Student Number

2020 Glenwood High School
YEAR 12-2021
HSC Assessment Task 1

MATHEMATICS EXTENSION 1

General Instructions

- Reading time - 5 minutes
- Working time - 60 minutes
- Write using black pen
- Approved calculators may be used.
- Show relevant mathematical reasoning and/or calculations

Total marks: 32

- Attempt Questions 1 and 2
- Allow about 30 minutes for each question
- Start a new writing booklet for each question

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Question 1 (18 Marks)

Start a new writing booklet. Your responses should include relevant mathematical reasoning and/or calculations.

a) Prove by mathematical induction that 3

$$1(1!) + 2(2!) + \cdots + n(n)! = (n+1)! - 1$$

for all positive integers

b) Use the principle of Mathematical induction to prove that $7^n + 2$ is divisible by 3 for all positive integers 3

c) Solve for x in the interval $0 \leq x \leq 2\pi$, the equation $\sin 2x = \tan x$ 3

d) $\cos\theta - \sqrt{3}\sin\theta = A\cos(\theta + \alpha)$, By expressing $\cos\theta - \sqrt{3}\sin\theta$ in the form $A\cos(\theta+\alpha)$, solve $\cos\theta - \sqrt{3}\sin\theta = 1$ for $0 \leq \theta \leq 2\pi$. 4

e)

i) Prove the trigonometric identity $sin\, 3A = 3\sin A - 4sin^3 A$ 2

ii) Hence show that the equation $6x - 8x^3 = 1$ has the roots $\sin\frac{\pi}{18}$, $\sin\frac{5\pi}{18}$, $\sin\frac{25\pi}{18}$ 2

iii) Hence find the exact value of $\sin\frac{\pi}{18} \times \sin\frac{5\pi}{18} \times \sin\frac{25\pi}{18}$ 1

End of Question 1

Question 2 (14 Marks)

Start a new writing booklet. Your responses should include relevant mathematical reasoning and/or calculations.

a) Use the principle of Mathematical induction to prove that $3^{3n} + 2^{n+2}$ is divisible by 5 for all positive integers 3

b) Consider the statement S(n):

$$1 + 5 + 9 + \ldots + 4n - 3 = 2n^2 - n$$

and the following attempted proof of this statement by induction.

Proof:

Assume the statement is true for $n = k$. That is,

$$1 + 5 + 9 + \ldots + 4\text{k} - 3 = 2k^2 - k$$

Next, we shall show it is true for $n = k + 1$ by noting that if

$$1 + 5 + 9 + \ldots + 4\text{k} - 3 = 2k^2 - k$$

is true, then

$$1 + 5 + 9 + \ldots + 4\text{k} - 3 + 4(\text{k}+1) - 3 = 2(k+1)^2 - (k+1)$$

$$1 + 5 + 9 + \ldots + 4\text{k} - 3 + 4\text{k} + 1 = 2(k+1)^2 - (k+1)$$

Substituting S(k)

$$2k^2 - k + 4\text{k} + 1 = 2(k+1)^2 - (k+1)$$

$$2k^2 + 3\text{k} + 1 = 2(k^2 + 2k + 1) - (k+1)$$

$$2k^2 + 3\text{k} + 1 = 2k^2 + 4k + 2 - k - 1$$

Now subtracting $(3k + 1)$ from both sides of this equation

LHS = RHS

The result is true for $n = (k+1)$ if true for $n = k$

Therefore, by the principle of Mathematical induction, the statement $P(n)$ is true.

Question 2 continues on next page

i) Give two reasons why the given proof is incorrect and does not prove $P(n)$. 1

ii) Provide a correct complete proof by induction of the statement $P(n)$. 2

c) Use the substitution $t = tan\frac{\theta}{2}$, to solve the equation 3

$2\sin x + \cos x = -1$ for $0 \leq x \leq 2\pi$.. Give your answers correct to two decimal places where necessary.

d) Find the amplitude of the graph $y = 2\sqrt{3}\cos\theta - 2\sin\theta$ 2

e) i) Show that $\sin 2x + \sin 4x = \sin 6x$ can be expressed as $\sin 3x \sin 2x \sin x = 0$ 2

ii) Hence, solve $\sin 2x + \sin 4x = \sin 6x$ for x in the domain $0 \leq \theta \leq \pi$. 1

END OF EXAMINATION

Q1 a)
$1(1!) + 2(2!) + \dots + n(n!) = (n+1)! - 1$

Prove the result is true for $n=1$

LHS RHS

$1(1!) = 1$ $(1+1)! - 1 = 1$

LHS = RHS $\therefore$ true for $n=1$

Assume it's true for $n=k$

$1(1!) + 2(2!) + \dots k(k!) = (k+1)! - 1$

Prove it's true for $n = k+1$

Prove $1(1!) + 2(2!) + \dots k(k!) + (k+1)(k+1)! = (k+2)! - 1$

LHS

$\underbrace{1(1!) + 2(2!) + \dots k(k!)}_{S(k)} + (k+1)(k+1)!$

$(k+1)! - 1 + (k+1)(k+1)!$

$(k+1)!\left[1 + (k+1)\right] - 1$

$(k+1)!\,(k+2) - 1$

$(k+2)! - 1 = RHS$

$\therefore$ The result is true for $n = k+1$ if it's true for $n=k$

$\therefore$ By the principle of mathematical induction, the result is true for all positive integers.

Q1 b)

Prove it's true for $n=1$

$7^1+2 = 7+2 = 9$, which is divisible by 3

Assume it's true for $n=k$

$7^k+2 = 3M$ where M is an integer.

Prove it's true for $n=k+1$

Prove $7^{k+1}+2 = 3N$, where N is an integer.

from assumption, $7^k = 3M-2$

$$7^{k+1}+2$$

$$= 7^k \times 7 + 2$$

$$= (3M-2)7+2$$

$$= 21M - 14 + 2$$

$$= 21M - 12$$

$= 3(7M-4)$ is divisible by 3 since M is an integer

1 c)

$\sin 2x = \tan x \qquad 0 \leq x \leq 2\pi$

$2\sin x \cos x = \dfrac{\sin x}{\cos x}$

$2\sin x \cos^2 x - \sin x = 0$

$\sin x \left(2\cos^2 x - 1\right) = 0$

$\sin x = 0$

$x = 0, \pi, 2\pi$

$2\cos^2 x - 1 = 0$

$\cos x = \pm \dfrac{1}{\sqrt{2}}$

Base angle $= \dfrac{\pi}{4}$

$x = \dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$

$= \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

$\therefore x = 0, \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, 2\pi.$

1 d)

$$\cos\theta - \sqrt{3}\sin\theta = R\cos(\theta+\alpha)$$

$$\cos\theta - \sqrt{3}\sin\theta = R[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$$

$$\cos\theta - \sqrt{3}\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$$\therefore R\cos\alpha = 1 \quad \text{——} \quad (1)$$

$$R\sin\alpha = \sqrt{3} \quad \text{——} \quad (2)$$

$(1)^2 + (2)^2$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1+3$$

$$R^2 = 4$$

$$R = 2$$

$(1) \div (2)$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

$$\therefore \cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \frac{\pi}{6}\right)$$

$$\cos\theta - \sqrt{3}\sin\theta = 1$$

$$2\cos\left(\theta + \frac{\pi}{6}\right) = 1 \qquad 0 \leq \theta \leq 2\pi$$

$$\cos\left(\theta + \pi/6\right) = \frac{1}{2} \qquad \frac{\pi}{6} \leq \theta + \frac{\pi}{6} \leq \frac{2\pi + \pi}{6}$$

Bare angle $= \frac{\pi}{3}$

$$\theta + \pi/6 = \frac{\pi}{3},\ 2\pi - \frac{\pi}{3},$$

$$\theta + \pi/6 = \frac{\pi}{3},\ \frac{5\pi}{3}$$

$$\theta = \frac{\pi}{3} - \frac{\pi}{6},\ \frac{5\pi}{3} - \frac{\pi}{6}$$

$$\theta = \frac{\pi}{6},\ \frac{3\pi}{2}$$

Q1) a)

$$\sin 3A = \sin(2A + A)$$

$$= \sin 2A \cos A + \cos 2A \sin A$$

$$= 2\sin A \cos A \cos A + (1 - 2\sin^2 A)\sin A$$

$$= 2\sin A \cos^2 A + \sin A - 2\sin^3 A$$

$$= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$$

$$= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$$

$$= 3\sin A - 4\sin^3 A$$

b)

$$6x - 8x^3 = 1$$

$$2(3x - 4x^3) = 1$$

Let $x = \sin x$

$$2(3\sin A - 4\sin^3 A) = 1$$

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$$3\sin A - 4\sin^3 A = \sin 3A$$

$$\therefore \ 2(\sin 3A) = 1$$

$$\sin 3A = \frac{1}{2}$$

Base angle $= \frac{\pi}{6}$

$$3A = \frac{\pi}{6}, \ \pi - \frac{\pi}{6}, \ 2\pi + \frac{\pi}{6}, \ldots$$

$$= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \ldots$$

$$A = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \ldots$$

The solution are $\sin\frac{\pi}{18}, \sin\frac{5\pi}{18}, \sin\frac{13\pi}{18}, \sin\frac{17\pi}{18}, \sin\frac{25\pi}{18}$

Distinct roots are $\sin\frac{\pi}{18}, \sin\frac{5\pi}{18} \times \sin\frac{25\pi}{18}$

c)

$$8x^3 - 6x + 1 = 0$$

$$\alpha\beta\gamma = \frac{-d}{a} = \frac{-1}{8}$$

$$\therefore \ \sin\frac{\pi}{18} \times \sin\frac{5\pi}{18} \times \sin\frac{25\pi}{18} = \frac{-1}{8}$$

Q2 a)

Prove it's true for $n=1$

$$3^{3(1)} + 2^{(1)+2}$$

$$= 3^3 + 2^3$$

$$= 27 + 8$$

$= 35$, which is divisible by 5

Assume it's true for $n=k$, where k is an integer.

$3^{3k} + 2^{k+2} = 5M$ where M is an integer.

Prove it's true for $n=k+1$

$\therefore$ Prove $3^{3(k+1)} + 2^{(k+1)+2} = 5N$

LHS $\quad 3^{3k} \times 3^3 + 2^k \times 2^3$

from assumption $\quad 3^{3k} = 5M - 2^{k+2}$

$$3^3(5M - 2^{k+2}) + 2^k \times 2^3$$

$$3^3 \times 5M - 3^3 \times 2^k \times 2^2 + 2^3 \times 2^k$$

$$3^3 \times 5M - 108 \times 2^k + 8 \times 2^k$$

$$3^3 \times 5M - 100 \times 2^k$$

$5(27M - 20 \times 2^k)$ which is divisible by 5 since M & k are integers

$\therefore$ The result is true for $n=k+1$ if true for $n=k$

By the principle of Mathematical Induction the result is true for all positive integers.

Q2 b)

a) $n=1$ is not proven

when proving for $n=k+1$, rather than LHS & RHS solved separately. They are done side by side and $(3k+1)$ is subtracted from both sides.

b). $1+5+9+ \dots +4n-3 = 2n^2-n$

Prove it's true for $n=1$

LHS $= 1$

RHS $= 2(1)^2 - 1 = 2-1 = 1$

$\therefore$ LHS $=$ RHS the result is true for $n=1$

Assume the result is true for $n=k$

$\therefore$ Assume $1+5+9 \dots + 4k-3 = 2k^2-k$

Prove the result is true for $n=k+1$

Prove $1+5+9+ \dots +4k+3 + 4(k+1)-3 = 2(k+1)^2-(k+1)$

LHS

$$\underbrace{1+5+9+ \dots +4k+3}_{S(k)} + 4(k+1) - 3$$

$= 2k^2 - k + 4(k+1) - 3$

$= 2k^2 - k + 4k + 4 - 3$

$= 2k^2 + 4k + 2 - k + 1$

$= 2(k+1)^2 - (k-1)$

$=$ RHS

$\therefore$ Result is true for $n=k+1$ if it's true for $n=k$

$\therefore$ By the principle of Mathematical Induction it's true for $n \geq 1$

Q2 c)

$2\sin x + \cos x = -1$

$2\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2} = -1$

$\frac{4t + 1 - t^2}{1+t^2} = -1$

$4t + 1 - t^2 = -1 - t^2$

$4t + 1 - \cancel{t^2} + 1 + \cancel{t^2} = 0$

$4t + 2 = 0$

$t = \frac{-1}{2}$

$\tan\frac{x}{2} = \frac{-1}{2}$ $\qquad 0 \leq x \leq 2\pi$

$\frac{x}{2} = \pi - 0.4636$ $\qquad 0 \leq x/2 \leq \pi$

$\frac{x}{2} = 2.6779$

$x = 5.36$

Test $x = \pi$

$2\sin\pi + \cos\pi = 2\times 0 + (-1) = -1$

$\therefore x = \pi$ is a solution

$\therefore x = 5.36, \pi$.

Q2 d)

$y = 2\sqrt{3}\cos\theta - 2\sin\theta$

$2\sqrt{3}\cos\theta - 2\sin\theta = R\cos(\alpha + \beta)$

$\alpha = \tan^{-1}(b/a) = \tan^{-1}(2/2\sqrt{3})$

$= \tan^{-1}(1/\sqrt{3})$

$= \pi/6$

Note :- Could skip this

$R = \sqrt{(2\sqrt{3})^2 + (-2)^2}$

$= \sqrt{12 + 4}$

$= \sqrt{16} = 4$

$\therefore y = 4\cos(\alpha + \pi/6)$

$\therefore$ Amplitude $= 4$

20)

$\sin 2x + \sin 4x = \sin 6x$

LHS

$A+B = 4x$

$A-B = 2x$

$2A = 6x$

$A = 3x$

$B = 4x - 3x = x$

$\therefore \sin 2x + \sin 4x = 2\sin 3x \cos x$

$\therefore 2\sin 3x \cos x = \sin 6x$

$2\sin 3x \cos x = 2\sin 3x \cos 3x.$

$2\sin 3x \cos x - 2\sin 3x \cos 3x = 0$

$2\sin 3x(\cos x - \cos 3x) = 0$

$2\sin 3x \times 2\sin x \sin 2x = 0$

$4\sin 3x \sin x \sin 2x = 0$

$\sin 3x \sin 2x \sin x = 0$

$3x = 0, \pi, 2\pi, 3\pi$

$2x = 0, \pi, 2\pi$

$x = 0, \pi$

$x = 0, \pi/3, \pi, \pi/2$

$A+B = 3x$

$A-B = x$

$2A = 4x$

$A = 2x$

$B = 3x - 2x = x$