YEAR 12

ASSESSMENT TASK 2

TERM 1, 2015

MATHEMATICS EXTENSION 1

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 2 hours.
- Write in black or blue pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 6 - 12, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 82

Section I: 5 marks

- Attempt Question 1 – 5.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 8 minutes for this section.

Section II: 77 Marks

- Attempt Question 6 - 12
- Answer on blank paper unless otherwise instructed. Start a new page for each new question.
- Allow about 1 hours & 52 minutes for this section.

Section 1 Multiple Choice (5 marks)

Attempt Questions 1 - 5 (I mark each).

Allow approximately 8 minutes for this section

1 What is the domain and range of $y = \cos^{-1}\left(\frac{5x}{2}\right)$?

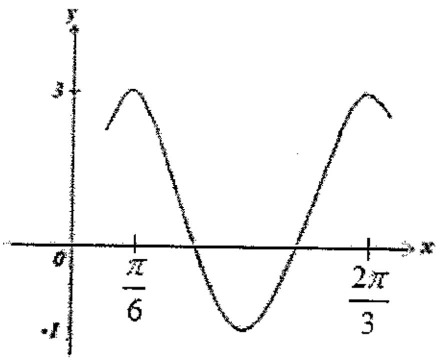
(A) Domain: $-\frac{2}{5} \le x \le \frac{2}{5}$, Range: $-\pi \le y \le \pi$

(B) Domain: $-1 \le x \le 1$, Range: $0 \le y \le \pi$

(C) Domain: $-1 \le x \le 1$, Range: $-\pi \le y \le \pi$

(D) Domain: $-\frac{2}{5} \le x \le \frac{2}{5}$, Range: $0 \le y \le \pi$

2 The graph below could have the equation



(A) $y = 2\cos\left(x + \frac{\pi}{6}\right) + 1$

(B) $y = 2\cos 4\left(x - \frac{\pi}{6}\right) + 1$

(C) $y = 2\cos 2\left(x - \frac{\pi}{6}\right) + 1$

(D) $y = 2\cos 4\left(x + \frac{2\pi}{3}\right) + 1$

3 What is the greatest coefficient in the expansion of $(5+2x)^{12}$?

(A) ${}^{12}C_3 \times 5^9 \times 2^3$

(B) ${}^{12}C_4 \times 5^8 \times 2^4$

(C) ${}^{12}C_5 \times 5^7 \times 2^5$

(D) ${}^{12}C_6 \times 5^9 \times 2^3$

4 What is the indefinite integral for $\int(\cos 3x - \sin^2 x + 2)\,dx$?

(A) $\frac{1}{3}\sin 3x - \frac{3}{2}x - \frac{1}{4}\sin 2x + c$

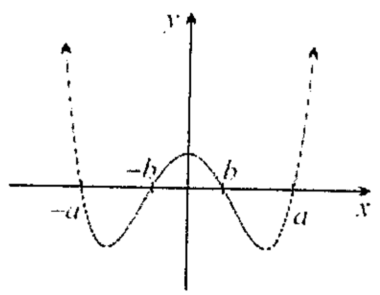
(B) $\frac{1}{3}\sin 3x - \frac{3}{2}x + \frac{1}{4}\sin 2x + c$

(C) $\frac{1}{3}\sin 3x + \frac{3}{2}x + \frac{1}{4}\sin 2x + c$

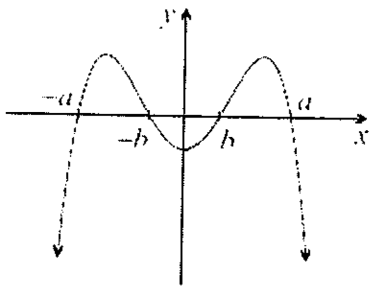
(D) $\frac{1}{3}\sin 3x + \frac{3}{2}x - \frac{1}{4}\sin 2x + c$

5. Which diagram best represents $P(x) = (x-a)^2(b^2 - x^2)$, where a > b?

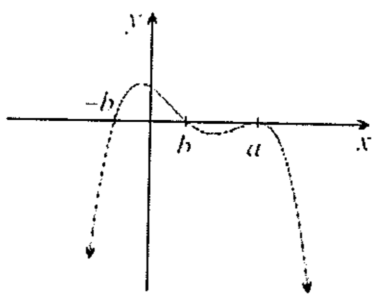
(A)



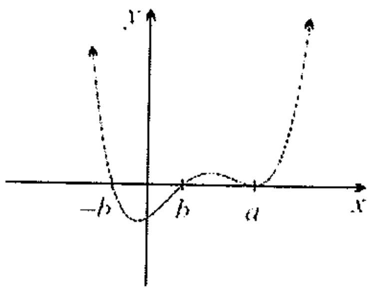
(B)



(C)



(D)



Section II Extended Response (77 marks)
Attempt Questions 6-12. Each question is worth 11 marks.
Allow approximately 1 hour 52 minute for this section.

Question 6 (Start a new page)

(a) Solve $\log_2 x + \log_2(x^2 - 4) - \log_2(x+2) = 3$. 3

(b) Express $0.5\dot{1}\dot{2}$ as a geometric series and hence write its equivalent fraction in simplest term. 2

(c) Use Simpson's approximation method with two sub-intervals to find an approximation to $\int_0^1 e^{x^2}\,dx$. Give your answer correct to 4 decimal places. 2

(d) Sketch the graph of $y = 3\cos^{-1}2x$. 2

(e) Evaluate $\cos\left[\sin^{-1}\left(\frac{-12}{13}\right)\right]$ exactly. 2

Question 7 (Start a new page)

(a) For the arbitrary integer p, find the sum of the series: 2

$2^{2p-1} + 2^{2p-2} + 2^{2p-3} + \dots\dots\dots\dots\dots + 2^{-2p}$

(b) Prove by induction that 5

$3^n > 1+2n$ for $n>1$

(c) The curve $y = sin3x$, is rotated about the x axis. Find the exact volume of the solid of revolution from $x = 0$ and $x = \frac{\pi}{3}$. 4

Question 8 (Start a new page)

(a) Find the following integral, $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x\,dx}{\sqrt{9-x^2}}$ using the substitution $x = 3\sin\theta$. 4

(b) Find the general solutions of the equation 3
$\tan\theta = \sin 2\theta$

(c) Find the stationary points on the curve $y = x^2e^{-2x}$ and sketch the graph. 4

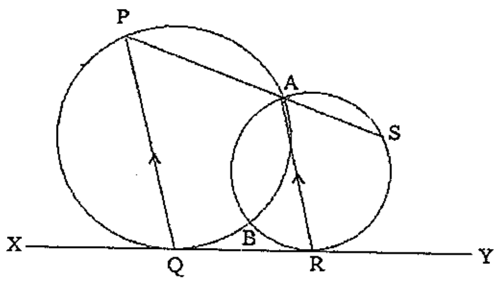
Question 9 (Start a new page)

(a) Andy borrowed \$50000 to purchase a car at 6% per annum reducible interest, calculated monthly. The loan is to be repaid in 60 equal monthly instalments.

(i) Calculate the monthly repayment he must pay to the nearest cent. 3

(ii) With the twelfth repayment, Andy receives \$4000, from an aunt and pays this to the bank towards the repayments of his loan. 3
How many more repayments will be needed to pay off his loan? (assuming the monthly repayment has not changed)

(b) Two circles meet at A and B. QR is a common tangent.



PQ is parallel to AR and PA is produced to meet the other circle at S.
Prove that:

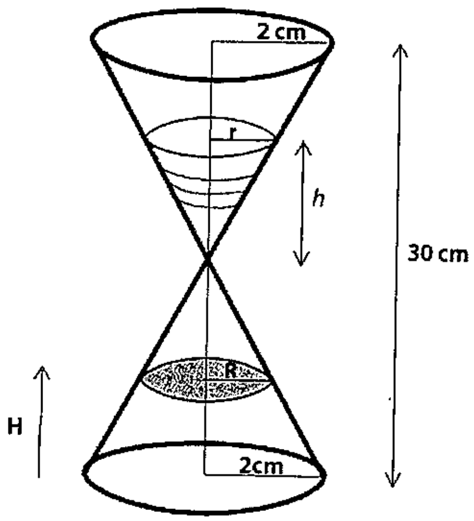
(i) QPSR is cyclic. 3

(ii) QA is parallel to RS 2

Question 10 (Start a new page)

(a) An hour glass has the shape of a double cone 30 *cm* high. The radii at both ends are 2 *cm*. Sand is flowing from the top cone into the bottom cone at the rate of $6\ cm^3/s$.

(i) Show that $r = \dfrac{2h}{15}$ 1



(ii) At what rate is the level of the sand in the top cone falling when the coffee is 6 *cm* deep? 2

(iii) How fast is the level of sand in the bottom cone rising at the instant when the sand in this cone is 6 *cm* deep? 3

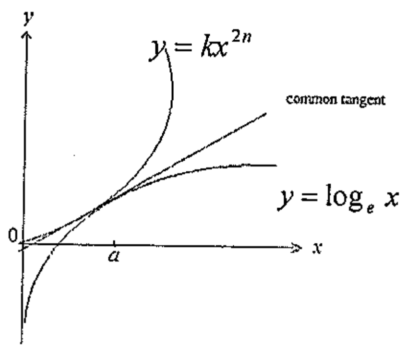
(b) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are on the parabola $x^2 = 4ay$.

(i) The equation of the tangent to $x^2 = 4ay$ at an arbitrary point $(2at, at^2)$ on the parabola is $y = tx - at^2$. (Do not prove this.) Show that the tangents at points P and Q meet at R, where R is the point $(a(p+q), apq)$. 2

(ii) As P varies, the point Q is always chosen so that $\angle POQ$ is a right angle, where O is the origin. Find the locus of R. 3

Question 11 (Start a new page)

(a)

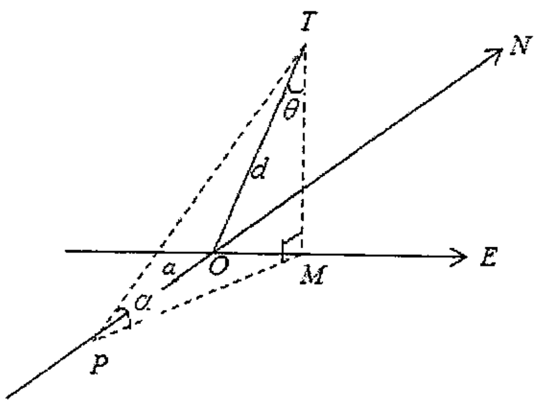


The graphs of the functions $y = kx^{2n}$ and $y = \log_e x$ have a common tangent at $x = a$ as shown in the diagram.

(i) By considering gradients, show that $a^{2n} = \dfrac{1}{2nk}$ 1

(ii) Express k as a function of n by eliminating a. 3

(b) A pole, OT, of length d metres stand on horizontal ground. The pole leans towards the east, making an angle θ with the vertical. From P, a metres south of O, the elevation of T is α.



(i) Prove that $PM = d\cos\theta\cot\alpha$. 2

(ii) Prove that

$$d^2 = \frac{a^2}{\cos^2\theta\cot^2\alpha - \sin^2\theta}$$ 2

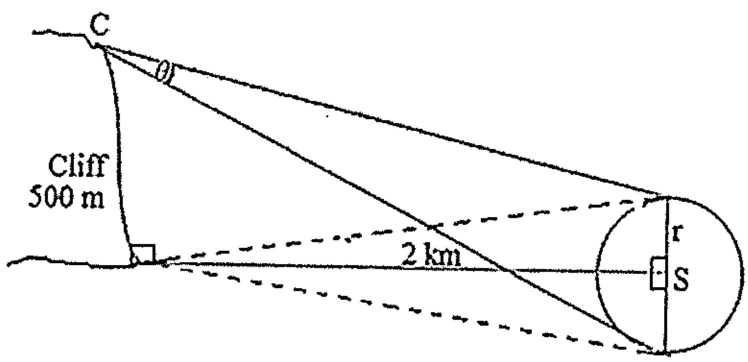
(c) Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+a)$, where $R > 0$ and $0 < x < \frac{\pi}{2}$ 3

Question 12 (Start a new page)

(a) Find the value of x which is independent of x in the expansion $\left(2x^3 + \frac{1}{x^3}\right)^{12}$. 3

(b) The sum of the cubes of three consecutive integers is given by $S_n = (n-1)^3 + n^3 + (n+1)^3$. Prove by method of induction that S_n is divisible by 3 for all $n \geq 1$. 4

(c) A sky diver falls into the sea forming circular waves at S. An observer standing on the cliff (C) is measuring the circular waves formed at S from a position C, 500 metres above sea level and 2 kilometres horizontally from the centre of the waves.



(i) At a certain time, the observer measures the angle θ, subtended by the diameter of the wave, to be 0.1 radians. What is the radius r, at this time? Give your answer correct to 1 decimal place. 2

(ii) At this time $\frac{d\theta}{dt} = 0.01$ radians per hour. Find the rate at which the radius of the wave is growing. Give your answer correct to 1 decimal place. 2

END OF EXAMINATION

1. Domain $-1 \le \frac{5x}{2} \le 1$

$= -\frac{2}{5} \le x \le \frac{2}{5}$

Range is $0 \le y \le \pi$

Answer D

2. Amplitude 2

Cos graph moved one unit up and $\frac{\pi}{6}$ to right

Answer B

3. $(5+2x)^{12}$

$T_{r+1} = {}^{12}C_r 5^{12-r}(2x)^r$ and $T_r = {}^{12}C_{r-1} 5^{13-r}(2x)^{r-1}$

$= {}^{12}C_r 5^{12-r} 2^r x^r \qquad = {}^{12}C_{r-1} 5^{13-r} 2^{r-1} x^{r-1}$

$\frac{T_{r+1}}{T_r} = \frac{{}^{12}C_r 5^{12-r} 2^r x^r}{{}^{12}C_{r-1} 5^{13-r} . 2^{r-1} . x^{r-1}}$

ratio of terms

$\frac{(13-r)\times 2}{r\times 5}$

$= \frac{26-2r}{5r}$.

$\therefore$ Greatest coefficient occurs when $r=3$

$T_4 = {}^{12}C_3 \times 5^9 \times 2^3 \times x^3$

has greatest cofficient when ${}^{12}C_3 \times 5^9 \times 2^3$

Answer is A

4. $\int \cos 3x - \sin^2 x + 2 \; dx$

$= \int \left(\cos 3x - \frac{1}{2}(1-\cos 2x) + 2\right) dx$

$= \frac{1}{3}\sin 3x - \frac{1}{2}x + \frac{1}{4}\sin 2x + 2x + C$

$= \frac{1}{3}\sin 3x + \frac{3}{2}x + \frac{1}{4}\sin 2x + c$

$\therefore$ Answer is C

5. It is a $-x^4$ graph parabola at $x=0$ and x-intercepts at $x=b$ and $x=-b$.

$\therefore$ graph is C

Question 6

(a) $\log_2 x + \log_2(x^2-4) - \log_2(x+2) = 3$

$$\log_2 \frac{x(x^2-4)}{x+2} = 3$$

$$\log_2 \frac{x(x+2)(x-2)}{x+2} = 3$$

$$\log_2 x(x-2) = 3$$

$$2^3 = x^2 - 2x$$

$$8 = x^2 - 2x$$

$$0 = x^2 - 2x - 8$$

$$0 = (x-4)(x+2)$$

$$x = 4 \quad \text{or} \quad x = -2$$

$\therefore$ only solution is $x = 4$

(b) $0.5121212\ldots$

$$0.5 + \underbrace{\frac{12}{1000} + \frac{12}{100000} \cdots\cdots}$$

$$0.5 + \quad a = \frac{12}{1000}, \quad r = \frac{1}{100}$$

$$\therefore S_\infty = \frac{a}{1-r}$$

$$= \frac{12}{1000} \times \frac{100}{99}$$

$$= \frac{12}{990}$$

$$= \frac{4}{330}$$

$\therefore$ total is $\frac{1}{2} + \frac{4}{330} = \frac{169}{330}$

(c)

x	0	$\frac{1}{2}$	1
e^{x^2}	$e^0 = 1$	$e^{1/4}$	e^1

Simpson's Rule

$$\frac{h}{3}\left[e^0 + 4 \times e^{1/4} + e^1\right]$$

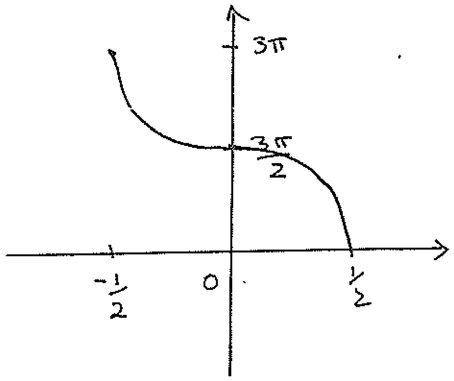
$$= \frac{0.5}{3}\left[1 + 5.136101667 + 2.718281828\right]$$

$$= \frac{0.5}{3}\left[8.854383495\right]$$

$$= 1.475730583$$

$$\approx 1.4757 \text{ sq units}$$

(d)



Domain $-\frac{1}{2} \le x \le \frac{1}{2}$

Range $0 \le y \le 3\pi$. ✓

✓

(e) $\cos\left[\sin^{-1}\left(\frac{-12}{13}\right)\right]$

$= \cos\left[-\sin\left(\frac{12}{13}\right)\right]$

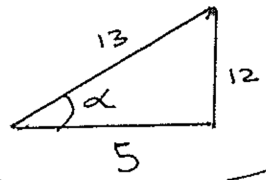
$= \cos(-\alpha)$

$= \cos\alpha$

$= \frac{5}{13}$

let $\sin^{-1}\left(\frac{12}{13}\right) = \alpha$

$\sin\alpha = \frac{12}{13}$



Question 7

(a)
$S(n) = 2^{2p-1} + 2^{2p-2} - - - - - - - 2^{-2p}$

$2S(n) = 2^{2p} + 2^{2p-1} + 2^{2p-2} - - - - - - 2^{-2p-1}$ ✓

$2S_n - S_n = 2^{2p} + 2^{-2p}$

$\therefore S_n = 2^{2p} - 2^{-2p}$ ✓

OR

$a = 2^{2p-1}$, $r = 2^{-1} = \frac{1}{2}$ and $l = 2^{-2p}$

$\therefore S(n) = \dfrac{a + rl}{1 - r}$

$= \dfrac{2^{2p-1} - 2^{-1} \cdot 2^{-2p}}{1 - \frac{1}{2}}$ ✓

$= \dfrac{2^{2p-1} - 2^{-2p-1}}{2^{-1}}$

$= 2^{2p} - 2^{-2p}$ ✓

1(b) Prove $3^n > 1 + 2n$ for $n > 1$

Step 1 Prove $n = 2$

$3^2 - 2(2) - 1 > 0$

$9 - 4 - 1 > 0$

true for $n = 2$ ✓

Step 2 Assume true for $n = k$

$3^k - 2k - 1 > 0$

Step 3 Prove true for $n = k+1$ ✓

$3^{k+1} - 2(k+1) - 1 > 0$

LHS

$= 3^{k+1} - 2k - 3$

$= 3 \cdot 3^k - 2k - 3$ ✓

$3(3^k - 2k - 1) + 4k$

which is > 0 since $3^k - 2k - 1 > 0$ ✓

as is $4k$

Step 4 true for $n = k+1$ if true for $n = 1$ and true for all positive integral $n > 1$

$\therefore$ proved by mathematical induction ✓

7(c) $V = \pi \int_0^{\pi/3} \sin^2 3x \, dx$ ✓

$V = \pi \int_0^{\pi/3} \frac{1}{2}(1 - \cos 6x) \, dx$

$V = \frac{\pi}{2} \int_0^{\pi/3} (1 - \cos 6x) \, dx$ ✓

$= \frac{\pi}{2} \left[x - \frac{1}{6} \sin 6x \right]_0^{\pi/3}$

$= \frac{\pi}{2} \left[\left(\frac{\pi}{3} - \frac{1}{6} \sin \frac{6\pi}{3} \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right]$ ✓

$= \frac{\pi}{2} \left[\left(\frac{\pi}{3} - \frac{1}{6} \sin 2\pi \right) - 0 \right]$

$= \frac{\pi}{2} \left[\frac{\pi}{3} \right]$

$= \frac{\pi^2}{6}$ unit^3. ✓

Question 8

(a) $$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x}{\sqrt{9-x^2}}\,dx = \int_0^{\pi/3} \frac{3\sin\theta\, 3\cos\theta}{\sqrt{9-9\sin^2\theta}}$$

let $x = 3\sin\theta$

$dx = 3\cos\theta$

$x = \frac{\sqrt{3}}{2}$

$\sin\theta = \sqrt{3}/2$

$\therefore \theta = \pi/3$

$$= \int_0^{\pi/3} \frac{9\sin\theta\cos\theta}{\sqrt{9\cos^2\theta}}.$$

$$= \int_0^{\pi/3} \frac{9\sin\theta\cos\theta}{3\cos\theta}$$

$$= \int_0^{\pi/3} 3\sin\theta.$$

$$= \Big[-3\cos\theta\Big]_0^{\pi/3}$$

$$= \Big[-3\times\tfrac{1}{2} + 3$$

$$= \frac{3}{2}.$$

(b) $\tan\theta = \sin 2\theta$

$\tan\theta - \sin 2\theta = 0$

$\frac{\sin\theta}{\cos\theta} - 2\sin\theta\cos\theta = 0$

$\times\cos\theta$ $\quad \sin\theta - 2\sin\theta\cos^2\theta = 0$

$\sin\theta(1 - 2\cos^2\theta) = 0$

$\sin\theta = 0$ or $1 - 2\cos^2\theta = 0$

$\sin\theta = \sin 0$ $\qquad \cos^2\theta = \frac{1}{2}.$

$\therefore \theta = n\pi$ $\qquad \cos\theta = \pm\sqrt{\frac{1}{2}}$

$\cos\theta = \frac{1}{\sqrt{2}}$ or $\cos\theta = -\frac{1}{\sqrt{2}}$

$\cos\theta = \cos\frac{\pi}{4}$ $\qquad \cos\theta = \cos\frac{3\pi}{4}$

$\therefore \theta = 2n\pi \pm \frac{\pi}{4}$ $\qquad \therefore \theta = 2n\pi \pm \frac{3\pi}{4}.$

8 (c) $y = x^2 e^{-2x}$

Stationary points occur when $y' = 0$

Product rule

let $u = x^2 \quad \therefore u' = 2x$

$v = e^{-2x} \quad \therefore v' = -2e^{-2x}$.

$\therefore y' = vu' + uv'$

$= e^{-2x} \cdot 2x + x^2 \cdot -2e^{-2x}$.

$= 2x e^{-2x} - 2x^2 e^{-2x}$

$= 2xe^{-2x}(1 - x)$

when $2xe^{-2x} = 0$ or $1 - x = 0$

$x = 0$. $\quad x = 1$

when $x = 0$, $y = 0$ $\therefore$ stationary point at $(0,0)$

when $x = 1$, $y = e^{-2}$ $\therefore$ stationary point at $(1, e^{-2})$.

Nature either second derivative or test points on either side of first derivative

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
y''	$\frac{-3e}{2}$	///	$\frac{1}{2e}$
	$-$ve		+ve

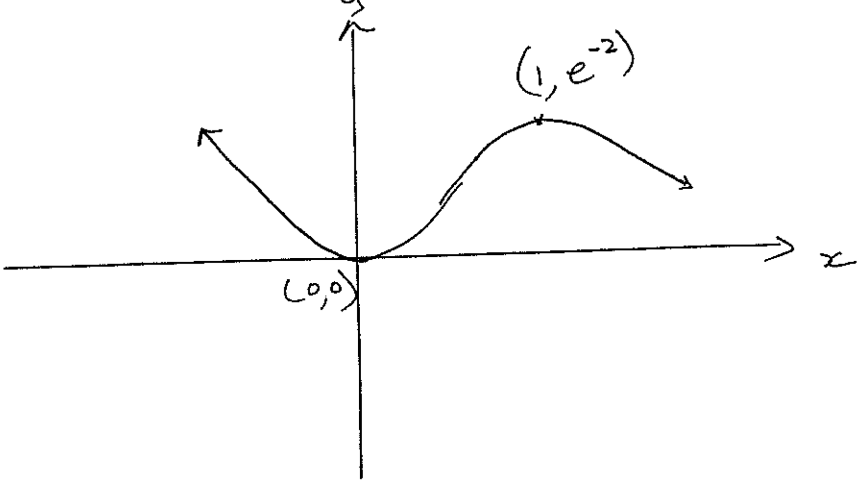
Minimum point

x	$\frac{1}{2}$	1	$\frac{3}{2}$
y''	$\frac{1}{2e}$	///	$\frac{-3}{2}e^{-3}$
	+ve		$-$ve

Maximum point

$\therefore$ at $(0,0)$ - there is minimum point

and at $(1, e^{-2})$ there is a maximum point.



Question 9

(a)(i) \$50 000   Interest per month = 6% = 0.5% = 0.005   payed in 60 monthly installments

$$A_1 = 50000 \times 1.005 - M$$

$$A_n = 50000 \times 1.005^n - M(1 + 1.005 + 1.005^2 \cdots\cdots 1.005^{n-1})$$

we know

$$A_{60} = 0$$

$$\therefore 0 = 50000 \times 1.005^{60} - M\underbrace{(1 + 1.005 + 1.005^2 \cdots\cdots 1.005^{59})}$$

$$S(n) = \frac{a(r^n - 1)}{r-1}$$

$$= \frac{1.(1.005^{60} - 1)}{1.005 - 0.005}$$

$$= \frac{1.005^{60} - 1}{0.005}$$

$$= 69.77003051$$

$$\therefore M = \frac{50000 \times 1.005^5}{69.77003051}$$

$$= 966.6400765$$

$$= \$966.64$$

(ii) $$A_{12} = 50000 \times 1.005 - 966.64\frac{(1.005^{12} - 1)}{1.005 - 1}$$

$$= 38325.95199$$

$$= \$38325.95$$

An extra \$4000 paid means (38325.95 − 4000) \$34325.95 remains to be paid off.

$$\therefore 34325.95 \times 1.005^n - 966.64\frac{(1.005^n - 1)}{0.005} = 0$$

$$34325.95 \times 1.005^n - \frac{966.64(1.005)^n}{0.005} + \frac{966.64}{0.005} = 0$$

$$1.005^n\left(34325.95 - \frac{966.64}{0.005}\right) = \frac{-966.64}{0.05}$$

$$1.005^n(-159002.05) = -193328$$

$$\therefore 1.005^n = \frac{-193328}{-159002.05}$$

$$= 1.215883695$$

$$n \log 1.005 = \log 1.215883695$$

$$\therefore n = 39.19188101$$

Therefore there are 40 more payments left.

9(b)(i) $\angle APQ = \angle RAS$ (corresponding angles as $PQ \parallel AR$)

$\angle RAS = \angle SRY$ (angle between the chord and tangent is equal to any angle in the alternat segment)

$\therefore \angle APQ = \angle SRY$

$\therefore QPSR$ is cyclic (the interior angle in a cyclic quadrilateral is equal to the opposite exterior angle.

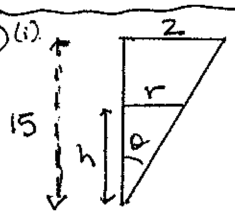
(ii) $\angle AQY = \angle APQ$ (interior angle in a cyclic quadrilateral is equal to the opposite exterior angle)

from part (i) $\angle AQY = \angle SRY$

$QA \parallel RS$ (corresponding angles in parallel lines.

Question 10

(a)(i)



1 $\begin{cases} \tan\theta = \dfrac{2}{15} \\ \tan\theta = \dfrac{r}{h} \end{cases}$

$\dfrac{r}{h} = \dfrac{2}{15}$

$\therefore r = \dfrac{2h}{15}$

or 1 $\dfrac{r}{2} = \dfrac{h}{15}$ + (corresponding sides of similar triangles are proportional).

$\therefore r = \dfrac{2h}{15}$

no explanation = 0 marks

(ii) $\dfrac{dV}{dt} = -6 \text{ cm}^3/\text{s}$ and $h = 6$ cm.

$$\left.\begin{aligned} V_{cone} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(\frac{2h}{15}\right)^2 h \\ &= \frac{4\pi h^3}{3\times 225} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \frac{dV}{dh} &= \frac{12\pi h^2}{3\times 225} \\ &= \frac{4\pi h^2}{225} \end{aligned}$$

1

$\dfrac{dV}{dt} = \dfrac{dV}{dh} \cdot \dfrac{dh}{dt}$ $\qquad$ $\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$

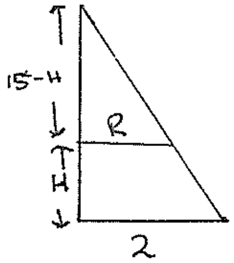
$-6 = \dfrac{4\pi(6^2)}{225} \times \dfrac{dh}{dt}$ $\qquad$ $\dfrac{dh}{dt} = \dfrac{225}{4\pi(6^2)} \times -6$

$\therefore \dfrac{dh}{dt} = \dfrac{-6\times 225}{4\pi(36)}$

$= \dfrac{-75}{8\pi}$

$\therefore$ level is falling at the rate of $\dfrac{75}{8\pi}$ cm/s 1

For bottom cone - similarity



$$\frac{R}{2} = \frac{15-H}{15}$$

$$\therefore R = \frac{2}{15}(15-H)$$

Amount of sand is:

$$V = \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi R^2 h$$

$$= \frac{\pi}{3}(4).15 - \frac{\pi}{3}R^2(15-H)$$

$$= 20\pi - \frac{\pi}{3}\left(\left(\frac{2}{15}(15-H)\right)^2(15-H)\right)$$

$$= 20\pi - \frac{\pi}{3} \times \frac{4}{225}(15-H)^3$$

$$= 20\pi - \frac{4\pi}{3\times225}(15-H)^3 \quad \text{- - - - - -} \quad 1$$

$$\frac{dV}{dH} = \frac{3\times 4\pi}{3\times225}(15-H)^2$$

$$= \frac{4\pi(15-H)^2}{225}$$

$$\frac{dV}{dt} = 6$$

$$\boxed{\frac{dV}{dt} = \frac{dV}{dH} \times \frac{dH}{dt}}$$

$$6 = \frac{4\pi(15-H)^2}{225} \times \frac{dH}{dt} \quad \text{- - - - - -} \quad 1$$

$$6 = \frac{4\pi(15-6)^2}{225} \times \frac{dH}{dt}$$

$$6 = \frac{4\pi(9^2)}{225} \times \frac{dH}{dt} \qquad \to 6 = \frac{36\pi}{25} \cdot \frac{dH}{dt}$$

$$6 = \frac{324\pi}{225} \cdot \frac{dH}{dt}$$

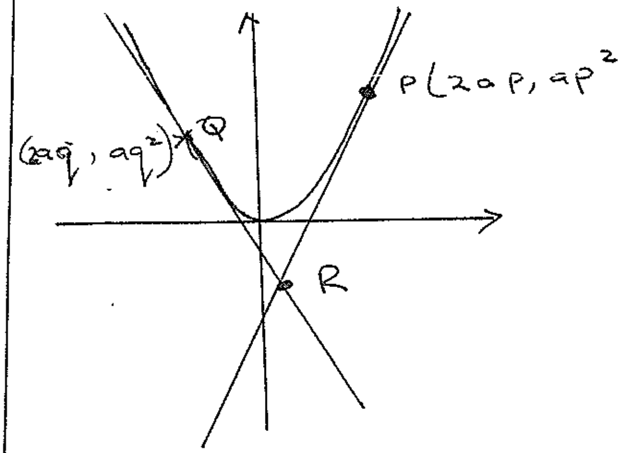
$$\therefore \frac{dH}{dt} = \frac{6\times225}{324\pi}$$

$$= \frac{25}{6\pi}$$

$\therefore$ Sand is rising at the rate of 1

$\frac{25}{6\pi}$ cm/s. $\doteqdot 1.326$ cm/s

4. (b)(i)



From equation $y = t(x - at^2)$.

tangent of P is

$y = px - ap^2$

tangent of Q is

$y = qx - aq^2$

R is the intersection

$px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p^2 - q^2)$

$x(p-q) = a(p+q)(p-q)$

$\therefore x = a(p+q)$ 1

when $x = a(p+q)$ then $y = p(a(p+q)) - ap^2$

$ap(p+q) - ap^2$

$= ap^2 + apq - ap^2$ 1

$= apq$

$\therefore R$ is $(a(p+q), apq)$

b(ii) If $\angle POQ = 90°$

then $M_{PO} = \dfrac{ap^2 - 0}{2ap - 0} = \dfrac{p}{2}$

and $M_{QO} = \dfrac{aq^2 - 0}{2aq - 0} = \dfrac{q}{2}$ 1

then $M_{PO} \times M_{QO} = -1$

$\dfrac{p}{2} \times \dfrac{q}{2} = -1$

$pq = -4$ 1

at R $x = a(p+q)$

$y = a\,pq$

$y = -4a$

$\therefore$ Locus of R is $y = -4a$. 1

a (i) $y = kx^{2n}$ then $y' = 2nkx^{2n-1}$

$y = \log_e x$ then $y' = \frac{1}{x}$

at point a both have same gradient

$$2nkx^{2n-1} = \frac{1}{x}$$

at point a

$$\left.\begin{aligned} 2nka^{2n-1} &= a^{-1} \\ a^{2n-1} &= \frac{1}{2nka} \\ a^{2n} \cdot a^{-1} &= \frac{1}{2nka} \\ \therefore a^{2n} &= \frac{1}{2nk} \end{aligned}\right\}$$ ✓

(ii) when graphs intersect at a

$$ka^{2n} = \log_e a$$

we know $a^{2n} = \frac{1}{2nk}$

add logs on both sides ✓

we know $\log_e a = ka^{2n}$

$$2n \log_e a = -\log_e 2nk$$
$$2n \cdot ka^{2n} = -\log_e 2nk$$
$$2nk \cdot \frac{1}{2nk} = -\log_e 2nk \quad ✓$$
$$1 = -\log_e 2nk$$
$$e^{-1} = 2nk$$
$$k = \frac{1}{2ne} \quad ✓$$

Alternatively

$$ka^{2n} = \log_e a$$
$$k \cdot \frac{1}{2nk} = \log_e a$$
$$\frac{1}{2n} = \log_e a$$
$$e^{-2n} = a$$

Substitute a

$$\left(e^{-2n}\right)^{2n} = \frac{1}{2nk}$$
$$e^{-4n^2} = \frac{1}{2nk}$$
$$k = \frac{1}{2n} \times e^{4n^2}$$
$$k = \frac{e^{4n^2}}{2n}$$

11(b) (i) $\sin\theta = \dfrac{OM}{d}$ $\quad \therefore OM = d\sin\theta$

$\cos\theta = \dfrac{TM}{d}$ $\quad \therefore TM = d\cos\theta$

$\tan\alpha = \dfrac{TM}{PM}$

$\therefore PM = \dfrac{TM}{\tan\alpha}$

$= TM\cot\alpha$

$= d\cos\theta\cot\alpha$

(ii) $a^2 = PM^2 - OM^2$

$a^2 = (d\cos\theta\cot\alpha)^2 - (d\sin\theta)^2$

$a^2 = d^2\cos^2\theta\cot^2\alpha - d^2\sin^2\theta$

$a^2 = d^2(\cos^2\theta\cot^2\alpha - \sin^2\theta)$

$$\therefore d^2 = \frac{a^2}{\cos^2\theta\cot^2\alpha - \sin^2\theta}$$

11 c $\cos x - \sqrt{3}\sin x = R\cos(x+a)$

$= R\cos x\cos a - R\sin x\sin a$

$R\cos a = 1$ and $R\sin a = +\sqrt{3}$ and $R^2 = 4$

$\tan a = \sqrt{3}$ $\qquad \therefore R = 2$

$\therefore \quad a = \dfrac{\pi}{3}$

$\therefore \cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$

Question 12

a). $T_{r+1} = {}^nC_r\, a^{n-r} b^r$ for $(a+b)^n$.

For $(2x^3 + x^{-3})^{12}$

$T_{r+1} = {}^{12}C_r (2x^3)^{12-r}(x^{-3})^r = {}^{12}C_r\left(2^{12-r}.\,x^{36-3r}.\,x^{-3r}\right)$

$= {}^{12}C_r.\,2^{12-r}.\,x^{36-6r}$

k is independent of x (ie k is a constant term)

$= kx^0$

$\therefore x^{36-6r} = x^0$

$36 - 6r = 0$

$r = 6$

$T_n = {}^{12}C_7\, 2^{12-7} = 2^5\, {}^{12}C_7 = 32 . \dfrac{12!}{5!7!} = 25344.$

Divisible by 3

$= (n-1)^3 + n^3 + (n+1)^3 = 3k$ where k and n are integers

<u>Step 1</u> for $n=1$

<u>LHS</u> $= 0^3 + 1^3 + 2^3 = 9$

<u>RHS</u> $= 3M = 3 \times 3 = 9$ - so true for $n=1$

<u>step 2</u> Assume S_n is true for $n=k$.

$S(k) = (k-1)^3 + k^3 + (k+1)^3 = 3k.$

$k^3 + (k+1)^3 = 3k - (k-1)^3$

<u>Step 3</u> prove for $n = k+1$

$k^3 + (k+1)^3 + (k+2)^3 = 3k - (k-1)^3 + (k+2)^3$

$= 3k - (k^3 - 3k^2 + 3k - 1) + k^3 + 6k^2 + 12k + 8$

$= 3k - k^3 + 3k^2 - 3k + 1 + k^3 + 6k^2 + 12k + 8$

$= 3k + 9k^2 + 9k + 9$

$= 3(k + 3k^2 + 3k + 3)$

$= 3P$ where $P = k + 3k^2 + 3k + 3$

$\therefore S(n)$ is true for $k+1$

step 4 Since S_n is true for $n=1$, it is true for $n=2$, $n=3$ and so on

By MI S_n is true for all $n = 1, 2 \ldots\ldots$

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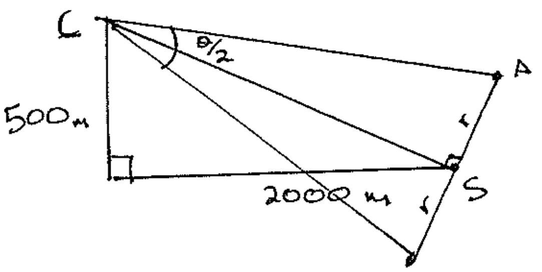
12 c (i).



$CS^2 = 500^2 + 2000^2$

$CS = \cancel{2052m}$

$= 2061.552813$

<u>In △ CAS</u>

$\tan\left(\frac{\theta}{2}\right) = \frac{r}{2061.552813}$

$\therefore r = \tan\left(\frac{\theta}{2}\right) \times 2061.552813$

If $\theta = 0.1$

$= \tan\left(\frac{0.1}{2}\right) \times 2061.552813$

$= \tan 0.05 \times 2061.552813$

$= 103.0776407$

$= 103.1$

$\therefore r = 103.1$ m

$$\frac{d\theta}{dt} = 0.01$$

$$\boxed{\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt}}$$

$$= 1031 \sec^2(0.05) \times 0.01$$

$$- \ 10.31000078$$

$$10.3 \ (1 \text{ dp})$$

we know $r = \tan\frac{\theta}{2} \times 2062$

$$\frac{dr}{d\theta} = \left(2062 \sec^2\frac{\theta}{2} \times \frac{1}{2}\right)$$

$$- \ 1031 \sec^2\frac{\theta}{2}$$

we know $\frac{d\theta}{dt} = \cancel{0.02}\ 0.01$

∴ radius of waves is growing at a rate of 10.3 m/hr