GLENWOOD

Year 12 Mathematics Extension 1

2019

Assessment Task 1

Full Name ……………………………………………………………………………..

Please circle your teacher's name **Samuel** **Siriwardena**

General Instructions

- Reading Time – 5 minutes
- Working Time – 55 minutes
- Write using a black pen
- Do not use liquid paper or tape
- Approved calculators may be used
- You may use the reference sheet provided during the examination.
- Diagrams are not to scale

Total marks – 30

Attempt both Questions.

Answer the questions in the space provided. These spaces provide guidance for the expected length of response.

Your responses should include relevant mathematical reasoning/ or calculations.

Question No.	1	2	Total
Marks	/9	/21	/30

Answer each question in the space provided.

QUESTION 1 (12 marks) *Marks*

a) Use mathematical induction to prove that for all positive integers of n: 3

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \cdots \ldots \ldots \ldots . + (n^2 + 1)n! = n(n + 1)!$$

b) Use mathematical induction to prove that for all positive integers of n: 3

$5^n + 2 \times 11^n$ is divisible by 3.

2

c) (i) Suppose that the statement

$1 + 3 + 5 + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + (2n - 1) = n^2 + 2$ is true for the positive integer $n = k$. Prove it is also true for $n = k + 1$.

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(ii) Explain, by giving mathematical reasoning, why we cannot conclude that the statement is true for all integers $n \geq 1$. 1

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End of Question 1

QUESTION 2 (21 marks)

a) (i) Express $\cos t - \sqrt{3}\sin t$ in the form $R\cos(t+\alpha)$, where α is in radians. 3

(ii) Hence or otherwise, find the all the solutions to the equation 2

$$\cos\theta - \sqrt{3}\sin\theta = \sqrt{2} \text{ for } 0 \le \theta \le 2\pi.$$

QUESTION 2

h) (i) Expr

(ii) Hence or ot

b) Find all the solutions to the equation 3

$\cos\theta = \sin 2\theta$, for $0 \leq \theta \leq 2\pi$.

i) Find all

$\cos\theta =$

c) Find all the solutions to the equation 3

$\sin 3x + sinx = \sin 2x$, for $0 \leq x \leq 2\pi$.

j) Find all

$\sin 3x +$

d) Solve the following equation using t formulae for $0 \leq x \leq 360°$. 3

$$5\cos x + 3\sin x = 4$$

k) Solve th

e) Prove that $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$ for $\sin\theta \neq 0$ and $\cos\theta \neq 0$. 2

l) Prove th

f) Solve the equation $sin2x = 2sin^2x$, for $0 \leq x \leq \pi$. 3

g) Using the formula, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, simplify: 2

$\frac{sin^3\theta + cos^3\theta}{sin\theta + cos\theta} - 1$ for $0 < \theta < \frac{\pi}{2}$.

m) Solve th

n) Using th

$\frac{sin^3\theta+c}{sin\theta+c}$

End of paper. ☺

Extra writing s

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Marks

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GLENWOOD

Year 12 Mathematics Extension 1

2019

Assessment Task 1

Full Name ....Solutions.........

Please circle your teacher's name **Samuel** **Siriwardena**

General Instructions

- Reading Time – 5 minutes
- Working Time – 55 minutes
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Total marks – 30

Attempt both Questions.

Answer the questions in the space provided. These spaces provide guidance for the expected length of response.

Your responses should include relevant mathematical reasoning/ or calculations.

Question No.	1	2	Total
Marks	/9	/21	/30

Answer each question in the space provided.

QUESTION 1 (12 marks) **Marks**

a) Use mathematical induction to prove that for all positive integers of n: 3

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \cdots \ldots \ldots \ldots + (n^2+1)n! = n(n+1)!$$

$n=1$, LHS $= 2 \times 1! = 2$, RHS $= 1 \times 2! = 2$

$\therefore$ The result is true for $n=1$

Assume the result is true for $n=k$.

$2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (k^2+1)k! = k(k+1)! \longrightarrow$ (A)

when $n=k+1$, the result to be proved,

$\underbrace{2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots (k^2+1)k!}_{\text{from (A)}} + [(k+1)^2+1](k+1)! = (k+1)(k+2)!$

LHS =

$= k(k+1)! + [(k+1)^2+1](k+1)!$

$= (k+1)! \{k + (k+1)^2 + 1\}$

$= (k+1)! \{k + k^2 + 2k + 1 + 1\}$

$= (k+1)! (k^2+3k+2)$

$= (k+1)! (k+2)(k+1)$

$= (k+1)(k+2)! =$ RHS.

$\therefore$ The result is true for $n=k+1$.

$\therefore$ By mathematical induction, the result is true for all positive integers of n.

b) Use mathematical induction to prove that for all positive integers of n: 3

$5^n + 2 \times 11^n$ is divisible by 3.

when $n=1$, $5^1 + 2\times 11^1 = 27$ is divisible by 3.

Assume the result is true for $n=k$.

$\therefore 5^k + 2\times 11^k = 3M$ where M is an integer.

$5^k = 3M - 2\times 11^k \rightarrow$ (A)

when $n=k+1$,

$5^{k+1} + 2\times 11^{k+1}$

$= 5\times 5^k + 2\times 11\times 11^k$

$= 5(3M - 2\times 11^k) + 22\times 11^k$ from (A)

$= 15M - 10\times 11^k + 22\times 11^k$

$= 15M + 12\times 11^k$

$= 3(5M + 4\times 11^k)$ is divisible by 3

$\therefore$ The result is true for $n=k+1$

$\therefore$ By mathematical induction, the result is true for all positive integers of n.

c) (i) Suppose that the statement 2

$1 + 3 + 5 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + (2n - 1) = n^2 + 2$ is true for the positive integer $n = k$. Prove it is also true for $n = k + 1$.

$1+3+5+ \ldots + (2n-1) = n^2+2$

when $n=k$, $1+3+5+ \ldots +(2k-1) = k^2+2$

when $n=k+1$, LHS $= 1+3+5+ \ldots +(2k-1)+(2k+1)$

$= k^2+2+2k+1$

$= k^2+2k+3$

$= k^2+2k+1+2$

$= (k+1)^2+2$

when $n=k+1$, RHS $= (k+1)^2+2$

$\therefore$ LHS = RHS

$\therefore$ The result is true for $n=k+1$.

(ii) Explain, by giving mathematical reasoning, why we cannot conclude that the statement is true for all integers $n \geq 1$. 1

when $n=1$, LHS $=1$, RHS $= 1^2+2 = 3$

$\therefore$ LHS $\neq$ RHS

$\therefore$ The result is not true for $n=1$

$\therefore$ we can't conclude that the statement is true for all integers $n \geq 1$

End of Question 1

QUESTION 2 (21 marks)

a) (i) Express $\cos t - \sqrt{3}\sin t$ in the form $R\cos(t+\alpha)$, where α is in radians. 3

$R\cos(t+\alpha) = R(\cos t\cos\alpha - \sin t\sin\alpha) = \cos t - \sqrt{3}\sin t$

By equating the coefficients of $\cos t$ and $\sin t$ on both sides,

$R\cos\alpha = 1 \rightarrow (1)$

$R\sin\alpha = \sqrt{3} \rightarrow (2)$

$(2)/(1)$ $\tan\alpha = \sqrt{3}$ $\therefore \alpha = \frac{\pi}{3}$

$(1)^2 + (2)^2$ $R^2(\cos^2\alpha + \sin^2\alpha) = 4$ $\therefore R = 2$

$\therefore \cos t - \sqrt{3}\sin t = 2\cos(t + \frac{\pi}{3})$

(ii) Hence or otherwise, find the all the solutions to the equation 2

$$\cos\theta - \sqrt{3}\sin\theta = \sqrt{2} \text{ for } 0 \le \theta \le 2\pi.$$

$\cos\theta - \sqrt{3}\sin\theta = 2\cos(\theta + \frac{\pi}{3})$ from part (i)

$\therefore 2\cos(\theta + \frac{\pi}{3}) = \sqrt{2}$ $0 \le \theta \le 2\pi$

$\cos(\theta + \frac{\pi}{3}) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$ $\frac{\pi}{3} \le \theta + \frac{\pi}{3} \le 2\pi + \frac{\pi}{3}$

$\therefore \theta + \frac{\pi}{3} = \frac{\pi}{4}$, or $\theta + \frac{\pi}{3} = 2\pi - \frac{\pi}{4}$, or $\theta + \frac{\pi}{3} = 2\pi + \frac{\pi}{4}$

$\theta = \frac{\pi}{4} - \frac{\pi}{3} = \frac{-\pi}{12}$ $\theta = 2\pi - \frac{\pi}{4} - \frac{\pi}{3}$ $\theta = 2\pi + \frac{\pi}{4} - \frac{\pi}{3}$

Not in the given domain $\theta = \frac{17\pi}{12}$ $\theta = \frac{23\pi}{12}$

$\therefore$ Solutions are $\theta = \frac{17\pi}{12}$ or $\frac{23\pi}{12}$

b) Find all the solutions to the equation 3

$\cos\theta = \sin 2\theta$, for $0 \le \theta \le 2\pi$.

$\sin 2\theta - \cos\theta = 0$

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ or $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ $\quad \theta = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore$ Solutions are $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$

c) Find all the solutions to the equation 3

$\sin 3x + \sin x = \sin 2x$, for $0 \le x \le 2\pi$.

$\sin 3x + \sin x = 2\sin 2x\cos x$ (using A+B formulae.)

$\therefore 2\sin 2x\cos x = \sin 2x$

$\therefore 2\sin 2x\cos x - \sin 2x = 0$

$\sin 2x(2\cos x - 1) = 0$

$\therefore \sin 2x = 0$ or $2\cos x - 1 = 0$

$\therefore 2x = 0, \pi$ or 2π $\quad \cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

$x = 0, \frac{\pi}{2}$ or π

$\therefore x = \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\therefore$ Solutions are $0, \pi/3, \frac{\pi}{2}, \pi$ or $\frac{5\pi}{3}$

d) Solve the following equation using t formulae for $0 \leq x \leq 360°$. 3

$$5\cos x + 3\sin x = 4$$

$5\left(\frac{1-t^2}{1+t^2}\right) + 3 \times \frac{2t}{(1+t^2)} = 4$

$5 - 5t^2 + 6t = 4(1+t^2)$

$9t^2 - 6t - 1 = 0$

$(3t-1)^2 - 2 = 0$

$(3t-1)^2 = 2$

$3t - 1 = \pm\sqrt{2}$

$t = \frac{1 \pm \sqrt{2}}{3}$

$\tan\frac{x}{2} = \frac{1+\sqrt{2}}{3}$, or $\tan\frac{x}{2} = \frac{1-\sqrt{2}}{3} = \tan(-7°52') = -\tan(7°52')$

$\tan\frac{x}{2} = \tan 38°49'$ or $\tan\frac{x}{2} = \tan(180° + 38°49')$ or $\frac{x}{2} = 180° - 7°52'$ or $\frac{x}{2} = 360° - 7°52'$ outside the domain

$\frac{x}{2} = 38°49'$ or $\frac{x}{2} = 218°49'$ outside the domain

$x = 77°38'$

$x = 2(180° - 7°52')$

$x = 344°16'$

$\therefore$ Solutions are $x = 77°38'$ or $344°16'$

e) Prove that $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$ for $\sin\theta \neq 0$ and $\cos\theta \neq 0$. 2

LHS $= \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$

$= \frac{\sin(3\theta - \theta)}{\frac{2\sin\theta\cos\theta}{2}}$

$= \frac{2\sin 2\theta}{\sin 2\theta}$

$= 2 =$ RHS.

f) Solve the equation $sin2x = 2sin^2x$, for $0 \leq x \leq \pi$. 3

$\sin 2x - 2\sin^2 x = 0$

$2\sin x \cos x - 2\sin^2 x = 0$

$2\sin x(\cos x - \sin x) = 0$

$\sin x = 0$ or $\cos x - \sin x = 0$

$x = 0, \pi,$ $\quad \tan x = 1 = \tan\frac{\pi}{4}$

$x = \frac{\pi}{4}$ or $x = \pi + \frac{\pi}{4}$ ← outside the domain.

$\therefore$ Solutions are $0, \frac{\pi}{4}, \pi$.

g) Using the formula, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, simplify: 2

$\frac{sin^3\theta + cos^3\theta}{sin\theta + cos\theta} - 1$ for $0 < \theta < \frac{\pi}{2}$.

$\frac{\cancel{(\sin\theta + \cos\theta)}(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)}{\cancel{(\sin\theta + \cos\theta)}} - 1$

$= 1 - \sin\theta\cos\theta - 1$

$= -\sin\theta\cos\theta$.

End of paper ☺

Section II Extra writing space

If you use this space, clearly indicate which question you are answering.

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