# GIRRAWEEN HIGH SCHOOL
## TASK 1 2021 *(December 2021)*

# MATHEMATICS
## EXTENSION 1

*Time allowed – 90 Minutes*

**DIRECTIONS TO CANDIDATES**

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be marked clearly Question 6, Question 7 etc. Each question is to be answered on a separate page in your answer booklet.
- You may ask for spare answer booklets if you need them.
- For Multiple choice: Fill in the circle corresponding to the correct answer on the multiple choice answer sheet in your answer booklet.

**For questions 1-5, fill in the response oval corresponding to the correct answer on your Multiple-choice answer sheet.**

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(1) $\dfrac{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}{\cot\frac{\theta}{2}+\tan\frac{\theta}{2}}=$

A) $\cos\theta$ B) $\sec\theta$ C) $\tan\theta$ D) $\cot\theta$

(2) $\cos 5x+\cos x$ may be written as:

A) $2\sin 3x\cos 2x$ B) $2\cos 3x\sin 2x$ C) $-2\sin 3x\sin 2x$ D) $2\cos 3x\cos 2x$

(3) If $\tan A=4, \tan B=\frac{3}{5}$, and $A$ and $B$ are acute angles, then $A-B=$

A) $45^0$ B) $30^0$ C) $60^0$ D) $135^0$

4. If $\tan A=-\frac{3}{4}$, $90^0<A<180^0$ and $\cos B=\frac{5}{13}$, $0^0<B<90^0$, the exact value of $\sin(A-B)$ is:

A) $\frac{33}{65}$ B) $\frac{63}{65}$ C) $\frac{62}{65}$ D) $\frac{48}{65}$

(5) What is the coefficient of the $x^3$ term in the expansion of $(2-3x)^8$?

A) 56 B) -1512 C)1512 D) -48384

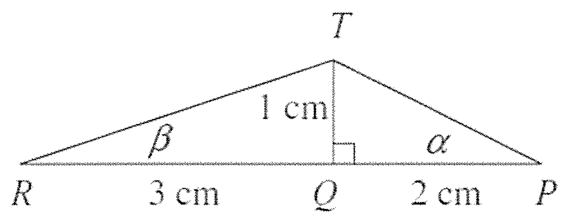
**Question 6 (12 marks)**

a) Show that

i) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$ **3**

ii) Hence, simplify $\sin^2 75^0-\sin^2 15^0$ **1**

b)




i) Use the diagram above diagram to find the exact value for $\sin(\alpha+\beta)$. **3**

ii) Hence show that the angles $\alpha$ and $\beta$ in the diagram above have sum $45^0$ **1**

c) Use the compound -angle and double -angle results to find the exact value of :

i) $\dfrac{\tan 110^0+\tan 25^0}{1-\tan 110^0 \tan 25^0}$ **2**

ii) $\sin\dfrac{8\pi}{9}\cos\dfrac{2\pi}{9}-\cos\dfrac{8\pi}{9}\sin\dfrac{2\pi}{9}$ **2**

**Question 7( 11 marks)**

a) Use $t=\tan\frac{\theta}{2}$ to find the exact value of $\dfrac{1-\tan^2\frac{3\pi}{8}}{1+\tan^2\frac{3\pi}{8}}$ **2**

b) Prove that $\cos\theta\left(\tan\theta-\tan\frac{\theta}{2}\right)=\tan\frac{\theta}{2}$ , using $t=\tan\frac{\theta}{2}$. **3**

c) i) If $t=\tan 75^0$, show that $\dfrac{2t}{1-t^2}=-\dfrac{1}{\sqrt{3}}$ **2**

ii) Hence show that $\tan 75^0=2+\sqrt{3}$ **1**

d) Solve $7\sin x-4\cos x=4$, for $0^0\le x\le 360^0$ ,by using the substitution

$t=\tan\frac{1}{2}x$. **3**

**Question 8(13 marks)**

Use the products– to- sums identities to show that

a) i) $\cos 36^0-\cos 84^0=\sqrt{3}\sin 24^0$ **2**

ii) $\dfrac{\sin 3\theta+\sin\theta}{\cos 3\theta+\cos\theta}=\tan 2\theta$ **3**

b) Solve the following equations for $0\le x\le 2\pi$

i) $\sin 2x+\sqrt{3}\cos x=0$ **3**

c) i) Express $\sqrt{3}\cos x+\sin x$ in the form $A\cos(x-\theta)$ where $A>0$. **3**

ii) Hence solve $\sqrt{3}\cos x+\sin x=-1$ for $0\le x\le 2\pi$ **2**

**Question 9 (20 marks)**

a) For $(1+x)^{12}$, find the coefficient of $x^8$. **1**

b) For $(1+2x)^6$, find the term in $x^4$. **1**

c) i) Expand $(1+2x)^5$ as far as the term $x^3$. **2**

ii) Hence, find the coefficient of $x^3$ in the expansion of $(2-3x)(1+2x)^5$. **2**

d) i) Find the general term in the expansion of $\left(2x^2 - x^{-1}\right)^{20}$. **3**

ii) Hence , find the term in $x^{-5}$. **2**

e) i) Find the coefficients of $x$ and $x^{-3}$ in the expansion of $\left(3x - \frac{a}{x}\right)^5$. **2**

ii) Hence, find the values of $a$ if these coefficients are in the ratio 2:1 **2**

f) i) Use the binomial expansion

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots\ldots + \binom{n}{n}x^n$ and by an appropriate substitution, prove that :

$$\binom{n}{0} + 2\times\binom{n}{1} + 4\times\binom{n}{2} + 8\times\binom{n}{3} + \ldots\ldots = 3^n$$ **2**

ii) Differentiate the identity $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots\ldots + \binom{n}{n}x^n$ ,then make

a substitution , to prove that

$$1\times\binom{n}{1} + 2\times\binom{n}{2} + \ldots\ldots + n\times\binom{n}{n} = n2^{n-1}$$ **3**

**Question 10 (12 marks)**

a) Prove by the method of mathematical induction that for all positive integer values of $n$:

$$5+11+19+.....+(8n-3)=n(4n+1)$$ **4**

b) Prove by mathematical induction that $2^{n+2}+3^{2n+1}$ is divisible by 7 for all positive integers $n$. **4**

c) Prove by mathematical induction that

$2\times 1!+5\times 2!+10\times 3!+......+(n^2+1)n!=n(n+1)!$ for all positive integers $n\geq 1$. **4**

**End of examination!!!**

Year 12 Ext 1. SOLUTIONS Task 1

multiple choice questions.

(1) $\dfrac{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}}{\cot\frac{\theta}{2} + \tan\frac{\theta}{2}}$ Let $t = \tan\frac{\theta}{2}$

$= \dfrac{\frac{1}{t} - t}{\frac{1}{t} + t} = \dfrac{1-t^2}{1+t^2} = \cos\theta$. (A)

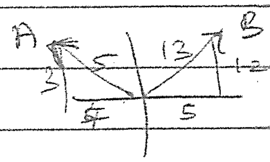
(2) $\cos 5x + \cos x = \frac{1}{2}\left[\cos(3x-2x) + \cos(3x+2x)\right]$

$\therefore \cos 5x + \cos x = 2\cos 3x \cos 2x$ (D)

(3) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B} = \dfrac{4 - \frac{3}{5}}{1 + 4\times\frac{3}{5}} = \frac{17}{5} \times \ldots = 1$

$\therefore A - B = 45^\circ$. (A)

(4) $\sin(A-B) = \sin A\cos B - \cos A \sin B$




$= \frac{3}{5}\times\frac{5}{13} - \left(-\frac{4}{5}\right)\times\frac{12}{13}$

$= \frac{15}{65} + \frac{48}{65} = +\frac{63}{65}$ (B)

$\sin A = \frac{3}{5}$

$\sin B = \frac{12}{13}$

(5) $(2-3x)^8 \Rightarrow$ coefficient of $x^3$ is

$^8C_3 \cdot (-3)^3 \cdot 2^5 =$

$= -48384$ (D)

Question 6

a) (i) $\sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B$

LHS $\sin(A+B).\sin(A-B) = (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B + \sin A\cos A\cos A\sin B - \sin A\cos B.\cos A\sin B - \cos^2 A\sin^2 B$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

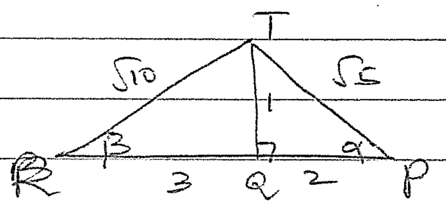
$= \sin^2 A - \cancel{\sin^2 A\sin^2 B} - \sin^2 B + \cancel{\sin^2 A\sin^2 B}$

$= \sin^2 A - \sin^2 B$.

(ii) $\sin^2 75^\circ - \sin^2 15^\circ = \sin(75^\circ+15^\circ).\sin(75^\circ-15^\circ)$

$= \sin 90^\circ.\sin 60^\circ = \dfrac{\sqrt{3}}{2}$

b) (i)




$\sin\alpha = \dfrac{1}{\sqrt{5}}$ and $\sin\beta = \dfrac{1}{\sqrt{10}}$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \dfrac{1}{\sqrt{5}}\times\dfrac{3}{\sqrt{10}} + \dfrac{2}{\sqrt{5}}\times\dfrac{1}{\sqrt{10}} = \dfrac{3}{\sqrt{50}} + \dfrac{2}{\sqrt{50}}$

$= \dfrac{5}{\sqrt{50}} = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\sin(\alpha+\beta) = \dfrac{1}{\sqrt{2}}$

(ii) $\therefore \alpha+\beta = 45^\circ$

c) (i) $\dfrac{\tan 110^\circ + \tan 25^\circ}{1-\tan 110^\circ\tan 25^\circ} = \tan 135^\circ = -1$

(ii) $\sin\dfrac{8\pi}{9}\cdot\cos\dfrac{2\pi}{9} - \cos\dfrac{8\pi}{9}.\sin\dfrac{2\pi}{9} = \sin\left(\dfrac{8\pi-2\pi}{9}\right) = \sin\dfrac{6\pi}{9}$

$= \sin\dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}$

Question 7

a) $t = \tan\frac{\theta}{2}$

$\therefore \dfrac{1-\tan^2\frac{3\pi}{8}}{1+\tan^2\frac{3\pi}{8}} = \cos\frac{6\pi}{8} = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

(2)

b) $\cos\theta\left(\tan\theta - \tan\frac{\theta}{2}\right) = \tan\frac{\theta}{2}$

Let $t = \tan\frac{\theta}{2}$

(3)

LHS $\cos\theta\left(\tan\theta - \tan\frac{\theta}{2}\right) = \left(\frac{1-t^2}{1+t^2}\right)\left(\frac{2t}{1-t^2} - t\right)$

$= \frac{1-t^2}{1+t^2} \times \left(\frac{2t - t + t^3}{1-t^2}\right)$

$= \frac{\cancel{1-t^2}}{\cancel{1+t^2}} \times \frac{t\,(\cancel{1+t^2})}{\cancel{1-t^2}} - t$

$= \tan\frac{\theta}{2} = RHS$

c) (i) $t = \tan 75^\circ$

$\tan 150^\circ = \frac{2\tan 75^\circ}{1-\tan^2 75^\circ} = \frac{2t}{1-t^2} = \frac{-1}{\sqrt{3}}$ (2)

(ii) $2\sqrt{3}t = -1 + t^2 \quad \therefore t^2 - 2\sqrt{3}t - 1 = 0$

$\therefore t = \frac{2\sqrt{3} \pm \sqrt{12+4}}{2}$

$= \sqrt{3} \pm 2$ (1)

$\therefore \tan 75^\circ = 2 + \sqrt{3}$ ( $75^\circ$ is acute $\angle$)

d) Let $t = \tan\frac{x}{2}$. Subtituting $t = \tan\frac{x}{2}$ gives

$\frac{14t}{1+t^2} - \frac{4-4t^2}{1+t^2} = 4$ , $x \neq 180^\circ$.

$\therefore 14t - 4 + 4t^2 = 4 + 4t^2 \Rightarrow 14t = 8 \quad \therefore t = \frac{4}{7}$

$\therefore \tan\frac{x}{2} \doteqdot \frac{4}{7} \Rightarrow x \doteqdot 59^\circ 29'$.

~~bat~~ check for $x = 180°$

LHS $= 7 \times 0 - 4 \times (-1) = 4 = $ RHS

$\therefore\ x = 180°$ is also a solution.

$\therefore\ x = 180°$ or $x \doteq 59°29'$

Questions

(i) LHS $\cos 36^\circ - \cos 84^\circ = \cos(A-B) - \cos(A+B)$

$A - B = 36$ and

$A + B = 84$

$\therefore A = 60$ and $B = 24$

$\therefore \cos 36^\circ - \cos 84^\circ = 2\sin 60^\circ \sin 24^\circ$

$= \sqrt{3}\sin 24 = RHS$

(2)

(ii) $\dfrac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} = \tan 2\theta$

LHS $\dfrac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} = \dfrac{2\sin 2\theta \cdot \cos\theta}{2\cos 2\theta \cdot \cos\theta} = \tan 2\theta$

$= RHS$

(2)

b) $\sin 2x + \sqrt{3}\cos x = 0$

$2\sin x\cos x + \sqrt{3}\cos x = 0$

$\cos x\,[2\sin x + \sqrt{3}] = 0$

$\therefore \cos x = 0$ or $2\sin x = -\sqrt{3}$

$\sin x = -\dfrac{\sqrt{3}}{2}$

$\therefore x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ $\qquad x = \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

$\therefore x = \dfrac{\pi}{2}, \dfrac{4\pi}{3}, \dfrac{3\pi}{2}, \dfrac{5\pi}{3}$

c) (i)

$$\sqrt{3}\cos x + \sin x = A\cos(x-\theta)$$

Expanding $\sqrt{3}\cos x + \sin x = A\cos x\cos\theta + A\sin x\sin\theta$

equating coefficients.
$$\sqrt{3} = A\cos\theta$$
$$1 = A\sin\theta$$

squaring and adding,
$$\therefore 4 = A^2 \qquad \therefore A = 2 \quad (A>0).$$

$$\cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6} \qquad (\theta \text{ in Quadrant I})$$

$$\therefore \sqrt{3}\cos x + \sin x = 2\cos\left(x-\frac{\pi}{6}\right).$$

(ii)

$$\sqrt{3}\cos x + \sin x = 2\cos\left(x-\frac{\pi}{6}\right) = -1, \quad 0\le x\le 2\pi$$

$$2\cos\left(x-\frac{\pi}{6}\right) = -1$$

$$\therefore \cos\left(x-\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$x-\frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\therefore x = \frac{2\pi}{3}+\frac{\pi}{6} \text{ or } \frac{4\pi}{3}+\frac{\pi}{6}$$

$$= \frac{5\pi}{6} \text{ or } \frac{9\pi}{6} = \frac{3\pi}{2}$$

Question 9

a) $(1+x)^{12}$ coefficient of $x^8 = {}^{12}C_8$ (1)

(b) $(1+2x)^6$ term in $x^4 = {}^6C_4(2x)^4$

$= 16 \times {}^6C_4\, x^4$ (1)

c) (i) $(2-3x)(1+2x)^5 = (2-3x)(\cdots + 10x + 40x^2 + 80x^3 + \cdots$

(2)

(ii) $\therefore$ coefficient of $x^3 = -120 + 160 = 40$ (2)

d) (i) $(2x^2 - x^{-1})^{20}$

General term $= {}^{20}C_r (2x^2)^{20-r} (-1)^r (x^{-1})^r$

$= (-1)^r \times {}^{20}C_r \cdot 2^{20-r}\, x^{40-3r}$ (3)

(ii) $40 - 3r = -5$

$\therefore 3r = 45 \quad \therefore r = 15$

$\therefore$ term in $x^{-5} = {}^{20}C_{15} (2x^2)^5 (-1)^{15} (x^{-1})^{15}$

$= -{}^{20}C_{15} \cdot 2^5\, x^{-5}$

e) $\left(3x - \frac{a}{x}\right)^5$

term of $x = {}^5C_2 \cdot (3x)^3 \left(-\frac{a}{x}\right)^2$

$= {}^5C_2 \cdot 3^3 a^2$

term of $x^3 = {}^5C_4 (3x)^{\cancel{4}} \left(-\frac{a}{x}\right)^4$

$= {}^5C_4 \cdot 3 \cdot a^4$

$\frac{{}^5C_2 \, 3^3 \cdot a^2}{{}^5C_4 \cdot 3 \cdot a^4} = \frac{2}{1} \Rightarrow \frac{10}{5} \cdot \frac{3^2}{a^2} = \frac{2}{1}$

$\therefore a^2 = 9$

$\therefore a = \pm 3$

f) (i) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$.

sub in $x = 2$

LHS $3^n$ and RHS $= {}^nC_0 + 2\,{}^nC_1 + 4\,{}^nC_2 + \cdots + \cdots$

$\therefore 3^n = {}^nC_0 + 2 \times {}^nC_1 + 4 \times {}^nC_2 + \cdots$ (2)

(ii) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots \cdots + {}^nC_n x^n$

differentiating both sides. $n(1+x)^{n-1} = {}^nC_1 + 2x\,{}^nC_2 + \cdots + n \cdot {}^nC_n x^{n-1}$

sub in $x = 1$, $n \cdot 2^{n-1} = {}^nC_1 + 2 \times {}^nC_2 + \cdots + n \times {}^nC_n$.

(2)

Question 10

a) $5 + 11 + 19 + \cdots + (8n-3) = n(4n+1)$ for all $n \geq 1$

Prove true for $n = 1$

LHS 5 RHS $= 1(5) = 5$

$\therefore$ true for $n = 1$

Assume true for $n = k$

$\therefore 5 + 11 + 19 + \cdots + (8k-3) = k(4k+1)$ —— ①

Prove true for $n = k+1$

ie $5 + 11 + 19 + \cdots + (8k-3) + (8k+5) = (k+1)(4k+5)$

LHS $\underbrace{5 + 11 + 19 + \cdots + (8k-3)} + (8k+5)$

$=$

$= k(4k+1) + 8k + 5$ from ①

$= 4k^2 + 9k + 5 = (k+1)(4k+5) = \text{RHS}$

$\therefore$ true for $n = k+1$.

$\therefore$ By the principle of M.I, it is true for $n = 2, 3, \ldots$ all positive integers of $n$

b) $2^{n+2} + 3^{2n+1}$ is divisible by 7

Prove true for $n=1$

$2^3 + 3^3 = 35$ is divisible by 7

$\therefore$ true for $n=1$

Assume true for $n=k$

$2^{k+2} + 3^{2k+1} = 7P$ ($P$ = integer) — (1)

Prove true for $n = k+1$

$2^{k+3} + 3^{2k+3} = 7Q$ ($Q$ = integer)

LHS $2^{k+3} + 3^{2k+3} = 2^{k+3} + 9 \times 3^{2k+1}$

$= 2^{k+3} + 9 \times (7P - 2^{k+2})$

$= 2 \cdot 2^{k+2} + 9 \times 7P - 9 \times 2^{k+2}$

$= 9 \times 7P - 7 \times 2^{k+2}$

$= 7\left[9P - 2^{k+2}\right] = 7Q.$

$\therefore$ By the principle of MI, it is true for $n = 1, 2, \ldots$ all positive integers of $n$.

c)

$2\times 1! + 5\times 2! + 10\times 3! + \dots + (n^2+1)n! = n(n+1)!$

Prove true for $n=1$

LHS $2\times 1! = 2$ RHS $= 1(2!) = 2$

$\therefore$ true for $n=1$

Assume true for $n=k$.

$2\times 1! + 5\times 2! + 10\times 3! + \dots + (k^2+1)k! = k(k+1)!$ — (1)

Prove true for $n=k+1$

$2\times 1! + 5\times 2! + \dots + (k^2+1)k! + [(k+1)^2+1](k+1)!$

$= (k+1)(k+2)!$

LHS $\underbrace{2\times 1! + 5\times 2! + \dots + (k^2+1)k!} + [(k+1)^2+1](k+1)!$

$= k(k+1)! + (k^2+2k+2)(k+1)!$ from (1)

$= (k+1)![k + k^2 + 2k + 2]$

$= (k+1)![k^2+3k+2] = (k+1)!(k+2)(k+1)$

$= (k+1)(k+2)! = $ RHS.

$\therefore$ By the principle of m.I, it is true for all $n = 1, 2, \dots n$.