GIRRAWEEN HIGH SCHOOL
TASK 1 2021 *(December 2020)*

MATHEMATICS

EXTENSION 1

Time allowed – 90 Minutes

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be marked clearly Question 6, Question 7 etc. Each question is to be returned on a separate page in your answer booklet.
- You may ask for spare answer booklets if you need them.
- For Multiple choice: Fill in the circle corresponding to the correct answer on the multiple choice answer sheet in your answer booklet.

For questions 1-5, fill in the response oval corresponding to the correct answer on your Multiple-choice answer sheet.

---

(1) Find the exact value of $\cos 35^0 \cos 5^0 + \sin 35^0 \sin 5^0$

A) $\frac{1}{2}$ B) $-\frac{1}{2}$ C) $\frac{\sqrt{3}}{2}$ D) $-\frac{\sqrt{3}}{2}$

(2) Given that $4\sin\theta + 3\cos\theta = -5$, what is the value of $\tan\frac{\theta}{2}$

A) -2 B) -1 C) 1 D)2

(3) What is the value of $\sin 2\theta$ given that $\sin\theta = \frac{\sqrt{3}}{4}$ and θ is obtuse?

A) $-\frac{\sqrt{39}}{8}$ B) $-\frac{\sqrt{3}}{2}$ C) $\frac{\sqrt{3}}{2}$ D) $\frac{\sqrt{39}}{8}$

(4) What is the coefficient of the x^3 term in the expansion of $(2-3x)^8$?

A) 56 B) -1512 C) 1512 D) -48384

(5) $2\cos 6\theta \sin 2\theta$ can be written as

A) $\cos 4\theta - \cos 8\theta$ B) $\cos 8\theta + \cos 4\theta$ C) $\sin 8\theta - \sin 4\theta$ D) $\sin 8\theta + \sin 4\theta$

Question 6 (19 marks)

a) Given that the angles A and B are acute, and that $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$,

find the value of $\sin\left(A - B\right)$. 3

b) Use compound angle formulae to find the exact value of

i) $2\sin 15^0 \cos 15^0$ 2

ii) $\cos 105^0$ 3

c) Given that $\sin A = \frac{2}{3}$, where $\frac{\pi}{2} < A < \pi$ and $\tan B = \frac{2}{3}$ where $\pi < B < \frac{3\pi}{2}$,

show that $\cos\left(A + B\right) = \dfrac{3\sqrt{5} + 4}{3\sqrt{13}}$ 4

d) Show that $\tan 35^0 + \tan 10^0 + \tan 35^0 \tan 10^0 = 1$ 4

e) By writing 3θ as $2\theta + \theta$ and using compound-angle and double-angle results,

Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. 3

Question 7 (16 marks)

a) Eliminate θ from the following pair of parametric equations.

$x = 2 + \cos\theta$ and $y = \cos 2\theta$ 3

b) Use the products– to- sums identities to show that

i) $2\sin 70^0 \sin 50^0 = \frac{1}{2} + \cos 20^0$ 2

ii) $\dfrac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} = \tan 2\theta$ 4

c) Prove the identities:

i) $\dfrac{1}{1 - \tan\theta} - \dfrac{1}{1 + \tan\theta} = \tan 2\theta$ 3

ii) $4\sin 5\theta \cos 3\theta \sin\theta = \cos\theta - \cos 3\theta + \cos 7\theta - \cos 9\theta$ 4

Question 8 (15marks)

a) Expand and simplify $\left(\frac{2}{x}+3x^2\right)^5$ 2

b) Find the term in $x^{10}y^2$ in the expansion of $\left(\frac{1}{2}x-3y^2\right)^{11}$ 3

c) i) Expand $(1+7x)^5$ as far as the term in x^2. 1

ii) Hence find the coefficient of x^2 in the expansion of $(1-5x)(1+7x)^5$. 3

d) Find the term independent of x in the expansion of $\left(2x+x^{-2}\right)^6$. 3

e) In the expansion of $(2+3x)^n$, the coefficients of x^5 and x^5 are in the ratio 4:9

Find the value of n. 3

Question 9 (18marks)

a) Using an appropriate subtitution in the expansion of $(1+x)^n$,

show that $\binom{n}{0}+2\times\binom{n}{1}+4\times\binom{n}{2}+8\times\binom{n}{3}+......=3^n$ 3

b) By expanding and differentiating $(1+x)^{2n}$, show that :

$1\times\binom{2n}{1}+2\times\binom{2n}{2}+.......+2n\times\binom{2n}{2n}=n\times 2^{2n}$

4

c) By letting $t=\tan\frac{x}{2}$, solve $\sin x+\cos x=-1$ in the domain $0\le x\le 2\pi$ 4

d) i) Express $\sqrt{3}\cos\theta-\sin\theta$ in the form $R\cos(\theta+\alpha)$, where $R>0$ and

$0<\alpha<\frac{\pi}{2}$ 4

ii) Hence or otherwise, solve $\sqrt{3}\cos\theta-\sin\theta=-1$ for $0\le\theta\le 2\pi$ 3

Question 10 (14marks)

a) Use the principle of mathematical induction to show that for all integers $n \geq 1$,

$$1 \times 2 + 2 \times 5 + 3 \times 8 + \ldots\ldots + n(3n-1) = n^2\left(n+1\right)$$ 3

b) Prove by mathematical induction that $4^n + 14$ is divisible by 6 for all positive integers $n \geq 1$. 4

c) Solve: $\left(n+2\right)! = 72n!$ 3

d) Use mathematical induction to prove that, for positive integers n,

$$1 + 3 + 9 + \ldots\ldots\ldots\ldots + 3^{n-1} = \frac{1}{2}\left(3^n - 1\right)$$ 4

.

End of examination!!!

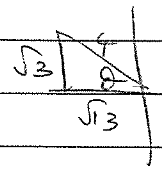
Mathematics Extension 1 Task 1 SOLUTIONS 2020

Multiple choice questions

1. C $\quad \cos(35^\circ - 5^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

2. A
$$\frac{4 \times 2t}{1+t^2} + \frac{3(1-t^2)}{1+t^2} = -5$$
$$8t + 3 - 3t^2 = -5 - 5t^2$$
$$2t^2 + 8t + 8 = 0 \Rightarrow (t+2)^2 = 0$$
$$t = -2$$

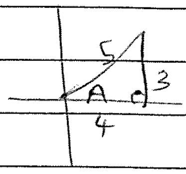
3. A
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2 \times \frac{\sqrt{3}}{4} \times \left(-\frac{\sqrt{13}}{4}\right)$$
$$= -\frac{\sqrt{39}}{8}$$

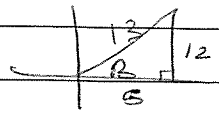


4. D $\quad = {}^8C_3 \times 2^5 \times (-3)^3 = -48384$

5. C $\quad 2\cos 6\theta \sin 2\theta = \sin 8\theta - \sin 4\theta.$

Question 6



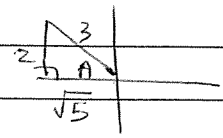


a) $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$

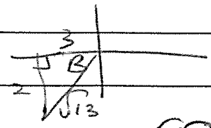
$$\sin(A-B) = \sin A\cos B - \cos A \sin B$$
$$= \frac{3}{5}\times\frac{5}{13} - \frac{4}{5}\times\frac{12}{13}$$
$$= \frac{15}{65} - \frac{48}{65} = \frac{-33}{65}$$ (3)

b) (i) $2\sin 15^\circ \cos 15^\circ = \sin 30^\circ = \frac{1}{2}$ (1)

(ii) $\cos(105^\circ) = \cos(60^\circ + 45^\circ) = \cos 60 \cos 45' - \sin 60' \sin 45'$
$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$ (3)



c) $\cos(A+B) = \cos A\cos B - \sin A\sin B$
$$= \left(-\frac{\sqrt{5}}{3}\times -\frac{3}{\sqrt{13}}\right) - \left(\frac{2}{3}\times -\frac{2}{\sqrt{13}}\right)$$
$$= \frac{3\sqrt{5}+4}{3\sqrt{13}}$$ (4)



d) $\tan 45^\circ = \tan(35^\circ + 10^\circ) = \frac{\tan 35^\circ + \tan 10^\circ}{1 - \tan 35^\circ \tan 10^\circ} = 1$

$\therefore \tan 35^\circ + \tan 10^\circ = 1 - \tan 35^\circ \tan 10^\circ$

$\therefore \tan 35^\circ + \tan 10^\circ + \tan 35^\circ \tan 10^\circ = 1$ (4)

Question 6 (continued)

e) LHS

$\cos 3\theta = \cos(2\theta+\theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta . \sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta = RHS.$

(3)

Question 7.

a) $\cos\theta = x-2$ and $y = \cos 2\theta$.

$\cos 2\theta = 2\cos^2\theta - 1$

$\therefore 2(x-2)^2 - 1 = y. \Rightarrow 2x^2 - 8x + 7.$

$\therefore y = 2x^2 - 8x + 7$

(3)

b) LHS

(i) $2\sin 70^\circ \sin 50^\circ = \cos 20^\circ - \cos 120^\circ$

$= \cos 20^\circ + \frac{1}{2} = RHS.$

(3)

LHS

(ii) $\dfrac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} = \dfrac{\frac{1}{2}[\sin(2\theta+\theta) + \sin(2\theta-\theta)]}{\frac{1}{2}[\cos(2\theta+\theta) + \cos(2\theta-\theta)]}$

$A = 2\theta$

$B = \theta$

$= \dfrac{\sin 2\theta \cos\theta}{\cos 2\theta \cos\theta} = \tan 2\theta = RHS$

(4).

Question 7 (continued).

c) (i) $\frac{1}{1-\tan\theta} - \frac{1}{1+\tan\theta} = \tan 2\theta$

LHS $\frac{1}{1-\tan\theta} - \frac{1}{1+\tan\theta} = \frac{1+\tan\theta - 1+\tan\theta}{1-\tan^2\theta}$

$= \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta = RHS.$

3

(ii) $4\sin 5\theta \cos 3\theta \sin\theta = \cos\theta - \cos 3\theta + \cos 7\theta - \cos 9\theta.$

LHS $4\sin 5\theta \cos 3\theta \sin\theta = 2\sin\theta\left[2\sin 5\theta\cos 3\theta\right]$

$= 2\sin\theta\left[\sin 8\theta + \sin 2\theta\right]$

$= 2\sin\theta\sin 8\theta + 2\sin\theta\sin 2\theta$

$= \cos 7\theta - \cos 9\theta + \cos\theta - \cos 3\theta$

$= \cos\theta - \cos 3\theta + \cos 7\theta - \cos 9\theta = RHS$

4

Question 8.

a) $\left(\frac{2}{x} + 3x^2\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^5 + {}^5C_1\left(\frac{2}{x}\right)^4(3x^2) + {}^5C_2\left(\frac{2}{x}\right)^3(3x^2)^2$

$+ {}^5C_3\left(\frac{2}{x}\right)^2(3x^2)^3 + {}^5C_4\left(\frac{2}{x}\right)(3x^2)^4 + {}^5C_5(3x^2)^5$

$= \frac{32}{x^5} + \frac{5\times 48}{x^2} + 10\times 8\times 9x + 10\times 4\times 27x^4$

$+ 810x^7 + 243x^{10}$

$= \frac{32}{x^5} + \frac{240}{x^2} + 720x + 1080x^4 + 810x^7 + 243x^{10}$

2

Question 8 (continued).

b) $\left(\frac{1}{2}x - 3y^2\right)^{11}$

$x^{10}y^2$ term $\Rightarrow {}^{11}C_1 \left(\frac{x}{2}\right)^{10}(-3y^2)^1 = \frac{-33}{2^{10}} x^{10} y^2$

$= \frac{-33}{1024} x^{10} y^2$

(3)

c) (i) $(1+7x)^5 = {}^5C_0 + {}^5C_1(7x) + {}^5C_2(7x)^2 + \cdots\cdots$

(1)

(ii) $(1-5x)\left({}^5C_0 + {}^5C_1(7x) + {}^5C_2(7x^2) + \cdots\cdots\right)$

$= (1-5x)(1 + 35x + 490x^2 + \cdots\cdots)$

coefficient of $x^2 \Rightarrow 490 - 5\times 35 = 315$

(3)

d) $(2x + x^{-2})^6$

$T_{k+1} = {}^6C_k \cdot (2x)^{6-k}(x^{-2})^k = {}^6C_k \cdot 2^{6-k} \cdot x^{6-3k}$

independent of $x \Rightarrow 6-3k = 0$

$\therefore k = 2$

$\therefore T_3 = {}^6C_2 \cdot 2^4 = 16 \times 15 = 240.$

(3)

e) $(2+3x)^n$

C_5 coefficient of $x^5 = {}^nC_5 \cdot 2^{n-5} \times 3^5$

C_6 coefficient of $x^6 = {}^nC_6 \cdot 2^{n-6} \times 3^6$

$$\frac{{}^nC_5 \times 2^{n-5} \times 3^5}{{}^nC_6 \times 2^{n-6} \times 3^6} = \frac{4}{9}$$

$$\frac{\cancel{n!}}{(n-5)!\,(5!)} \times \frac{6!\,(n-6)!}{\cancel{n!}} \times \frac{2}{3} = \frac{4}{9}$$

$$\frac{6}{n-5} \times \frac{2}{3} = \frac{4}{9} = \frac{4}{n-5}$$

$\therefore 4n - 20 = 36$

$4n = 56 \quad \therefore n = 14$

3

Question 9

a) (i) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$ (1)

(ii) substitute $x=2$ in the expansion $\Rightarrow$

$3^n = {}^nC_0 + 2\times{}^nC_1 + 4\times{}^nC_2 + \ldots$ (2)

b) $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \ldots + {}^{2n-1}C_{2n-1} x^{2n-1}$

$+ {}^{2n}C_{2n} x^{2n}$.

Differentiating both sides,

$2n(1+x)^{2n-1} = {}^{2n}C_1 + 2\times{}^{2n}C_2 + \ldots + 2n\times{}^{2n}C_{2n} x^{2n-1}$

substitute $x=1$ both sides,

$2n\times 2^{2n-1} = {}^{2n}C_1 + 2\times{}^{2n}C_2 + \ldots + 2n\times{}^{2n}C_{2n}$.

$\therefore n\times 2^{2n} = {}^{2n}C_1 + 2\times{}^{2n}C_2 + \ldots + 2n\times{}^{2n}C_{2n}$ (3)

c) $\sin x + \cos x = -1$ $\quad 0 \leq x \leq 2\pi$.

sub. $t = \tan\frac{x}{2} \Rightarrow \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = -1$

$\therefore 2t + 1 - t^2 = -1 - t^2 \Rightarrow 2t = -2$

$t = -1$

$\therefore \tan\frac{x}{2} = -1 \Rightarrow \frac{x}{2} = \frac{3\pi}{4}$ $\quad \therefore x = \frac{3\pi}{2}$

Check $x = \pi$. $\Rightarrow$ LHS $\sin\pi + \cos\pi = 0 + (-1) = -1 =$ RHS

$\therefore x = \pi$ is a solution. $\therefore x = \frac{3\pi}{2}, \pi$.

(3)

d) (i) $\sqrt{3}\cos\theta - \sin\theta = R\cos(\theta+\alpha)$

$\sqrt{3}\cos\theta - \sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

equating coefficients of $\cos\theta$,

$R\cos\alpha = \sqrt{3}$ — (1)

$R\sin\alpha = 1$ — (2)

$(1)^2 + (2)^2 \Rightarrow R^2 = 4 \quad \therefore R = 2 \quad (R>0)$

$\sin\alpha = \frac{1}{2}$ and $\cos\alpha = \frac{\sqrt{3}}{2}$

$\therefore \alpha$ is in quadrant 1 and $\alpha = \frac{\pi}{6}$

$\therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta + \frac{\pi}{6}\right)$

(ii) $2\cos\left(\theta+\frac{\pi}{6}\right) = -1$

$\therefore \cos\left(\theta+\frac{\pi}{6}\right) = \frac{-1}{2}$ $\qquad 0 \leq \theta \leq 2\pi$

$\therefore \theta + \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}$ $\qquad \frac{\pi}{6} \leq \theta + \frac{\pi}{6} \leq \frac{13\pi}{6}$

$\therefore \theta = \frac{2\pi}{3} - \frac{\pi}{6}$ or $\frac{4\pi}{3} - \frac{\pi}{6}$

$= \frac{\pi}{2}$ or $\frac{7\pi}{6}$

Question 10

a) $1\times 2 + 2\times 5 + 3\times 8 + \cdots + n(3n-1) = n^2(n+1)$ for $n \geq 1$

step 1 Prove true for $n=1$

LHS $1\times 2 = 2 = 1^2\times 2 = 2 =$ RHS

$\therefore$ true for $n=1$

step 2 Assume true for $n=k$

ie $1\times 2 + 2\times 5 + \cdots + k(3k-1) = k^2(k+1)$. (1)

step 3 Prove true for $n=k+1$

ie $1\times 2 + 2\times 5 + \cdots + k(3k-1) + (k+1)(3k+2) = (k+1)^2(k+2)$

LHS. $\underbrace{1\times 2 + 2\times 5 + \cdots + k(3k-1)} + (k+1)(3k+2)$

$= k^2(k+1) + (k+1)(3k+2)$ from (1)

$= (k+1)(k^2+3k+2) = (k+1)(k+1)(k+2)$

$= (k+1)^2(k+2) =$ RHS.

$\therefore$ true for $n=k+1$.

step 4.

Since true for $n=1$, by PMI, true for all integers $n \geq 1$

b) $4^n + 14$ is divisible by 6 for $n \geq 1$.

step1 : show true for $n=1$

$4^1 + 14 = 18 = 6 \times 3$ ∴ true for $n=1$

step2 Assume true for $n=k$

i.e. $4^k + 14 = 6P$ (P = integer) — (1)

step3 Prove true for $n = k+1$.

i.e. $4^{k+1} + 14 = 6Q$ (Q = integer).

LHS $4^{k+1} + 14 = 4 \times 4^k + 14$

$= 4(6P - 14) - 14$

$= 24P - 42 = 6(4P - 7)$

$= 6Q$, Q = integer

∴ true for $n = k+1$.

step4. Since true for $n=1$, by PMI, true for all $n \geq 1$.

c) $(n+2)! = 72n!$

$(n+2)(n+1) \times n! = 72n!$

∴ $(n+2)(n+1) = 72 = n^2 + 3n + 2$

∴ $n^2 + 3n - 70 = 0$

$(n+10)(n-7) = 0$ ∴ $n = 7$ ($\because n \geq 0$).

d) $1+3+9+\dots\dots+3^{n-1} = \frac{1}{2}(3^n-1)$

for positive integers

$n \geq 1$

step1 ~~Prove~~ true for $n=1$

LHS $=1$ RHS $=\frac{1}{2}(3-1)=1$

$\therefore$ true for $n=1$

step2 Assume true for $n=k$

ie $1+3+9+\dots+3^{k-1} = \frac{1}{2}(3^k-1)$ (1)

step3 Prove true for $n=k+1$.

ie $1+3+9+\dots+3^{k-1}+3^k = \frac{1}{2}(3^{k+1}-1)$.

LHS $\underbrace{1+3+9+\dots\dots+3^{k-1}}+3^k$

$= \frac{1}{2}(3^k-1)+3^k = 3^k(\frac{1}{2}+1)-\frac{1}{2}$

$= \frac{3^k \times 3}{2}-\frac{1}{2} = \frac{1}{2}[3^{k+1}-1]$

$=$ RHS.

step4

Since true for $n=1$, by PMI, true for all integers $n \geq 1$