QUESTION 1 (9 Marks)

(a) If $f(x) = x\ln(e^{\sqrt{x}})$, find $f'(x)$ (1)

(b) Find $f(x)$, given that $f''(x) = \frac{3}{\sqrt{x}}$, $f'(4) = 7$ and $f(4) = 20$ (3)

(c) Determine (2)

$$\lim_{x\to 0}\frac{6x + 6x\cos 6x}{\sin 6x\cos 6x}$$

(d) Use Simpson's rule with 3 function values to estimate (3)

$$\int_{\pi/2}^{\pi} x^2\sin^2 x\,dx$$

Express your answer to three decimal places.

QUESTION 2 (9 Marks)

(a) Evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \tan^{-1}(-\frac{1}{\sqrt{3}})$ (2)

(b) (i) Show that the equation of the tangent, ℓ, to $y = e^{3x}$ at $x = 1$ is (2)

$y = e^3(3x - 2)$

(ii) If ℓ cuts the x –axis at P and the y-axis at Q, find the coordinates of P and Q. (2)

(iii) Show that the area bound by the y –axis, the curve $y = e^{3x}$ and ℓ is $\frac{1}{6}(5e^3 - 2)$ square units. (3)

QUESTION 3 (9 Marks)

(a) Find $\int \frac{x^2-5}{x^2+3}\,dx$ (2)

(b) A function is defined by $f(x) = \frac{3x}{x^2+1}$

(i) Find any turning points and hence their nature. (3)

(ii) Hence graph $y = \frac{3x}{x^2+1}$ clearly showing what happens to y as x grows large. (2)

(iii) Find the exact area of the region bounded by the curve, the y –axis, and the line $y = \frac{3}{2}$ (2)

QUESTION 4 (9 Marks)

(a) Use the substitution $u = 4 - x^2$ to show that: (3)

$$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx = 2 - \sqrt{3}$$

R70

(b) (i) Express $\sqrt{3}\sin 2\theta - \cos 2\theta$ in the form $R\sin(2\theta - \alpha)$, $0 \le \alpha \le \frac{\pi}{2}$ (2)

(ii) Hence solve: $\sqrt{3}\sin 2\theta - \cos 2\theta = 1 \quad ; \quad 0 \le \theta \le \pi.$ (2)

(c) For what values of m is the line with equation $mx + y = 3$ a tangent to the circle $x^2 + y^2 = 5$? (2)

QUESTION 5 (9 Marks)

(a) Consider the function $f(x)=\frac{1}{2}\cos^{-1}(3x-1)$

(i) State the domain and range of $f(x)$. (1)

(ii) Hence sketch the graph of $y=f(x)$. (1)

(b) In the diagram below, two circles of equal radius r units are drawn such that their centres O and P are r units apart. The two circles intersect at A and B.

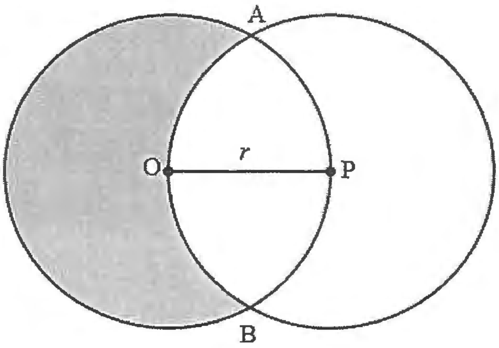


Diagram not to scale

(i) Show that quadrilateral $AOBP$ is a rhombus. (1)

(ii) Hence, or otherwise, find the area of the shaded region in terms of r. (2)

(c) (i) Differentiate with respect to x: $[\tan^{-1}(\frac{x}{3})]^2$ (2)

(ii) Hence find the exact value of (2)

$$\frac{1}{\pi}\int_0^{\sqrt{3}} \frac{\tan^{-1}(\frac{x}{3})}{x^2+9}\,dx$$

QUESTION 6 (9 Marks)

(a) Given $f(x) = \sin^{-1}(x^2 - 1)$.

(i) Find $f'(x)$ (1)

(ii) Hence write down the domain of $f'(x)$ (1)

(b) Find the exact volume of the solid obtained by rotating the region bounded by the curves $y = 1 - x^2$ and $y = 1 - x$ about the x –axis. (2)

(c) Prove that $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\left(\frac{x}{a}\right)$, $|x| < a$ (2)

(d) The cost (C) and revenue (R) of a refrigerator manufacturer can be modelled respectively by the equations (3)

$$C = 87\ 000 + 150x \quad \text{and}$$

$$R = x^2 - 80\ 500,$$

where x is the number of units produced in 1 week.

If production in one particular week is 500 units and is increasing at a rate of 300 units per week, find the rate at which the profit is changing.

QUESTION 7 (9 Marks)

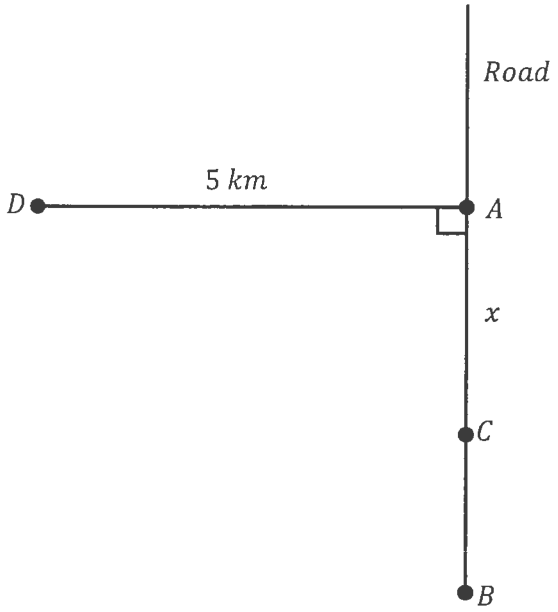
(a) Find the values of x for which the following inequations are simultaneously satisfied:

$x + \frac{1}{|x|} < 0$ and $x^2 - x - 2 > 0$ (3)

(b) The region under the graph $y = \sqrt{x}e^x$ between $x = 1$ and $x = 3$ is rotated about the x –axis. Using the trapezoidal rule with five function values, estimate the volume of the solid formed, to four significant figures. (3)

(c) A motorist is stranded in the desert 5 kilometres from point A, which is a point on a long straight road nearest to him, as shown in the diagram below. He wishes to get to a point B, on the road, which is 5 kilometres from A. He can travel at 16 km/h on the desert and 39 km/h on the road.

Let the distance from A to C be x kilometres and the time taken to reach his destination be T.



(i) Show that the time, T, is given by $T = \frac{39\sqrt{25+x^2}-16x+80}{624}$ (1)

(ii) Hence find the point C at which the motorist must join the road to get to B in the shortest possible time and the time taken to achieve this. (2)

END OF EXAMINATION

Y12 MATH EXT 1 ASSESSMENT TASK 1
TERM 4, 2012

MATHEMATICS Extension 1 : Question 1

Suggested Solutions	Marks	Marker's Comments
Q1(a) $f(x) = x\ln(e^{\sqrt{x}})$ $= x\sqrt{x}\ln e$ $= x\sqrt{x}$ $= x^{3/2}$ $\boxed{1}$	$\frac{1}{2}$	$f'(x) = \ln(e^{\sqrt{x}}) +$ $x.\frac{e^{\sqrt{x}}}{e^{\sqrt{x}}} \times \frac{1}{2\sqrt{x}}$ $= \ln(e^{\sqrt{x}}) + \frac{x}{2\sqrt{x}}$ 1
$\therefore f'(x) = \frac{3}{2}\sqrt{x}$	$\frac{1}{2}$	$= \ln e^{\sqrt{x}} + \frac{\sqrt{x}}{2}$ $=$ OMG!!!
(b) $f''(x) = \frac{3}{\sqrt{x}} = 3x^{-\frac{1}{2}}$	$(\frac{1}{2})$	
$\therefore f'(x) = \int 3x^{-\frac{1}{2}}dx = \frac{3}{\frac{1}{2}}x^{\frac{1}{2}} + c_1 = 6\sqrt{x} + c_1$		1
but $f'(4) = 7$ so $7 = 6\sqrt{4} + c_1$ $7 = 6 \times 2 + c_1$ $7 = 12 + c_1$ $\therefore c_1 = -5$		
$\therefore f'(x) = 6\sqrt{x} - 5$	1	
so $f(x) = \int(6x^{\frac{1}{2}} - 5)\,dx$ $= 4x^{3/2} - 5x + c_2$	$\frac{1}{2}$	$\frac{6}{\frac{3}{2}} = \frac{6\times2}{3} = 4$
but $f(4) = 20$ $\therefore 20 = 4\times4^{\frac{3}{2}} - 20 + c_2$ $\Rightarrow c_2 = 8$ $\therefore f(x) = 4x^{\frac{3}{2}} - 5x + 8$	$\frac{1}{2}$	$20 = 32 - 20 + c_2$ $c_2 = 8$ $\boxed{3}$

\\~~CALLISTO\StaffHome$\WOHJRAH M Fac Admin\Assessment info\Suggested Mk solns template_V4_half Ls.doc~~

Suggested Solutions	Marks	Marker's Comments

(e) Many Approaches $\lim_{x\to 0} \frac{6x(1+\cos 6x)}{\sin 6x \cos 6x}$

I: $\lim_{x\to 0} \frac{6x}{\sin 6x} \times \lim_{x\to 0} \frac{(1+\cos 6x)}{\cos 6x}$

$= 1 \times \frac{(1+\cos 0)}{\cos 0}$

$= 1 \times \frac{(1+1)}{1}$

$\underline{= 2}$

(II): $\lim_{x\to 0} \left(\frac{6x}{\sin 6x \cos 6x} + \frac{6x \cos 6x}{\sin 6x \cos 6x} \right)$

$= \lim_{x\to 0} \left(\frac{12x}{\sin 12x} + \frac{6x}{\sin 6x} \right)$ cos6x≠0 for x=0

$= 1 + 1$

$\underline{= 2}$

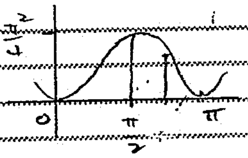
$\boxed{2}$

(III): $\lim_{x\to 0} \frac{6x}{\frac{1}{2}\sin 12x} \cdot (2\cos^2 3x)$, $= \lim_{x\to 0} \frac{12x}{\sin 12x} \times 2\cos^2 3x$

$= 1 \times 2\cos^2 0$

$\underline{= 2}$

(d) $I = \int_{\frac{\pi}{2}}^{\pi} x^2 \sin^2 x \, dx$



$h = \frac{\pi - \frac{\pi}{2}}{2} = \frac{\pi}{4}$ ✓

	a		b
x	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$x^2 \sin^2 x$	$\frac{\pi^2}{4}$ ✓ 2.467...	$\frac{9\pi^2}{32}$ ✓ 2.2258	0
	y_1	y_2	y_3

$I \doteq \frac{h}{3}\left[y_1 + 4y_2 + y_3 \right]$

$= \frac{\frac{\pi}{4}}{3}\left[\frac{\pi^2}{4} + 4 \times \frac{9\pi^2}{32} + 0 \right]$ ✓

$= \frac{\pi}{12}\left[\frac{\pi^2}{4} + \frac{9\pi^2}{8} \right]$

$= \frac{\pi}{12} \times \frac{11\pi^2}{8}$ $\frac{\pi}{12} \times 13.5707...$

$= \frac{11}{96}\pi^3$ ✓

$= 3.5528...$ 'display'

$\underline{\therefore I = 3.553 \ (3dp)}$

$I \doteq \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$

$= \frac{\pi - \frac{\pi}{2}}{6}\left[f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi) \right]$

$= \frac{\pi}{12}\left[\frac{\pi^2}{4} + 4 \times \frac{9\pi^2}{32} + 0 \right]$

$= \frac{\pi}{12}\left[\frac{11\pi^2}{8} \right]$

$= \frac{11}{96}\pi^3$

$= 3.5528...$

$I = 3.553 \ (3dp)$

$h = \pi/4 \ (\frac{1}{2})$

$\frac{h}{3} = \frac{b-a}{6} = \frac{\pi}{12} \ (\frac{1}{2})$

1, 4, 1 $(\frac{1}{2})$

Need the values

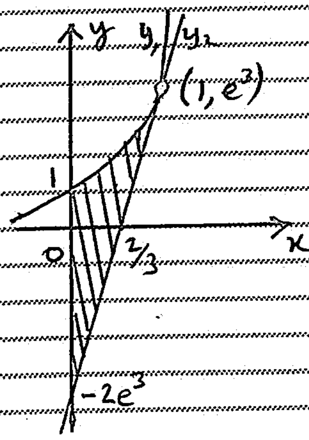
← 'display' $(\frac{1}{2})$

$\frac{1}{2}$ (1)

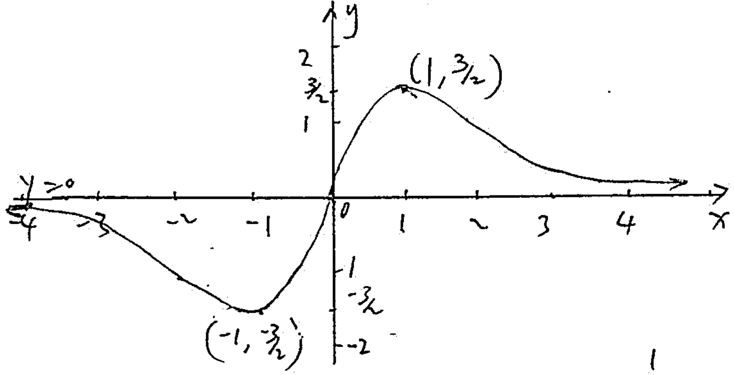
$\boxed{3}$

Note: NOT 'AREA' question !!

TERM 4 ASST 1. 2012 **MATHEMATICS Extension 1 : Question** ..2....

Suggested Solutions	Marks	Marker's Comments
a) $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\frac{5\pi}{6}$ $\quad(0\le\cos^{-1}x\le\pi)$	1	Too many students looked for something harder.
$\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=\frac{-\pi}{6}$ $\quad\left(-\frac{\pi}{2}<\tan^{-1}x<\frac{\pi}{2}\right)$	1	
$\therefore \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=\frac{5\pi}{6}-\frac{\pi}{6}=\underline{\underline{\frac{2\pi}{3}}}$		
b) On $y=e^{3x}$, when $x=1$, $y=e^3$	½	Nealy all students got full marks here
$\frac{dy}{dx}=3e^{3x}$ $\quad\therefore m=3e^3$ when $x=1$.	½	
$\therefore$ Tangent is $(y-e^3)=3e^3(x-1)$	½	
$y=3e^3x-3e^3+e^3$		
$\underline{\underline{y=e^3(3x-2)}}$	½	
c) At P, $y=0$ $\therefore 3x-2=0$, $x=2/3$ $\quad\therefore \underline{\underline{P=(2/3,0)}}$	1	Nealy all correct.
At Q, $x=0$ $\therefore$ $y=-2e^3$ $\quad\therefore \underline{\underline{Q=(0,-2e^3)}}$	1	
d) 		Poorly done. Diagrams awful.
Area between lines $=\int_0^1 (y_1-y_2)\,dx$		Many people made it difficult by splitting up the area into triangles etc or working off the y axis. Whereas it is a straight forward example of its type.
$=\int_0^1 e^{3x}-e^3(3x-2)\,dx$	1	
$=\left[\frac{e^{3x}}{3}-\frac{3x^2}{2}e^3+2e^3x\right]_0^1$	1	
$=\frac{e^3}{3}-\frac{3e^3}{2}+2e^3-\frac{1}{3}=\frac{5}{6}e^3-\frac{1}{3}.$		
$\therefore$ Area is $\underline{\frac{1}{6}(5e^3-2)\text{ sq units.}}$	1	

Y11 T4 2012 **MATHEMATICS Extension 1 : Question** 3.......

Suggested Solutions	Marks	Marker's Comments
a) $\int \frac{x^2+3-8}{x^2+3}\,dx = \int\left(1 - \frac{8}{x^2+3}\right)dx$	1m	Some forgot c $-\frac{1}{2}m$
$= x - \frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$	1m	students have problems with $\frac{8}{\sqrt{3}}$. $-\frac{1}{2}m$
b) $f(x) = \frac{3x}{x^2+1}$		
i) $f'(x) = \frac{(x^2+1)3 - 3x(2x)}{(x^2+1)^2} = \frac{3-3x^2}{(x^2+1)^2}$	1m	just getting $\int dx = x$ did not score any marks.
S.P. $-3x^2+3=0 \quad \therefore x^2=1$ $\therefore x = \pm 1$	$\frac{1}{2}$m	
when $x=1$, $y=3/2$ $x=-1$, $y=-3/2$	$\frac{1}{2}$m	many forgot y values $-\frac{1}{2}m$
Test max/min $\begin{array}{\|c\|c\|c\|c\|c\|c\|}\hline x & -2 & -1 & 0 & 1 & 2 \\ \hline f' & -9/25 & 0 & 3 & 0 & 9/25 \\ \hline\end{array}$		many students got the wrong values of $f'(1)$ r $f'(-1)$, then
$\therefore$ rel min at $(-1, -3/2)$ rel max at $(1, 3/2)$	$\frac{1}{2}$m $+\frac{1}{2}$m	they won't get the $\frac{1}{2}$m for each T.P
ii) 	2m	Forgot write $y=0$ on graph $-\frac{1}{2}m$. T.P $\frac{1}{2}m$ (0,0) o shape 1m $y=0$ $\frac{1}{2}m$
iii) Area $= \frac{3}{2} - \int_0^1 \frac{3x\,dx}{x^2+1} = \frac{3}{2} - \frac{3}{2}\Big[\ln(x^2+1)\Big]_0^1$	$1+\frac{1}{2}$m	Forgot unit² $-\frac{1}{2}m$
$= 3/2 - \frac{3}{2}\ln 2$ unit² #	$\frac{1}{2}$m	$3/2 \rightarrow \frac{1}{2}m$ $\frac{3}{2} - \int_0^1 \frac{3x\,dx}{x^2+1}$ got 1m

1/3

MATHEMATICS Extension 1 : Question 4

Suggested Solutions	Marks	Marker's Comments
(a) let $u = 4 - x^2$ $\frac{du}{dx} = -2x$ $du = -2x\,dx$ $-\frac{1}{2}du = x\,dx$	$\frac{1}{2}$	
when $x = 0$, $u = 4$ when $x = 1$, $u = 3$	$\frac{1}{2}$	
$\therefore \int_0^1 \frac{x}{\sqrt{4-x^2}}dx = \int_4^3 \frac{-\frac{1}{2}du}{\sqrt{u}}$	$\frac{1}{2}$	
$= -\frac{1}{2}\int_4^3 u^{-\frac{1}{2}} du$	$\frac{1}{2}$	* If you never worked out the new limits – 1mk
$= -\frac{1}{2}\left[2\sqrt{u}\right]_4^3$	$\frac{1}{2}$	
$= -(\sqrt{3} - \sqrt{4})$ $= -\sqrt{3} + 2$	$\frac{1}{2}$	
$= 2 - \sqrt{3}$		
or // $= \frac{1}{2}\int_3^4 u^{-\frac{1}{2}} du$ $= \frac{1}{2}\left[2\sqrt{u}\right]_3^4$ $= \sqrt{4} - \sqrt{3}$ $= 2 - \sqrt{3}$		* If you change the limits but then changed everything back to x – $\frac{1}{2}$mk
(b) (i) $\sqrt{3}\sin 2\theta - \cos\theta = R\sin(2\theta - \alpha)$ $= R\sin 2\theta\cos\alpha - R\cos 2\theta\sin\alpha$		$\frac{1}{2}$
$\therefore \sqrt{3} = R\cos\alpha \qquad 1 = R\sin\alpha$		
$\tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = \frac{\pi}{6}$ as $0 \le \alpha \le \frac{\pi}{2}$	$\frac{1}{2}$	

2/3

MATHEMATICS Extension 1 : Question 4

Suggested Solutions	**Marks**	**Marker's Comments**
$R = \sqrt{\sqrt{3}^2+1^2}$ $= \sqrt{4}$ $= 2$ $R > 0$ as α is acute, its in first quad	$\frac{1}{2}$	
$\therefore \sqrt{3}\sin 2\theta - \cos 2\theta = 2\sin(2\theta - \frac{\pi}{6})$		$\frac{1}{2}$ * lost this mark * no conclusion
(ii) $\sqrt{3}\sin 2\theta - \cos 2\theta = 1$ $\therefore 2\sin(2\theta - \frac{\pi}{6}) = 1$ $\sin(2\theta - \frac{\pi}{6}) = \frac{1}{2}$	$\frac{1}{2}$	
$2\theta - \frac{\pi}{6} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$	$\frac{1}{2}$	* If they only said $2\theta - \frac{\pi}{6} = \frac{\pi}{6}$ then; max 1mk
$2\theta = \frac{2\pi}{6}$ or $\frac{6\pi}{6}$ $\theta = \frac{\pi}{6}$ or $\frac{\pi}{2}$ $\frac{1}{2}$ $\frac{1}{2}$		* If they used general solutions and made a mistake $= 1\frac{1}{2}$mks
(c) Method ONE $mx + y = 3$ ---- ① $x^2 + y^2 = 5$ ---- ② sub ① into ② $x^2 + (3-mx)^2 = 5$ $x^2 + 9 - 6mx + m^2x^2 = 5$ $x^2(1+m^2) - 6mx + 4 = 0$	$\frac{1}{2}$	
line is a tangent to a circle, then there's only one soln to quad. eqn ie $\Delta = 0$	$\frac{1}{2}$	* lost $\frac{1}{2}$mk if no explanation!!
$\therefore (-6m)^2 - 4(1+m^2)4 = 0$ $36m^2 - 4(4+4m^2) = 0$	$\frac{1}{2}$	
$36m^2 - 16 - 16m^2 = 0$ $20m^2 = 16$ $m^2 = 16/20$		* lost $\frac{1}{2}$mk if

only wrote $m = \frac{2}{\sqrt{5}}$

$m = \pm\frac{2}{\sqrt{5}}$ $\frac{1}{2}$

3/3

MATHEMATICS Extension 1 : Question 4

Suggested Solutions	Marks	Marker's Comments
(c) METHOD TWO		
the perpendicular distance from a tangent to a circle equals the radius.	1/2	
$\therefore \sqrt{5} = \left\lvert \frac{ax+by+c}{\sqrt{a^2+b^2}} \right\rvert$ $\sqrt{5} = \left\lvert \frac{0\times m + 0\times 1 - 3}{\sqrt{m^2+1}} \right\rvert$ $\sqrt{5} = \frac{3}{\sqrt{m^2+1}}$	$\frac{1}{2}$	
$5 = \frac{9}{m^2+1}$ $m^2+1 = 9/5$	$\frac{1}{2}$	
$m^2 = 4/5$ $m = \pm\frac{2}{\sqrt{5}}$	1/2	
* If the students differentiated		= 0 mks.

MATHEMATICS Extension 1 : Question 5

Suggested Solutions | **Marks** | **Marker's Comments**

a) $f(x) = \frac{1}{2}\cos^{-1}(3x-1)$

Domain $-1 \le 3x-1 \le 1$

$\therefore D = \{x : 0 \le x \le {}^{2}/_{3}\}$

Range $0 \le \cos^{-1}(3x-1) \le \pi$

$0 \le \frac{1}{2}\cos^{-1}(3x-1) \le {}^{\pi}/_{2}.$

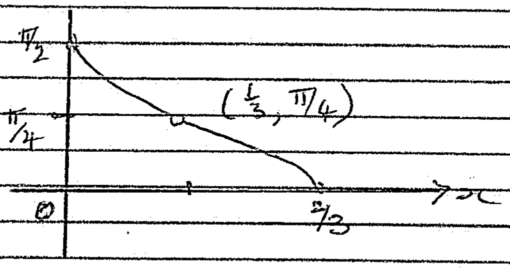
$R = \{y : 0 \le y \le {}^{\pi}/_{2}\}$

①

Domain $\left(\frac{1}{2}\right)$

Range $\left(\frac{1}{2}\right)$

b)



①

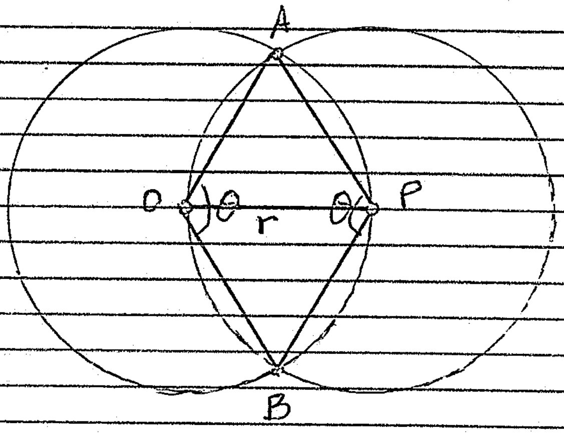
correct end points

correct shape

pass through $\left(\frac{1}{3}, {}^{\pi}/_{4}\right)$

gradient $\neq 0$ at $\left(\frac{1}{3}, {}^{\pi}/_{4}\right)$

(i)



$OA = OB = r$ (equal radii of circle)

$PA = PB = r$ (equal radii of circle)

$\therefore OA = OB = PA = PB$ (equal circles)

$\therefore AOBP$ is a rhombus (4 equal sides)

①

$\left(\frac{1}{2}\right)$ all sides equal to r and circles equal.

$\left(\frac{1}{2}\right)$ rhombus reason.

Geometric statements not in "Geometry notes" NOT ACCEPTED.

(ii) Various solutions

$AO = OP = AP = r$

$\therefore \angle AOP = {}^{\pi}/_{3}$ (angle of equilateral triangle)

$\angle AOB = 2 \times {}^{\pi}/_{3} = {}^{2\pi}/_{3}. = \angle APB = \theta$

$\therefore$ Segment AOB = segment APB.

$= \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}r^2\left({}^{2\pi}/_{3} - \sin {}^{2\pi}/_{3}\right)$

$= \frac{1}{2}r^2\left({}^{2\pi}/_{3} - {}^{\sqrt{3}}/_{2}\right)$

Area (shaded) $= \pi r^2 - 2 \times \frac{1}{2} r^2\left({}^{2\pi}/_{3} - {}^{\sqrt{3}}/_{2}\right)$

$= \pi r^2 - \frac{2\pi}{3}r^2 + \frac{\sqrt{3}}{2}r^2$

Area $= \frac{\pi r^2}{3} + \frac{\sqrt{3}}{2}r^2 \;\; u^2$

$\left(\frac{1}{2}\right)$ angle with reason.

$\left(\frac{1}{2}\right)$ sub into formula.

$\left(\frac{1}{2}\right)$ $\sin {}^{2\pi}/_{3} = \frac{\sqrt{3}}{2}.$

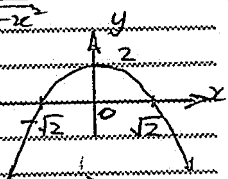
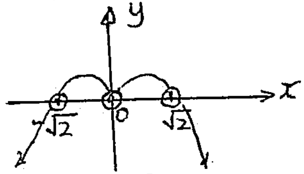
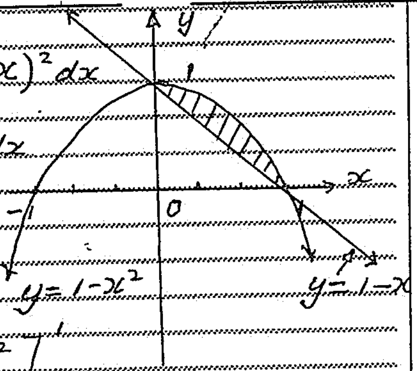
$\left(\frac{1}{2}\right)$ correct answer.

MATHEMATICS Extension 1 : Question 5

Suggested Solutions	Marks	Marker's Comments
c) (i) $\frac{d}{dx}\left(\tan^{-1}(x/3)\right)^2 = 2\tan^{-1}(x/3) \times \frac{1/3}{1+\left(\frac{x}{3}\right)^2}$ $= \frac{6\tan^{-1}(x/3)}{x^2+9}$	②	(½) each part of answer + (½) + (½) (½) simplified answer.
(ii) $\frac{1}{\pi}\int_0^{\sqrt{3}} \frac{\tan^{-1}(x/3)}{x^2+9}\,dx$ $= \frac{1}{6\pi}\left[\left(\tan^{-1}\left(\frac{x}{3}\right)\right)^2\right]_0^{\sqrt{3}}$ $= \frac{1}{6\pi}\left[\left(\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right)^2 - \left(\tan^{-1}0\right)^2\right]$ $= \frac{1}{6\pi}\left[\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)^2\right]$ $= \frac{1}{6\pi}\left[\left(\frac{\pi}{6}\right)^2\right]$ $= \frac{1}{6\pi} \times \frac{\pi^2}{36}$ $= \frac{\pi}{216}$	②	(½) $\frac{1}{6\pi}$ (½) $\left(\tan^{-1}\left(\frac{x}{3}\right)\right)^2$ (½) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$ (½) correct answer.

Y12 MATH EXT I ASSESSMENT TEST I TERM 4 2012

MATHEMATICS Extension 1 : Question 6

Suggested Solutions	Marks	Marker's Comments
a) $f(x) = \sin^{-1}(x^2-1)$ (i) $f'(x) = \frac{1}{\sqrt{1-(x^2-1)^2}} \times 2x$ $= \frac{2x}{\sqrt{1-(x^4-2x^2+1)}}$ $= \frac{2x}{\sqrt{2x^2-x^4}}$ $= \frac{2x}{\lvert x\rvert\sqrt{2-x^2}} = \begin{cases} \frac{2}{\sqrt{2-x^2}}, & x>0 \\ \frac{-2}{\sqrt{2-x^2}}, & x<0 \end{cases}$	$\frac{1}{2}$ $\frac{1}{2}$	Students failed to learn the correct formula for the derivative of $\sin^{-1}f(x)$. $\boxed{1}$ Note: $\sqrt{x^2}=\lvert x\rvert$
(ii) Domain: $2-x^2>0$ $\therefore -\sqrt{2}<x<\sqrt{2}$  $\therefore$ Domain $= \{x: -\sqrt{2}<x<\sqrt{2},\ x\neq 0\}$	$\frac{1}{2}$	Alternatively:- $2x^2-x^4>0$ $x^2(2-x^2)>0$  $\{-\sqrt{2}<x<0\}\cup\{0<x<\sqrt{2}\}$ $\boxed{1}$
b) $V=\pi\int_0^1 (1-x^2)^2-(1-x)^2\,dx$ $=\pi\int_0^1 x^4-3x^2+2x\,dx$  $=\pi\left[\frac{x^5}{5}-\frac{3x^3}{3}+\frac{2x^2}{2}\right]_0^1$ $=\pi\left[\left(\frac{1^5}{5}-1^3+1^2\right)-(0)\right]$ $\therefore V=\frac{\pi}{5}$ Thus, the volume of the solid of revolution is $\frac{\pi}{5}$ cubic units	1 $\frac{1}{2}$ $\frac{1}{2}$	b) Students failed to learn correct formula and how to apply it correctly for the difference between 2 volumes • Some students misread the question. • $\frac{1}{2}$ mark if students vaguely knew that volume had π, a squared term & that the question involved the difference. • $\frac{1}{2}$ mark for integrating an expression of equal difficulty.

1

MATHEMATICS Extension 1 : Question 6

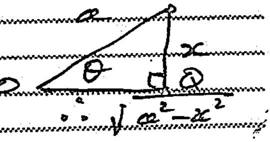
Suggested Solutions | **Marks** | **Marker's Comments**

(c) RTP $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$ for $|x|<a$, $a>0$

APPROACH I: for $-a<x<a \Rightarrow -1<\frac{x}{a}<1$

Let $\theta = \sin^{-1}\frac{x}{a}$ $\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$

so $\sin\theta = \frac{x}{a}$ $\quad \frac{1}{2}$



$a^2 = x^2 + OQ^2$ (Pyth Thm)

$OQ = \sqrt{a^2-x^2}$

For case: $0 \le x < a$

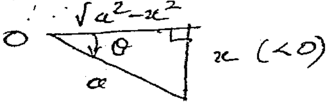
$0 \le \theta < \pi/2$ $\quad \frac{1}{2}$

$\therefore \tan\theta = \frac{x}{\sqrt{a^2-x^2}}$ $\quad \frac{1}{2}$

$\therefore \theta = \tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$

$\frac{1}{2}$

For case: $-a<x\le 0$, $\sin^{-1}$ since $\tan^{-1}$ are ODD FNs or .



$\therefore \tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$ for $|x|<a$

$\boxed{2}$

APPROACH II

consider $y = \tan^{-1}\frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a}$

$$\therefore \frac{dy}{dx} = \frac{1}{1+\frac{x^2}{a^2-x^2}} \times \frac{\sqrt{\ } + x\cdot\frac{x}{\sqrt{\ }}}{a^2-x^2} - \frac{1}{\sqrt{a^2-x^2}}$$

$$= \frac{(a^2-x^2)}{a^2} \times \frac{a^2}{(a^2-x^2)\sqrt{a^2-x^2}} - \frac{1}{\sqrt{a^2-x^2}}$$

$$= \frac{1}{\sqrt{a^2-x^2}} - \frac{1}{\sqrt{a^2-x^2}} = 0$$

$\therefore y$ is a constant for $|x|<a$

let $x=0$ $\quad y = 0 - 0 = 0$

$\Rightarrow \tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$ for $|x|<a$

APPROACH III Let $\alpha = \tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$, $\beta = \sin^{-1}\frac{x}{a}$

consider $\sin(\alpha-\beta)$

etc !!

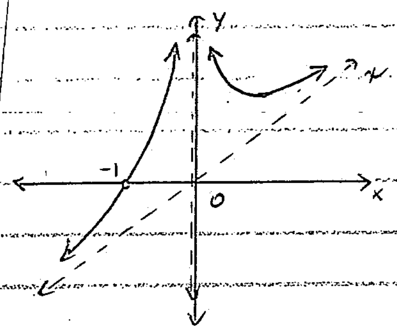
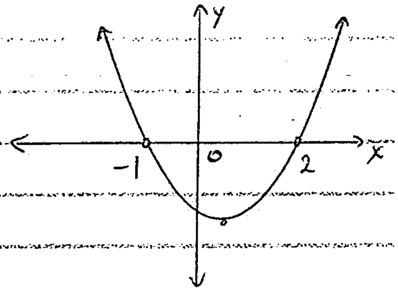
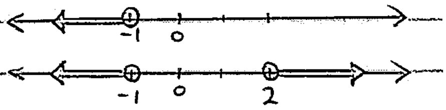
MATHEMATICS Extension 1 : Question 6

Suggested Solutions	Marks	Marker's Comments
(d) Profit $P(x) = R(x) - C(x)$ $= x^2 - 80\,500 - (87\,000 + 150x)$		
$P(x) = x^2 - 150x - 167\,500$	$\frac{1}{2}$	
Data: $\frac{dx}{dt} = 300$ when $x = 500$		
RTF Rate $\left(\frac{dP}{dt}\right)$ when $x = 500$		
Now $\frac{dP}{dt} = \frac{dP}{dx} \times \frac{dx}{dt}$	$\frac{1}{2}$	
$= (2x - 150) \times \frac{dx}{dt}$	$\frac{1}{2}$	
∴ when $x = 500$ 850 $\frac{dP}{dt} = (2 \times 500 - 150) \times 300$		$\boxed{3}$
$= 255\,000$	1	
∴ Rate is \$255 000 / week at 500 units		$\frac{1}{2}$

Suggested Solutions	Marks	Marker's Comments
OR PROFIT $P(x) = R(x) - C(x)$ $= x^2 - 80\,500 - (87\,000 + 150x)$		
$P(x) = x^2 - 150x - 167\,500$		
DATA: $\frac{dx}{dt} = 300$ when $x = 500$		
∴ $x = \int 300\,dt = 300t + c$		
take $t = 0$ $x = 500$ $\Rightarrow c = 500$		
∴ $x = 300t + 500$		
∴ $P(t) = (300t + 500)^2 - 150(300t + 500) - 167\,500$		
$= 90000t^2 + 255000t + 7500$		
$\frac{dP}{dt} = 2(300t + 500) \times 300 - 150 \times 300$ $=$		$180000t + 255000$
∴ at $t = 0$ $\frac{dP}{dt} = 2 \times 500 \times 300 - 150 \times 300$		
$= 255\,000$		
∴ Rate is (\$) 255 000 / week at 500 units		

\\CALLISTO\StaffHome$\WOH\JRAH M Fac Admin\Assessment info\Suggested Mk solns template_V4_half Ls.doc

MATHEMATICS EXTENSION 1: Question 7

Suggested Solutions	Marks	Marker's Comments
(a) METHOD 1: SOLVE GRAPHICALLY		
$x+\frac{1}{\|x\|}<0$	1	Solve $x+\frac{1}{\|x\|}<0$
$x^2-x-2>0$ $(x-2)(x+1)>0$	1	Solve $x^2-x-2>0$ (Students who solved algebraically usually mishandled the absolute value.)
 Satisfied $x<-1$		
 Satisfied $x<-1$ or $x>2$.		
∴ Both inequalities satisfied when $x<-1$.	1	Combine solutions from both inequalities
METHOD 2: SOLVE ALGEBRAICALLY		
$x+\frac{1}{\|x\|}<0$ $\{x\neq 0\}$		Students who did not state their cases clearly often confused themselves.
Case 1: $x>0$ $x+\frac{1}{x}<0$ $x^2+1<0$ ∴ No $\mathbb{R}$ soln		
Case 2: $x<0$ $x+\frac{1}{(-x)}<0$ $x-\frac{1}{x}<0$ $x^2-1>0$ ✱ ∴ $x<-1$ or $x>1$ But $x<0$, ∴ $x<-1$ only.		✱ Note change of inequality direction as $x<0$
$x^2-x-2>0$ $(x-2)(x+1)>0$ $x<-1$ or $x>2$.		
		
∴ Both inequalities satisfied when $x<-1$.		

MATHEMATICS EXTENSION 1: Question

Suggested Solutions	Marks	Marker's Comments
(b) $y = \sqrt{x}\, e^{x}$ $y^2 = xe^{2x}$ $\therefore V = \pi\int_1^3 xe^{2x}\,dx$	1	Exact expression for volume
$\begin{array}{\|c\|c\|c\|c\|c\|c\|} \hline x & 1 & 1\frac{1}{2} & 2 & 2\frac{1}{2} & 3 \\ \hline y^2 & e^2 & \frac{3e^3}{2} & 2e^4 & \frac{5e^5}{2} & 3e^6 \\ \hline \end{array}$ $\therefore h = \frac{1}{2}$ $V \approx \pi \times \frac{\left(\frac{1}{2}\right)}{2}\left[e^2 + 2\left(\frac{3e^3}{2} + 2e^4 + \frac{5e^5}{2}\right) + 3e^6\right]$ $\approx \frac{\pi e^2}{4}\left(1 + 3e + 4e^2 + 5e^3 + 3e^4\right)$ $V \approx 1758 \cdot 0277 \cdots$	1	Application of Trapezoidal Rule (several used all or part of Simpson's Rule instead)
$\therefore$ Volume is approx 1758 units^3 (4 significant figures).	1	Approximated solution
NB $(a^2+b^2+c^2+d^2) \neq (a+b+c+d)^2$		Many students elected to square results after taking the sum, rather than squaring first
Furthermore, Volume $\neq \pi \times$ Area		Others calculated the area under the curve and then multiplied by π, forgetting that $V = \pi\int_a^b y^2\,dx$

MATHEMATICS EXTENSION 1: Question 7

<table>
<tr><th>Suggested Solutions</th><th>Marks</th><th>Marker's Comments</th></tr>
<tr><td>(c)(i) Desert distance CD = $\sqrt{25+x^2}$
Road distance BC = $5-x$</td><td>$\frac{1}{2}$</td><td>Desert travel time</td></tr>
<tr><td>$\therefore\ T = \frac{\sqrt{25+x^2}}{16} + \frac{5-x}{39}$ —①</td><td>$\frac{1}{2}$</td><td>Road travel time</td></tr>
<tr><td>$= \frac{39\sqrt{25+x^2}+16(5-x)}{16\times 39}$
$= \frac{39\sqrt{25+x^2}-16x+80}{624}$ as required</td><td></td><td></td></tr>
<tr><td>(ii) $\frac{dT}{dx} = \frac{x}{16\sqrt{25+x^2}} - \frac{1}{39}$ from ①</td><td>$\frac{1}{2}$</td><td>Derivative</td></tr>
<tr><td>S.P. exist when $\frac{dT}{dx}=0$
$\frac{x}{16\sqrt{25+x^2}} - \frac{1}{39} = 0$
$39x = 16\sqrt{25+x^2}$
$1521x^2 = 256(25+x^2)$
$1265x^2 = 6400$
$x^2 = \frac{1280}{253}$</td><td></td><td></td></tr>
<tr><td>Since $0 \le x \le 5$, $x = \sqrt{\frac{1280}{253}}$</td><td>$\frac{1}{2}$</td><td>Stationary point</td></tr>
<tr><td><table><tr><td>x</td><td>2</td><td>$\sqrt{\frac{1280}{253}}$</td><td>3</td></tr><tr><td>$\frac{dT}{dx}$</td><td>−0·002</td><td>0</td><td>0·007</td></tr></table>∴ Relative minimum.</td><td>$\frac{1}{2}$</td><td>Test nature of SP
(Many students wrongly assumed domain as $x>0$.)</td></tr>
<tr><td>Since this is the only SP & the function is continuous in the domain $0 \le x \le 5$, the relative minimum is also the absolute minimum.
∴ Quickest journey occurs when $x = \sqrt{\frac{1280}{253}}$, which results in a time of 0·413 hours (24·8 minutes).</td><td>$\frac{1}{2}$</td><td>Resultant time</td></tr>
</table>