FORT STREET HIGH SCHOOL

Year 12 Mathematics Extension 1

Task 1

2023

Student Name:	
Student Number:	

Teacher	Barton	Kaur	Hussein
	▢	▢	▢

General Instructions

- Reading time – 5 minutes
- Working time – 75 minutes
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided
- Start each question in the correct answer booklet
- For all questions show relevant mathematical reasoning and/or calculations
- Marks may be deducted for poorly set work

Question	Total	Marks
Q1	/11	
Q2	/10	
Q3	/13	
Q4	/10	
Total	**/44**	
	Percent	

Outcomes being assessed:

Syllabus Outcomes	**Assessment Area Description and Marking Guidelines**
ME11-5	uses concepts of permutations and combinations to solve problems involving counting or ordering
ME12-2	applies concepts and techniques involving vectors and projectiles to solve problems
ME11-6	uses appropriate technology to investigate, organise and interpret information to solve problems in a range of contexts
ME12-7	evaluates and justifies conclusions, communicating a position clearly in appropriate mathematical forms

Question 1 - Write your answer in the question 1 answer booklet (11 marks)

(a) Find the magnitude and directional angle (to the nearest degree) of the vector $-5\underset{\sim}{i} + 2\underset{\sim}{j}$. **3**

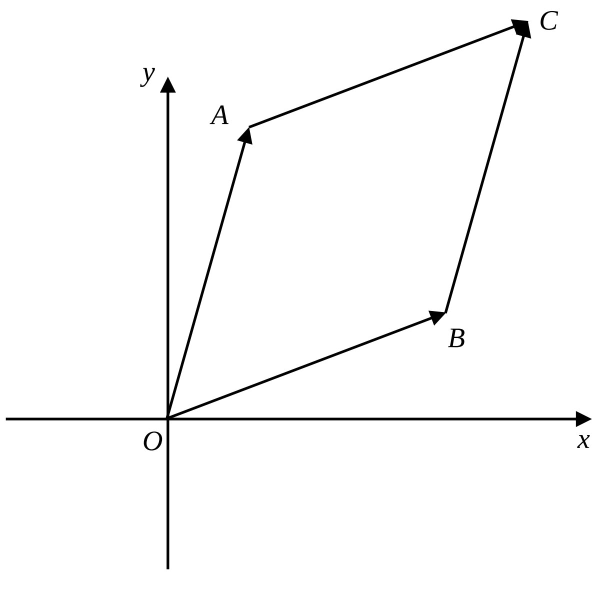
(b) Considering the word SAARANG. How many ways can the letters be uniquely arranged? **1**

(c) $A\ (1,4)$, $B\ (0,1)$, $C\ (-1,-2)$ and $D\ (-3,1)$, Using vectors,

(i) prove that the points A, B, and C are collinear. **2**

(ii) find, to the nearest degree, the angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$. **2**

(d) Using vectors to prove that the diagonals of a rhombus intersect each other at right angle. **3**



Question 2 - Write your answer in the question 2 answer booklet (10 marks)

(a) State the difference between the direction vector and a position vector. **1**

(b) Six people are to be selected from nine people to sit on a round table.

How many possible arrangements are there? **2**

(c) Find the constant term in the expansion of $(x-\frac{1}{3x^3})^{20}$. **2**

(d) A student is given 16 randomly chosen positive integers and is asked to divide them by 5. Show that at least 4 of the numbers will have the same remainder. **2**

(e) Vectors $\underset{\sim}{a}=\begin{bmatrix} x \\ 2 \end{bmatrix}$ and $\underset{\sim}{b}=\begin{bmatrix} 3 \\ y \end{bmatrix}$ are perpendicular to each other. Given that $|\underset{\sim}{a}|=2\sqrt{5}$,

find all possible values of x and y. **3**

Question 3 - Write your answer in the question 3 answer booklet (13 marks)

(a) Ten university students share a house. Every Monday morning, they all leave the house one at a time. If their ages are as follows: one is 17 years old, three are 18 years old, two are 19 years old, three are 20 years old, and one is 21 years old.

(i) How many possible arrangements are there of them leaving the house with no further restrictions? Do not evaluate this. **1**

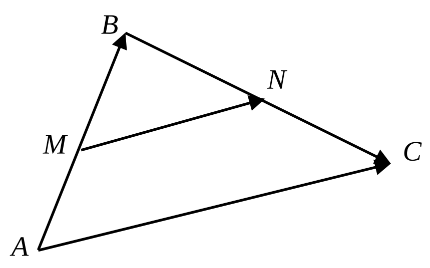
(ii) What is the probability that they leave the house in ascending age order? Leave your answer in factorial notation. **2**

(b) How many four digit numbers greater than 4500 can be formed from 3, 4, 5, 6, 7 if no digits can be repeated. **2**

(c) If $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$ Show that $\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$ **2**

(d) In triangle ABC, M is the midpoint of side AB and N is the midpoint of side BC as shown in the diagram.

Given $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$, using vector notation show that vector MN is parallel to vector AC. **2**



(e) Let $\underset{\sim}{a} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - \underset{\sim}{j}$

(i) Find the projection of $\underset{\sim}{a}$ on to $\underset{\sim}{b}$. **2**

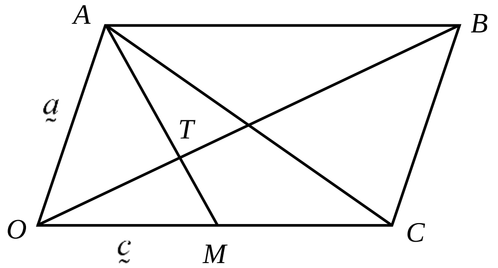
(ii) Given $\underset{\sim}{c} = \underset{\sim}{a} - \text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}$, what is the value of $\underset{\sim}{b} \bullet \underset{\sim}{c}$, giving a reason for your answer. **2**

Question 4 - Write your answer in the question 4 answer booklet (10 marks)

(a) Frank would like to make a bracelet for his partner. The bracelet will be made up of the letter beads: a, e, i, o, d, f, g, h, k, m, and v.
If no more than 2 vowels can be next to each other, how many different arrangements of the beads can he have for the bracelet. **3**

(b) The diagram below shows a parallelogram $OABC$. Let M be the midpoint of OC.
The point T lies at the intersection of of AM and OB such that $AT:TM = 2:1$.
Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OM} = \underset{\sim}{c}$.

Find the ratio $OT:TB$ **4**



(c) Use combinatoric argument to prove that

$\binom{n}{m}\binom{n-m}{k-m} = \binom{n}{k}\binom{k}{m}$ where $n > k > m$. (An algebraic solution will not receive any marks). **3**

End of Paper.

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Solutions

General Instructions

- Reading time – 5 minutes
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Question 1 - Write your answer in the question 1 answer booklet (11 marks)

(a) Find the magnitude and directional angle (to the nearest degree) of the vector $-5\underset{\sim}{i}+2\underset{\sim}{j}$. **3**

$\lvert -5\underset{\sim}{i}+2\underset{\sim}{j}\rvert = \sqrt{25+4}$ $= \sqrt{29}$ Base angle: $\tan\alpha = \frac{2}{5}$ $\alpha = 21.801\ldots^\circ$  $\therefore \theta = 180^\circ - \alpha$ $\doteqdot 158^\circ$ (to nearest degree)	1 – correctly finding the magnitude 1 – finding a base angle 1 – finding the correct angle

Marker's Comments:

Generally well answered. Most students were not clear on the angle and did not communicate how they were measuring the angle. Some students used true bearings or compass bearings. Although this would not be appropriate as no context was given, students were not penalised for this in this assessment unless their compass bearing notation was incorrect.

(b) Considering the word SAARANG. How many ways can the letters be uniquely arranged? **1**

No. of arregements $= \frac{7!}{3!}$ $= 840$	1 – for correct solution

Marker's Comments:

Answered well by most students.

(c) $A\ (1,4)$, $B\ (0,1)$, $C\ (-1,-2)$ and $D\ (-3,1)$, Using vectors,

(i) prove that the points A, B, and C are collinear. **2**

$\vec{AB} = \begin{bmatrix} 0-1 \\ 1-4 \end{bmatrix}$; $\vec{AC} = \begin{bmatrix} -1-1 \\ -4-2 \end{bmatrix}$ $= \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ $\quad = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$ $\therefore \vec{AC} = 2\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ $= 2\vec{AB}$ $\therefore \vec{AC}$ and $\vec{AB}$ are in the same direction and share point A. $\therefore$ they are collinear.	1 – correctly finding vectors AB and AC 1 – For correct relationship and reasoning.

Marker's Comments:

Most students identified the length of two vectors correctly but did not provide the correct reasoning. Most students gave reason that as the two vectors AB and BC are equal, but this is not the reason the points are collinear. It is essential to highlight direction and shared point.

A few students found the gradient of the intervals. But the question specifically asked to use vectors to demonstrate the property.

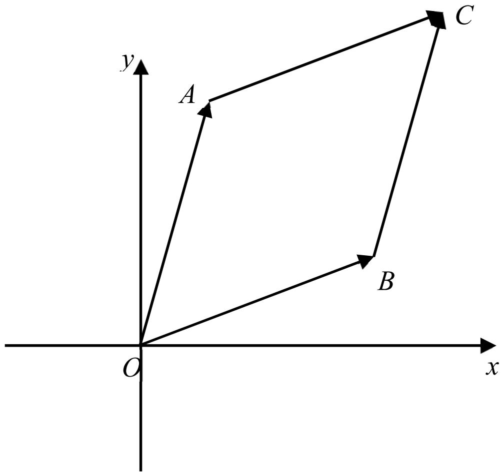
(ii) find, to the nearest degree, the angle between $\vec{AB}$ and $\vec{AD}$. **2**

$\vec{AB} = (-1,-3)$; $\vec{AD} = (-4,-3)$ $\cos\theta = \dfrac{\vec{AB}\cdot\vec{AD}}{\lvert\vec{AB}\rvert\lvert\vec{AD}\rvert}$ $= \dfrac{4+9}{\sqrt{10}\times 5}$ $= \dfrac{13}{5\sqrt{10}}$ $\theta \doteqdot 35^\circ$	1 – Correct substitution 1 – Final angle.

Marker's Comments:

Answered will by most students.

(d) Using vectors to prove that the diagonals of a rhombus intersect each other at right angle. **3**



let $\overrightarrow{OA} = \underset{\sim}{a}$ & $\overrightarrow{OB} = \underset{\sim}{b}$ $\lvert \underset{\sim}{a} \rvert = \lvert \underset{\sim}{b} \rvert$ property of rhombus $\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$; $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$ $\lvert \overrightarrow{OC} \rvert \cdot \lvert \overrightarrow{AB} \rvert = (\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$ $= \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b}$ $= -\lvert \underset{\sim}{a} \rvert^2 + \lvert \underset{\sim}{b} \rvert^2$ $= 0$ $\therefore$ diagonals intersect at right angles.	1 – Finding vectors OC and AB in terms of vectors OA and OB. 1 – Dot product 1 – correct use of length of sides equal

Marker's Comments:

Most students could make a start but struggled to correctly conclude the dot product. Many students skipped some working out when performing the dot product. This is not recommended as markers might consider this working out an essential part of the solution and therefore marks would be deducted.

Question 2 - Write your answer in the question 2 answer booklet (10 marks)

(a) State the difference between the direction vector and a position vector. **1**

The position vector is a vector relative to a fixed point, usually the origin. A direction vector is a free vector.	1 – correct reasoning
Marker's Comments: Generally, well answered.	

(b) Six people are to be selected from nine people to sit on a round table.

How many possible arrangements are there? **2**

$\binom{9}{6} \times 5! = 10080$	1 – 5! 1 - $\binom{9}{6}$
Marker's Comments: Generally, well answered.	

(c) Find the constant term in the expansion of $(x - \frac{1}{3x^3})^{20}$. **2**

$T_{k+1} = \binom{20}{k}(x)^{20-k}\left(-\frac{1}{3}\right)^k (x^{-3})^k$ $= \binom{20}{k}\left(-\frac{1}{3}\right)^k x^{20-4k}$ Let $20-4k=0 \Rightarrow 4k=20$ $k=5$ $\therefore$ the constant term $= \binom{20}{5}\left(-\frac{1}{3}\right)^5$ $= -\frac{5168}{81}$ (or $\frac{15504}{243}$)	1 – Correct general term 1 – the constant term (accept $\binom{20}{5}\left(-\frac{1}{3}\right)^5$)
Marker's Comments: Generally, well answered.	

(d) A student is given 16 randomly chosen positive integers and is asked to divide them by 5. Show that at least 4 of the numbers will have the same remainder. **2**

When dividing by 5, there are five possible remainders: 0, 1, 2, 3 or 4 $\frac{16}{5} = 3.2$ Therefore there must be at least four numbers with the same remainder.	1 – calculation 1 - reasoning
Marker's Comments: Students are reminded to explain that remainders 0, 1, 2, 3, and 4 are the pigeonholes (marks were not deducted this time). Overall, answered well.	

(e) Vectors $\underset{\sim}{a} = \begin{bmatrix} x \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 3 \\ y \end{bmatrix}$ are perpendicular to each other. Given that $|\underset{\sim}{a}| = 2\sqrt{5}$, find all possible values of x and y. **3**

$\underset{\sim}{a} = \begin{bmatrix} x \\ 2 \end{bmatrix}$; $\lvert\underset{\sim}{a}\rvert = 2\sqrt{5} \Rightarrow x^2 + 4 = 20$ $x = \pm 4$ $\underset{\sim}{b} = \begin{bmatrix} 3 \\ y \end{bmatrix}$; $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ $3x + 2y = 0 \Rightarrow y = -\frac{3}{2}x$ $\therefore$ when $x = 4$, $y = -6$ when $x = -4$, $y = 6$	1 – correct use of dot product 1 – for one set of values or only x values or only y values 1 – for all values.
Marker's Comments: Generally, answered well.	

Question 3 - Write your answer in the question 3 answer booklet (13 marks)

(a) Ten university students share a house. Every Monday morning, they all leave the house one at a time. If their ages are as follows: one is 17 years old, three are 18 years old, two are 19 years old, three are 20 years old, and one is 21 years old.

(i) How many possible arrangements are there of them leaving the house with no further restrictions? Do not evaluate this. **1**

<table>
<tr><td>$10!$</td><td>1 – correct answer</td></tr>
<tr><td colspan="2">Marker's Comments:
Generally Answered well</td></tr>
</table>

(ii) What is the probability that they leave the house in ascending age order? Leave your answer in factorial notation. **2**

<table>
<tr><td>$P(\text{ascending order}) = \dfrac{1 \times 3! \times 2! \times 3! \times 1}{10!}$
$= \dfrac{3! \times 3! \times 2!}{10!}$</td><td>1 – correctly finding the number of arrangements in ascending order

1 – correct probability</td></tr>
<tr><td colspan="2">Marker's Comments:
Not answered very well. Many students did not use the correct numerator.</td></tr>
</table>

(b) How many four digit numbers greater than 4500 can be formed from 3, 4, 5, 6, 7 if no digits can be repeated. **2**

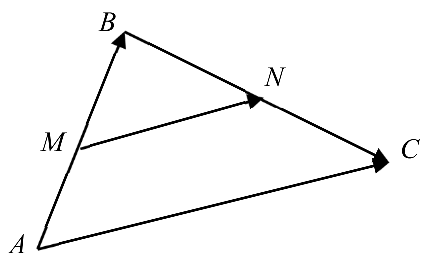
<table>
<tr><td>$\boxed{4}\ \boxed{\ }\ \boxed{\ }\ \boxed{\ }$
$1 \times 3 \times 3 \times 2 = 18$
$\boxed{5,6,7}\ \boxed{\ }\ \boxed{\ }\ \boxed{\ }$
$3 \times 4 \times 3 \times 2 = 72$
$\therefore$ a total of 90 4 digit nos greater than 4500 can be formed</td><td>1 – one of the arrangements

1 – for both arrangements added</td></tr>
<tr><td colspan="2">Marker's Comments:
This question was not answered very well, cases were incorrectly applied or students generated a full list of cases rather than using counting techniques</td></tr>
</table>

(c) If $(1+x)^n = \sum_{r=0}^{n}\binom{n}{r}x^r$ Show that $\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$ **2**

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$ Differentiating both sides: $n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots + n\binom{n}{n}x^{n-1}$ Sub $x=1$: $n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}$ $= \sum_{r=1}^{n} r\binom{n}{r}$ as required	1 – correctly differentiating 1 – correctly substituting
Marker's Comments: This question was not answered very well, and it was not attempted by many students.	

(d) In triangle ABC, M is the midpoint of side AB and N is the midpoint of side BC as shown in the diagram.



Given $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$, using vector notation show that vector MN is parallel to vector AC. **2**

$\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN}$ $= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$ $= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$ $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$ $\therefore \overrightarrow{MN} = \frac{1}{2}\overrightarrow{AC}$ $\therefore$ the vectors are parallel	1 – correctly finding vector MN in terms of vectors AB and BC 1 – identifying the relationship between the two vectors.
Marker's Comments: Generally answered well but students were not always setting out their work correctly or they used an over complicated approach by attempting to use the dot product. A mark was deducted if vectors were introduced but not defined.	

(e) Let $\underset{\sim}{a} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - \underset{\sim}{j}$

(i) Find the projection of $\underset{\sim}{a}$ on to $\underset{\sim}{b}$. **2**

<table>
<tr>
<td>

$$\text{Proj}_{\underset{\sim}{b}}\,\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|^2}\,\underset{\sim}{b}$$

$$= \frac{(2)(4)+(5)(-1)}{\sqrt{17}}\,(4\underset{\sim}{i} - \underset{\sim}{j})$$

$$= \frac{12}{\sqrt{17}}\,\underset{\sim}{i} - \frac{3}{\sqrt{17}}\,\underset{\sim}{j}$$

</td>
<td>1 – correctly substitution

1 – final answer</td>
</tr>
<tr>
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Marker's Comments:

Generally answered well but a few students recalled projection formula incorrectly or applied it incorrectly.

</td>
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</table>

(ii) Given $\underset{\sim}{c} = \underset{\sim}{a} - \text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}$, what is the value of $\underset{\sim}{b} \bullet \underset{\sim}{c}$, giving a reason for your answer. **2**

<table>
<tr>
<td>

$\underset{\sim}{b} \cdot \underset{\sim}{c} = 0$ as $\underset{\sim}{c}$ is perpendicular to $\underset{\sim}{b}$.

</td>
<td>1 – $\underset{\sim}{b} \cdot \underset{\sim}{c} = 0$

1 - reasoning</td>
</tr>
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Marker's Comments:

Generally answered well but most students didn't recognise $c_{\sim}$ as perpendicular to $b_{\sim}$ from the $c_{\sim} = a_{\sim} - proj_{b_{\sim}} a_{\sim}$ and proceeded to calculate $b_{\sim} \cdot c_{\sim}$

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</table>

Question 4 - Write your answer in the question 4 answer booklet (10 marks)

(a) Frank would like to make a bracelet for his partner. The bracelet will be made up of the letter beads: a, e, i, o, d, f, g, h, k, m, and v.

If no more than 2 vowels can be next to each other, how many different arrangements of the beads can he have for the bracelet. **3**

Total unrestricted arrangements = $10!$

can not have 3 or 4 vowels next to each other.

case 1 3 vowels:

$\binom{4}{3}\times 3!$

7 const 6 const

1 2 3 4 5 6

$\binom{4}{3}3!\times 7\times 6\times 6!$

Case 2 4 vowels:

$4!$

1 7 2 6 3 4 5

$4!\times 7!$

$\therefore$ total no. of arrangements $= \dfrac{10! - \binom{4}{3}3!(7)(6)6! - 4!7!}{2}$

$= 1391\,040$

1 – Case 1

1 – Case 2

1 – final answer

Marker's Comments:

In counting questions, students are **strongly advised** to explain why, through either a diagram or a written explanation, they are using a particular multiplication process. Expressions with no explanation that are wrong are very unlikely to achieve part marks.

Many Students tried to count the number of times the no vowels together then 2 vowels together and then add the two together . All students who attempted this method did not manage to get the correct answer. The more successful method required to calculate 3 vowels together and 4 vowels together and then subtract from total possible arrangements.

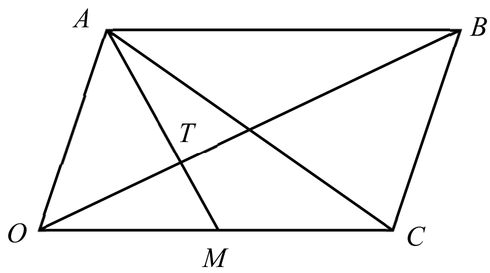
Bracelet with n objects is $\dfrac{(n-1)!}{2}$ as a bracelet can be flipped as well as rotated.

(b) The diagram below shows a parallelogram $OABC$. Let M be the midpoint of OC.

The point T lies at the intersection of AM and OB such that $AT : TM = 2 : 1$.

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OM} = \underset{\sim}{c}$.

Find the ratio $OT : TB$ **4**



$$\overrightarrow{OT} = \overrightarrow{OA} + \overrightarrow{AT}$$
$$= \underset{\sim}{a} + \frac{2}{3}\overrightarrow{AM}$$
$$\overrightarrow{AM} = \underset{\sim}{c} - \underset{\sim}{a}$$
$$\Rightarrow \overrightarrow{OT} = \underset{\sim}{a} + \frac{2}{3}(\underset{\sim}{c} - \underset{\sim}{a})$$
$$= \frac{1}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{c}$$
$$= \frac{1}{3}(\underset{\sim}{a} + 2\underset{\sim}{c})$$
$$\text{Now: } \overrightarrow{OB} = (\underset{\sim}{a} + 2\underset{\sim}{c})$$
$$\therefore \overrightarrow{OT} = \frac{1}{3}\overrightarrow{OB}$$
$$\Rightarrow \overrightarrow{OT} : \overrightarrow{TB} = 1 : 2$$

1 – initial logical step to lead to a solution

1 – correct reasoning to partially lead to a final solution

1 – relationship between vectors OT and OB

1 – correct final ratio

Marker's Comments:

Very well don.

Main error was to use $\underset{\sim}{a} - \underset{\sim}{c}$ instead of $\underset{\sim}{c} - \underset{\sim}{a}$ for vector AM

(c) Use combinatoric argument to prove that

$\binom{n}{m}\binom{n-m}{k-m}=\binom{n}{k}\binom{k}{m}$ where $n>k>m$. (An algebraic solution will not receive any marks). **3**

Senario: select a team of k people from a total of n. Then selecting a subset of m people from k. Case 1 • select the subset of m from the total of n people • As 'm' members of the set 'k' people have already been selecte, there are 'k-m' yet to be choosen from the remaining 'n-m' $\therefore$ total no. of arrangements $=\binom{n}{m}\binom{n-m}{k-m}$ Case 2 • select the 'k' members from 'n' • Then select the subset 'm' members from 'k' $\therefore$ total arrangements $=\binom{n}{k}\binom{k}{m}$ $\therefore \binom{n}{m}\binom{n-m}{k-m}=\binom{n}{k}\binom{k}{m}$	1 – correctly finding the magnitude

Marker's Comments:

Many students did not describe what the LHS and RHS were actually counting and instead just turned each expression into a selection process and did not explain how they were counting the same thing.

End of Paper.