FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 1
Task 2 Half Yearly 2005
ANSWERS COVER SHEET

Name: ____

QUESTION	MARK	HE1	HE2	HE3a	HE3b	HE3c	HE3d	HE4	HE5	HE6	HE7
Question 1	/19	✓									✓
Question 2	/39	✓									✓
Question 3											
a)	/5	✓	✓								✓
b) , c)	/6	✓									✓
d)	/3	✓		✓							✓
e)	/5	✓									✓
Total	/19										
Question 4	/19	✓									✓
TOTAL											
	/96	/96	/5	/3							/96

GIRRAWEEN HIGH SCHOOL

YEAR 12 HALF YEARLY EXAMINATION
Task 2

2005

MATHEMATICS
Extension 1

Time allowed - Two hours

(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a new page.
- A table of standard integrals is provided.

Question 1. (19 marks)

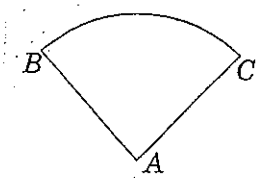
a) For what values of x is $|x-1|<x$? 3

b) Find the term independent of x in the expansion $\left[\frac{1}{x^3}+2x^5\right]^{16}$ 3

c) Sketch the following: (i) $y=\tan 2x \quad (0\le x\le 2\pi)$ 3

(ii) $y=3\sin 2x \quad (-\pi\le x\le \pi)$ 3

d) The headlights of a car cast a beam of light in the shape of a sector of a circle, as shown in the diagram. The arc BC of the circle subtends an angle θ at the centre A, with a radius r. The perimeter of the beam (AB + BC + CA) is 12 metres.



Show the area of the sector ABC is given by $A=\frac{720\theta}{(\theta+2)^2}$ 3

e) Find the acute angle between $x+2y-5=0$ and $x-3y+3=0$. 4

Question 2. (39 marks)

a) Differentiate:

(i) $y=\sin^2 3x$ (ii) $y=\tan(\ln 5x)$ 6

(iii) $y=\frac{e^x}{\tan x}$ (iv) $y=\ln\frac{\sqrt{x}}{x^2+1}$ 6

b) Find the equation of the tangent to the curve $y=-\cos x$ at $x=\frac{\pi}{3}$. 4

c) Find:

(i) $\int(\sin x-4x^3)dx$ (ii) $\int_0^{\frac{\pi}{8}}\sec^2 2t\,dt$ (iii) $\int_2^3\frac{3x}{x^2-1}dx$ 9

(iv) $\int_0^1 x^3e^{2x^4}dx$ (v) $\int_0^{\frac{\pi}{3}}\cos^2 2x\,dx$ 6

d) The area enclosed between $y=\frac{2}{\sqrt{x-7}}$ and the x axis between x = 8 and x = 10 is rotated about the x axis. Find the volume formed. 4

e) (i) Differentiate $y=\tan(x^2-3)$ 2

(ii) Use your result in part (i) to find $\int x\sec^2(x^2-3)dx$ 2

Question 3. **(19 marks)**

a) Use Mathematical Induction to prove, $1\times 1!+2\times 2!+3\times 3!+\ldots+n\times n! = (n+1)!-1$ (for positive integers n) 5

b) Prove $\dfrac{\sin\theta}{\sin\theta+\cos\theta}-\dfrac{\sin\theta}{\sin\theta-\cos\theta}=\tan 2\theta$ 3

c) Solve $\sin^2\theta=\frac{1}{2}\sin 2\theta \qquad (0\le\theta\le 2\pi)$ 3

d) The probability of a certain family having brown eyed children is $\frac{2}{3}$.
If the family plans to have 4 children, what is the probability that, at most, two will have brown eyes? Give your answer correct to 2 decimal places. 3

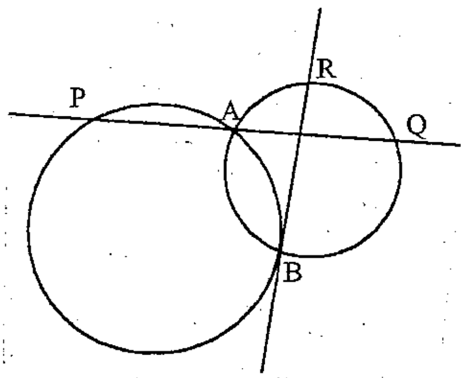
e) Sketch the curve $y=\dfrac{x^2+1}{x}$, showing all main features, including any asymptotes. 5

Question 4. **(20 marks)**

a) Two circles cut at A and B. A line through A meets one circle at P and the other at Q.

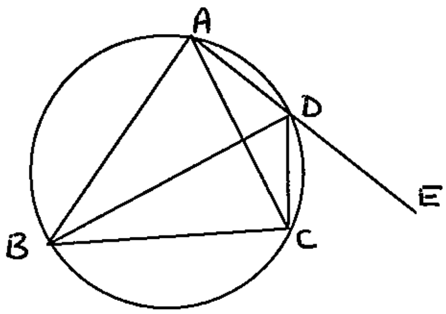
BR is a tangent to circle ABP and R lies on the circle ABQ.

(i) Copy the diagram
(ii) Prove that PB//QR. 6



b) ABC is a triangle where BC = AC:
(i) Copy the diagram
(ii) Give reasons why $\angle CDE=\angle ABC$ 1

(iii) Hence show that DC bisects $\angle BDE$ 4



c) (i) On the same set of axes sketch the curves
$y=2x+1$ and $y=\ln(x+1)$ 3

(ii) Show that the minimum vertical distance between the curves
$y=2x+1$ and $y=\ln(x+1)$ occurs when $x=-\frac{1}{2}$ 5

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Year12 Extension 1 Half yearly 2005

Question 1.

a) $|x-1| < x$

$\pm(x-1) < x$

$\therefore x-1 < x$ or $-(x-1) < x$

no solution $\quad x-1 > -x$

$2x > 1$

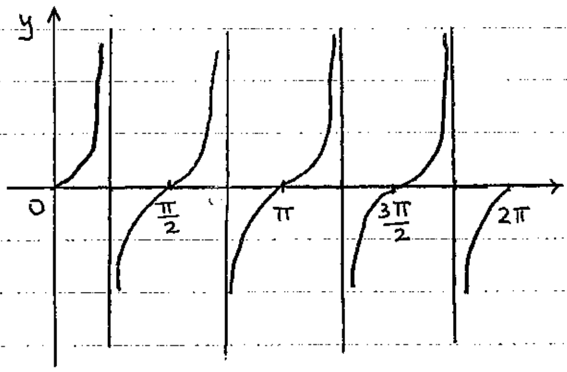
$x > \frac{1}{2}$

b) $\left(\frac{1}{x^3} + 2x^5\right)^{16}$

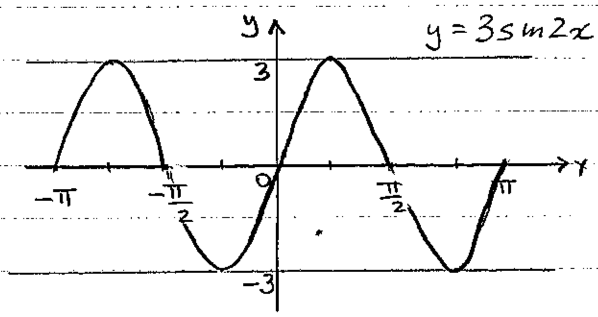
Term is $^{16}C_6 \left(\frac{1}{x^3}\right)^{10} \left(2x^5\right)^6$

$\therefore$ term is $^{16}C_6 \cdot 2^6$ (512512)

c) i)



ii)



d) Arc $BC = r\theta$

Perimeter $= 2r + r\theta$

$\therefore 12 = r(2+\theta)$

$\frac{12}{2+\theta} = r$

Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \cdot \frac{144}{(2+\theta)^2} \cdot \theta$

$A = \frac{72\theta}{(2+\theta)^2}$

e) $x + 2y - 5 = 0$

$2y = -x + 5$

$y = -\frac{1}{2}x + \frac{5}{2} \qquad m_1 = -\frac{1}{2}$

$x - 3y + 3 = 0$

$-3y = -x - 3$

$y = \frac{1}{3}x + 1 \qquad m_2 = \frac{1}{3}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-\frac{1}{2} - \frac{1}{3}}{1 + (-\frac{1}{2})\cdot\frac{1}{3}}\right|$

$= \left|\frac{-\frac{5}{6}}{1 - \frac{1}{6}}\right|$

$= |-1|$

$\therefore \tan\theta = 1$

acute angle is 45°

Question 2

a) i) $y = \sin^2 3x$

$\frac{dy}{dx} = 2\sin 3x \cdot \cos 3x \cdot 3$

$= 6\sin 3x \cos 3x$

ii) $y = \tan \ln 5x$

$\frac{dy}{dx} = \sec^2 \ln 5x \cdot \frac{1}{5x} \cdot 5$

$= \frac{\sec^2 \ln 5x}{x}$

iii) $y = \dfrac{e^x}{\tan x}$

$$\frac{dy}{dx} = \frac{\tan x \cdot e^x - e^x \cdot \sec^2 x}{\tan^2 x}$$

$$= \frac{e^x(\tan x - \sec^2 x)}{\tan^2 x}$$

iv) $y = \ln\left(\dfrac{\sqrt{x}}{x^2+1}\right)$

$$y = \ln\sqrt{x} - \ln(x^2+1)$$
$$= \tfrac{1}{2}\ln x - \ln(x^2+1)$$

$$\frac{dy}{dx} = \frac{1}{2x} - \frac{1}{x^2+1} \cdot 2x$$
$$= \frac{1}{2x} - \frac{2x}{x^2+1}$$

OR $\dfrac{1-3x^2}{2x(x^2+1)}$

b) $y = -\cos x$ $\quad x = \frac{\pi}{3}$

$y = -\frac{1}{2}$

$$\frac{dy}{dx} = \sin x$$

at $x = \frac{\pi}{3}$ $\quad \dfrac{dy}{dx} = \sin\frac{\pi}{3}$

$= \sqrt{3}/2$

Tangent:

$$y - \left(-\tfrac{1}{2}\right) = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$
$$y + \tfrac{1}{2} = \frac{\sqrt{3}x}{2} - \frac{\sqrt{3}\pi}{6}$$

$$6y + 3 = 3\sqrt{3}x - \sqrt{3}\pi$$
$$0 = 3\sqrt{3}x - 6y - 3 - \sqrt{3}\pi$$

c) i) $\int (\sin x - 4x^3)\,dx$

$$= -\cos x - x^4 + C$$

ii) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2 2t\,dt$

$$= \left[\frac{\tan 2t}{2}\right]_0^{\pi/8}$$
$$= \frac{\tan\frac{\pi}{4} - 0}{2}$$
$$= \frac{1}{2}$$

iii) $\displaystyle\frac{3}{2}\int_2^3 \frac{2x}{x^2-1}\,dx$

$$= \left[\frac{3}{2}\ln(x^2-1)\right]_2^3$$
$$= \frac{3}{2}[\ln 8 - \ln 3]$$
$$= \frac{3}{2}\ln\frac{8}{3}$$

iv) $\displaystyle\int_0^1 x^3 e^{2x^4}\,dx = \frac{1}{8}\int_0^1 e^{2x^4} \cdot 8x^3$

$$= \frac{1}{8}\left[e^{2x^4}\right]_0^1$$
$$= \frac{1}{8}\left[e^2 - e^0\right]$$
$$= \frac{1}{8}\left[e^2 - 1\right]$$

v. $\displaystyle\int_0^{\frac{\pi}{3}} \cos^2 2x\,dx$

Now $\cos 4x = 2\cos^2 2x - 1$

$\therefore \cos^2 2x = \frac{1}{2}(\cos 4x + 1)$

$$\therefore = \frac{1}{2}\int_0^{\pi/3} (\cos 4x + 1)\,dx$$

$$= \frac{1}{2}\left[\frac{\sin 4x}{4} + x\right]_0^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{\sin\frac{4\pi}{3}}{4} + \frac{\pi}{3} - 0\right]$$

$$= \frac{1}{2}\left(\frac{-\frac{\sqrt{3}}{2}}{4} + \frac{\pi}{3}\right)$$

$$= -\frac{\sqrt{3}}{16} + \frac{\pi}{6}$$

d) $V = \pi \int_8^{10} y^2 \, dx$

$= \pi \int_8^{10} \left(\frac{4}{x-7} \right) dx$

$= 4\pi \left[\ln(x-7) \right]_8^{10}$

$= 4\pi \left[\ln 3 - \ln 1 \right]$

$= 4\pi \ln 3$ unit^3

e) $y = \tan(x^2-3)$

$\frac{dy}{dx} = \sec^2(x^2-3) \cdot 2x$

$= 2x \sec^2(x^2-3)$

$\int x \sec^2(x^2-3) \, dx = \frac{1}{2} \int 2x \sec^2(x^2-3) \, dx$

$= \frac{1}{2} \tan(x^2-3) + C$

<u>Question 3</u>

a) Prove

$1 \times 1! + 2 \times 2! + 3 \times 3! \cdots + n \times n!$

$= (n+1)! - 1$

- for $n=1$ $\quad 1 \times 1! = (1+1)! - 1$

$1 = 2! - 1$

$= 1$

true

- assume true for $n=k$

$S_k = (k+1)! - 1$

- prove true for $n=k+1$

ie. Prove $S_{k+1} = (k+2)! - 1$

Now: $S_k + T_{k+1} = S_{k+1}$

$\therefore$ LHS $= (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)! \left[1 + (k+1) \right] - 1$

$= (k+1)! \, (k+2) - 1$

$= (k+2)! - 1$

$=$ RHS

- If true for $n=k$ then proved true for $n=k+1$. Since true for $n=1$, must be true for $n=2$ and so on for all n.

b) $\frac{\sin\theta}{\sin\theta + \cos\theta} - \frac{\sin\theta}{\sin\theta - \cos\theta}$

$= \frac{\sin\theta(\sin\theta - \cos\theta) - \sin\theta(\sin\theta + \cos\theta)}{(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)}$

$= \frac{\sin^2\theta - \sin\theta\cos\theta - \sin^2\theta - \sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta}$

$= \frac{-2\sin\theta\cos\theta}{-\cos 2\theta}$

$= \frac{\sin 2\theta}{\cos 2\theta}$

$= \tan 2\theta$

c) $\sin^2\theta = \frac{1}{2}\sin 2\theta$

$2\sin^2\theta = \sin 2\theta$

$\cancel{2}\sin^2\theta = \cancel{2}\sin\theta\cos\theta$

$\sin^2\theta - \sin\theta\cos\theta = 0$

$\sin\theta(\sin\theta - \cos\theta) = 0$

$\sin\theta = 0 \; ; \; \sin\theta = \cos\theta$

$\tan\theta = 1$

$\therefore \theta = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{5\pi}{4}$

d) $p = \frac{2}{3}$ $q = \frac{1}{3}$

P(at most 2 with brown eyes)

$P(X=2) + P(X=1) + P(X=0)$

$= {}^4C_2 \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + {}^4C_1 \left(\frac{2}{3}\right)^1 \cdot \left(\frac{1}{3}\right)^3$

$+ \; {}^4C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^4$

$= 0{\cdot}2963 + 0{\cdot}0988 + 0{\cdot}0123$

$\doteqdot 0{\cdot}41$ $\left(\frac{33}{81}\right)$

e) $y = \frac{x^2+1}{x}$

$y = x + \frac{1}{x}$

$\frac{dy}{dx} = 1 - x^{-2}$ $\left(1 - \frac{1}{x^2}\right)$

$\frac{d^2y}{dx^2} = 2x^{-3}$ $\left(\frac{2}{x^3}\right)$

Stat points $\frac{dy}{dx} = 0$

$\therefore \; 1 - \frac{1}{x^2} = 0$

$x^2 - 1 = 0$

$x = \pm 1$

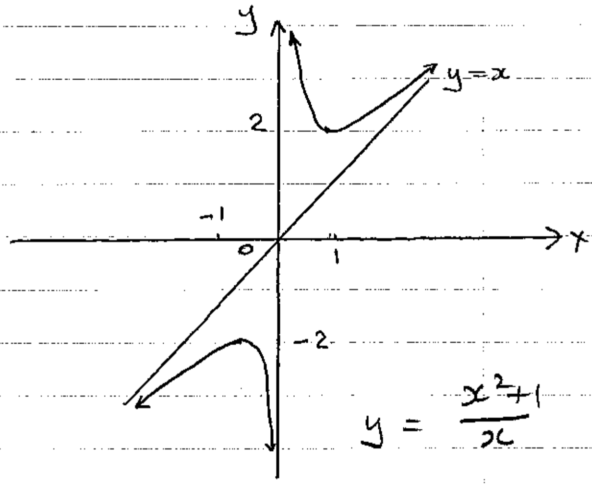
$x = 1$ $\quad$ $x = -1$

$y = 2$ $\quad$ $y = -2$

Limits: as $x \to \infty$ $y \to x$

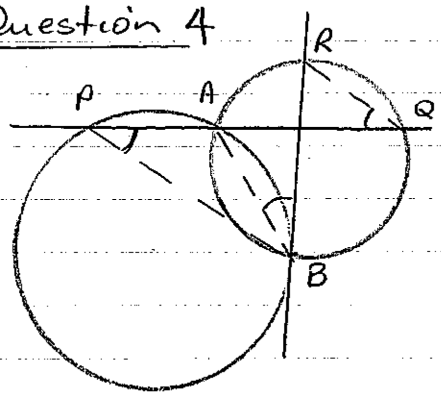
asymptotes $y = x$ and $x = 0$

No inflexion points as $\frac{2}{x^3} \neq 0$



Question 4

i)



ii) $\angle RBA = \angle APB$ (angle in alternate segment)

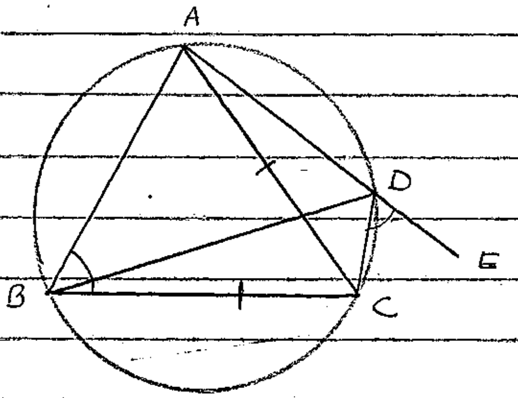
$\angle RBA = \angle AQR$ (angles standing on arc AR are equal)

$\therefore \angle APB = \angle AQR$

$\therefore PB \parallel QR$ (equal alternate angles $\angle APB, \angle AQR$)

b)

i)



ii) $\angle ABC = \angle CDE$
(exterior $\angle$ of cyclic quad equal to interior opposite $\angle$)

iii) $\angle ABC = \angle BAC$ (equal $\angle$'s opposite equal sides of an isosceles triangle)

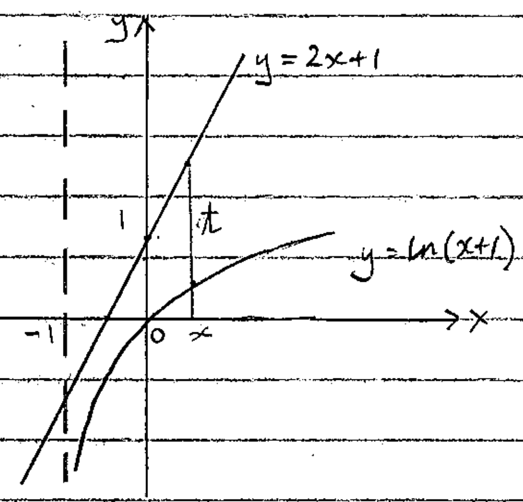
$\angle BAC = \angle BDC$ (angles standing on the same arc BC)

$\therefore \angle ABC = \angle BDC$

$\therefore \angle CDE = \angle BDC$

$\therefore$ DC bisects $\angle BDE$

c i)



ii) distance $t = 2x+1 - \ln(x+1)$

$$\frac{dt}{dx} = 2 - \frac{1}{x+1}$$

$$= 2 - (x+1)^{-1}$$

$$\frac{d^2t}{dx^2} = (x+1)^{-2}$$

$$= \frac{1}{(x+1)^2} \text{ (always +ve)}$$

$\therefore$ concave up

When $\frac{dt}{dx} = 0$

$$2 - \frac{1}{x+1} = 0$$

$$2(x+1) - 1 = 0$$

$$2x + 2 - 1 = 0$$

$$2x = -1$$

$$x = -\tfrac{1}{2}$$

$\therefore$ Distance between curves is a min at $x = -\frac{1}{2}$