.

After twelve months the population of the nest was 3000 wasps.

Use the fact that $\dfrac{C}{P(C-P)} = \dfrac{1}{P} + \dfrac{1}{C-P}$ to show that the carrying capacity of the nest is approximately 3750 wasps.

Question 7 (15 marks) Use the pages labelled Question 7 in the answer booklet

(a) (i) By using the substitution $u = \frac{y}{x}$, show that the differential equation *2*

$$\frac{dy}{dx} - \frac{y}{x} = \frac{x^3 \cos x}{3y^2}$$

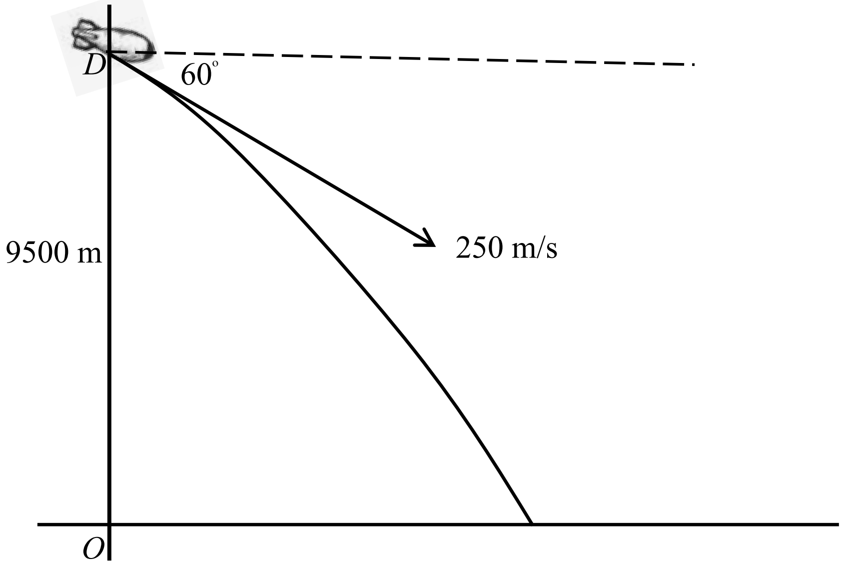
can be transformed to

$$\frac{du}{dx} = \frac{\cos x}{3u^2}$$

(ii) Hence find the general solution to the differential equation *2*

$$\frac{dy}{dx} - \frac{y}{x} = \frac{x^3 \cos x}{3y^2}$$

(b) During an army exercise, a dummy bomb is released from a point D, 9500 metres above the ground at a velocity of 250 m/s and with an angle of incidence of 60°.



In this question, neglect any air resistance and take the acceleration due to gravity to be 10 m/s^2.

(i) Show that the position vector of the dummy bomb, at time t seconds, is given by *3*

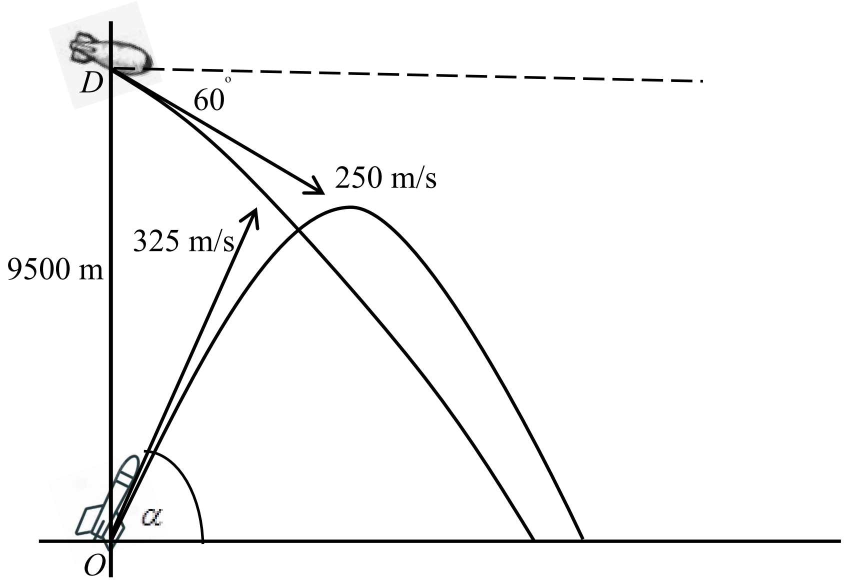
$$\underset{\sim}{r}_D = \begin{pmatrix} 125t \\ 9500 - 125\sqrt{3}\,t - 5t^2 \end{pmatrix}$$

(ii) Calculate how far the dummy bomb has travelled horizontally when it strikes the ground, correct to the nearest metre. *2*

Question 7 (b) continues on page 6

Question 7 (b) (continued)

At the same time the dummy bomb is released, a surface to air missile is launched from the point O that is directly below the dummy bomb, on the ground.



The missile is launched with a velocity of 325 m/s at an angle of elevation of α.

The position vector of the missile at time t seconds is

$$\underset{\sim}{r}_M = \begin{pmatrix} 325t\cos\alpha \\ 325t\sin\alpha - 5t^2 \end{pmatrix} \qquad \text{(Do NOT prove this)}$$

(iii) Show that in order for the missile to intercept the dummy bomb, it must be launched with an angle of projection $\alpha = \cos^{-1}\left(\dfrac{5}{13}\right)$. *1*

(iv) Show that the missile and dummy bomb collide after approximately 18 seconds. *2*

(v) Using this approximation and the dot product, calculate the angle of collision of the missile and the dummy bomb, correct to the nearest degree. *3*

End of Question 7

Question 8 (8 marks) Use the pages labelled Question 8 in the answer booklet

(a) By considering the product rule, or otherwise, find the general solution of *2*

$$\frac{dy}{dx}\sin x + y\cos x = 1$$

(b) Frosty the snowman is made from two uniform spherical snowballs, initially of radii $2R$ and $3R$. The smaller snowball (which is his head) stands on top of the larger snowball.

As each snowball melts, its volume decreases at a rate which is directly proportional to its surface area, the constant of proportionality being the same for each snowball. During melting, the snowball remains spherical and uniform.

(i) Show that the rate of change of either snowball can be modelled by the differential equation *1*

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -4k\pi r^2$$

where r is the radius of the snowball after t minutes and k is the constant of proportionality.

(ii) Show that Frosty's height after t minutes is given by *3*

$$H = 2(5R - 2kt)$$

(iii) Hence, or otherwise, show that when Frosty is half his initial height, the the ratio of his volume to his initial volume is 37:224. *2*

End of paper

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BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 1 H3: ASSESSMENT TASK 3 2024 SOLUTIONS

Solution	Marks	Comments
SECTION I		
1 **B** – A first order linear differential equation can be written in the form $$\frac{dy}{dx}+f(x)y=g(x)$$ Option B cannot be written is this form as y is to the power of 3. *Note:* A: is in the correct form with $f(x)=0$ and $g(x)=x^2+x+1$ C: can be rearranged to the correct form, $e^{2x}\frac{dy}{dx}-4e^{3x}y=\cos x$ $$\frac{dy}{dx}-4e^{x}y=e^{-2x}\cos x$$ D: the variables have been separated, it can be rearranged to the correct form, $$2(y-4x^2)dx+xdy=0$$ $$xdy=2(4x^2-y)dx$$ $$\frac{dy}{dx}=2\frac{(4x^2-y)}{x}dx$$ $$\frac{dy}{dx}+\frac{2}{x}y=8x$$	1	
2 **D** – $$\frac{dy}{dt}=ky$$ $$\int\frac{dy}{y}=k\int dt$$ $$\ln\lvert y\rvert=kt+c$$ $$\lvert y\rvert=e^{kt+c}$$ $$\lvert y\rvert=e^{kt}\times e^{c}$$ $$\lvert y\rvert=Ae^{kt}$$ $\therefore\ y=2e^{kt}$ could be an equation for y	1	
3 **A** – greatest height occurs when $\dot{y}=0$ $$\ddot{y}=-10$$ $$\frac{d\dot{y}}{dt}=-10$$ $$\int_{30\sin 30°}^{0}d\dot{y}=-10\int_0^t dt$$ $$0-30\sin 30°=-10t$$ $$t=3\sin 30°$$ $$=1.5$$ $\therefore$ the ball takes 1.5 seconds to reach its greatest height	1	
4 **A** – Since $\frac{dy}{dx}=y^2-1$, the slopes will be negative when $y^2-1<0$ $$-1<y<1$$ The only slope field that satisfies this condition is A.	1	
5 **C** – A and B both have an initial vertical velocity of zero, so will take the same time to hit the ground. This also means that they will hit the ground with the same vertical velocity. However as both A and B travel with a constant horizontal velocity, B will travel twice as far as A, horizontally. Thus C must be the FALSE statement *Note: whilst B is travelling twice the speed of A horizontally, their vertical speeds are the same,*	1	

SECTION II		
Solution	**Marks**	**Comments**
QUESTION 6		
6 (a) $y = e^{2x}$ $y' = 2e^{2x}$ $y'' = 4e^{2x}$ $y'' - y' - ky = 0$ $4e^{2x} - 2e^{2x} - ky = 0$ $e^{2x}(2-k) = 0$ $e^{2x} \neq 0 \quad \therefore 2-k=0$ $k = 2$	**2**	**2 marks** • Correct solution **1 mark** • Finds both the first and second derivative of the given function
6 (b) $\frac{dh}{dt} = -\sqrt{h}$ $-\int_1^0 \frac{dh}{\sqrt{h}} = \int_0^t dt$ $t = \int_0^1 h^{-\frac{1}{2}} dh$ $t = \left[2h^{\frac{1}{2}}\right]_0^1$ $t = 2$ $\therefore$ it takes 2 minutes for the water flow to stop	**2**	**2 marks** • Correct solution **1 mark** • Finds a correct primitive function
6 (c) $\frac{dy}{dx} = x^2 e^{-y}$ $\int_1^y e^y dy = \int_0^x x^2 dx$ $\left[e^y\right]_1^y = \frac{1}{3}\left[x^3\right]_0^x$ $e^y - e = \frac{x^3}{3}$ $e^y = \left(\frac{x^3}{3} + e\right)$ $y = \ln\left(\frac{x^3}{3} + e\right)$	**3**	**3 marks** • Correct solution **2 marks** • Finds a correct primitive function **1 mark** • Separates the variables to create an equation involving integrals

Solution	Marks	Comments
6 (d) $\frac{dP}{dt}=\frac{P}{5}\left(\frac{C-P}{C}\right)$ $5\int_{1000}^{3000}\frac{C}{P(C-P)}dP=\int_0^{12}dt$ $5\int_{1000}^{3000}\left(\frac{1}{P}+\frac{1}{C-P}\right)dP=[t]_0^{12}$ $5\Big[\ln P-\ln(C-P)\Big]_{1000}^{3000}=12$ $\left[\ln\left(\frac{P}{C-P}\right)\right]_{1000}^{3000}=2.4$ $\ln\left(\frac{3000}{C-3000}\times\frac{C-1000}{1000}\right)=2.4$ $\frac{3(C-1000)}{C-3000}=e^{2.4}$ $3C-3000=e^{2.4}(C-3000)$ $C=3000\left(\frac{1-e^{2.4}}{3-e^{2.4}}\right)$ $=3747.833...$ $\therefore$the carrying capacity of the nest is approximately 3750 wasps	**4**	**4 marks** • Correct solution **3 marks** • Simplifies the logs in the equation, or equivalent merit **2 marks** • Integrates both sides correctly, or equivalent merit **1 mark** • Attempts to separate the variables in the differential equation, or equivalent merit
QUESTION 7		
7 (a) (i) let $u=\frac{y}{x}\Rightarrow y=ux$ $\frac{dy}{dx}=u+x\frac{du}{dx}$ $\frac{dy}{dx}-\frac{y}{x}=\frac{x^3\cos x}{3y^2}$ $u+x\frac{du}{dx}-u=\frac{x^3\cos x}{3u^2x^2}$ $\frac{du}{dx}=\frac{\cos x}{3u^2}$	**2**	**2 marks** • Correct solution **1 mark** • Finds a substitution for $\frac{dy}{dx}$ or equivalent merit
7 (a) (ii) $\frac{du}{dx}=\frac{\cos x}{3u^2}$ $3\int u^2du=\int\cos x\,dx$ $u^3=\sin x+c$ $u=(\sin x+c)^{\frac{1}{3}}$ $\frac{y}{x}=(\sin x+c)^{\frac{1}{3}}$ $y=x(\sin x+c)^{\frac{1}{3}}$	**2**	**2 marks** • Correct solution **1 mark** • Finds an expression for u^3

Solution	Marks	Comments
7 (b) (i) Initial conditions: $\dot{x}_D = 250\cos(-60°)$ $\dot{y}_D = 250\sin(-60°)$ $= 125$ $= -125\sqrt{3}$ $x_D = 0$ $y_D = 9\ 500$ $\ddot{x} = 0$ $\ddot{y} = -10$ $\frac{d\dot{x}_D}{dt} = 0$ $\frac{d\dot{y}_D}{dt} = -10$ $\int_{125}^{\dot{x}_D} d\dot{x}_D = 0$ $\int_{-125\sqrt{3}}^{\dot{y}_D} d\dot{y}_D = -10\int_0^t dt$ $\dot{x}_D - 125 = 0$ $\dot{y}_D + 125\sqrt{3} = -10t$ $\dot{x}_D = 125$ $\dot{y}_D = -125\sqrt{3} - 10t$ $\frac{dx_D}{dt} = 125$ $\frac{dy_D}{dt} = -125\sqrt{3} - 10t$ $\therefore \underset{\sim}{r}_D = \begin{pmatrix} 125t \\ 9500 - 125\sqrt{3}\,t - 5t^2 \end{pmatrix}$ $\int_0^{x_D} dx_D = 125\int_0^t dt$ $\int_{9500}^{y_D} dy_D = -\int_0^t \left(125\sqrt{3} + 10t\right)dt$ $x_D = 125t$ $y_D - 9500 = -\left[125\sqrt{3}\,t + 5t^2\right]_0^t$ $y_D = 9500 - 125\sqrt{3}\,t - 5t^2$	3	**3 marks** • Correct solution **2 marks** • Finds expressions for both the horizontal and vertical components of velocity or finds the expression for either the horizontal or vertical component of displacement **1 mark** • Finds the initial horizontal and vertical velocities
7 (b) (ii) Dummy bomb hits the ground when $y_D = 0$ $0 = 9500 - 125\sqrt{3}\,t - 5t^2$ $t^2 + 25\sqrt{3}\,t - 1900 = 0$ $t = \frac{-25\sqrt{3} + \sqrt{1875 + 7600}}{2}$ $(t > 0)$ $= 27.01917074$ $= 27$ seconds (to nearest second) When $t = 27.01917074...$ $x_D = 125 \times 27.01917074...$ $= 3377$ metres (to nearest metre)	2	**2 marks** • Correct solution **1 mark** • Calculating the time it takes for the bomb to hit the ground *Note: no penalty for using the rounded answer (x = 3375)*
7 (b) (iii) Particles collide when $x_D = x_M$ $125t = 325t\cos\alpha$ $\cos\alpha = \frac{125}{325}$ $\alpha = \cos^{-1}\left(\frac{5}{13}\right)$	1	**1 mark** • Correct solution
7 (b) (iv) Particles collide when $y_D = y_M$ $9500 - 125\sqrt{3}\,t - 5t^2 = 325t\sin\alpha - 5t^2$ $\left(325\sin\alpha + 125\sqrt{3}\right)t = 9500$ $t = \frac{9500}{300 + 125\sqrt{3}}$ $= 18.39280385...$ $\therefore$they collide after approximately 18 seconds	2	**2 marks** • Correct solution **1 mark** • Attempts to equate $y_D = y_M$

Solution	Marks	Comments
7 (b) (v) $\underset{\sim}{v}_D=\begin{pmatrix}125\\-125\sqrt{3}-10t\end{pmatrix}$ $\quad$ $\underset{\sim}{v}_M=\begin{pmatrix}325\cos\alpha\\325\sin\alpha-10t\end{pmatrix}$ When $t = 18$; $\underset{\sim}{v}_D=\begin{pmatrix}125\\-125\sqrt{3}-180\end{pmatrix}$ $\quad$ $\underset{\sim}{v}_M=\begin{pmatrix}125\\120\end{pmatrix}$ $\cos\theta=\dfrac{\underset{\sim}{v}_D\cdot\underset{\sim}{v}_M}{\lvert\underset{\sim}{v}_D\rvert\,\lvert\underset{\sim}{v}_M\rvert}$ $=\dfrac{(125)(125)+\left(-125\sqrt{3}-180\right)(120)}{\sqrt{125^2+\left(125\sqrt{3}+180\right)^2}\sqrt{125^2+120^2}}$ $=-0.44359...$ $\theta=116.3332...^\circ$ $\therefore$ the missile and the dummy bomb collide at an angle of 116°	3	**3 marks** • Correct solution **2 marks** • Uses the dot product in a valid attempt to finding the angle or equivalent merit **1 mark** • Finds expressions for the velocity vectors of both the bomb and the missile
QUESTION 8		
8 (a) $\dfrac{dy}{dx}\sin x+y\cos x=1$ $\dfrac{d}{dx}(y\sin x)=1$ $y\sin x=\int dx$ $y\sin x=x+c$ $y=\dfrac{x+c}{\sin x}$	2	**2 marks** • Correct solution **1 mark** • Uses the product rule to simplify the LHS to the derivative of a single function
8 (b) (i) $\dfrac{dV}{dt}=-kS$ $\dfrac{d}{dt}\left(\dfrac{4}{3}\pi r^3\right)=-4k\pi r^2$	1	**1 mark** • Correct solution
8 (b) (ii) $\dfrac{d}{dt}\left(\dfrac{4}{3}\pi r^3\right)=-4k\pi r^2$ $\dfrac{d}{dt}(r^3)=-3kr^2$ $\dfrac{d}{dr}(r^3)\times\dfrac{dr}{dt}=-3kr^2$ $3r^2\times\dfrac{dr}{dt}=-3kr^2$ $\dfrac{dr}{dt}=-k$ **Head Snowball** Let S_H be the radius of the head snowball $\int_{2R}^{S_H}dr=-k\int_0^t dt$ $S_H-2R=-kt$ $S_H=-kt+2R$ **Body Snowball** Let S_B be the radius of the body snowball $\int_{3R}^{S_B}dr=-k\int_0^t dt$ $S_B-3R=-kt$ $S_B=-kt+3R$ $H=2(S_H+S_B)$ $=2(-kt+2R-kt+3R)$ $=2(5R-2kt)$	3	**3 marks** • Correct solution **2 marks** • Finds an expression for the radius of Frosty's head or equivalent merit **1 mark** • Establishes that $\dfrac{dr}{dt}=-k$

Solution	Marks	Comments
8 (b) (iii) Frosty initially has a height of $10R$, so when $H = 5R$ $5R = 2(5R - 2kt)$ $\quad S_H = -\frac{5R}{4} + 2R$ $\quad S_B = -\frac{5R}{4} + 3R$ $5R = 10R - 4kt$ $\quad = \frac{3R}{4}$ $\quad = \frac{7R}{4}$ $kt = \frac{5R}{4}$ $\frac{\text{volume at half height}}{\text{initial volume}} = \frac{\frac{4}{3}\pi\left(\frac{3R}{4}\right)^3 + \frac{4}{3}\pi\left(\frac{7R}{4}\right)^3}{\frac{4}{3}\pi(2R)^3 + \frac{4}{3}\pi(3R)^3}$ $= \frac{\left(\frac{3}{4}\right)^3 + \left(\frac{7}{4}\right)^3}{2^3 + 3^3}$ $= \frac{27 + 343}{64(8 + 27)}$ $= \frac{37}{224}$	**2**	**2 marks** • Correct solution **1 mark** • Finds the value of *kt* when Frosty is half his original height • Finds the initial volume of Frosty