Name:	
Class:	

YEAR 12

ASSESSMENT TASK 2
TERM 1, 2016

MATHEMATICS
EXTENSION 1

Time Allowed – 120 Minutes
(Plus 5 minutes Reading Time)

General Instructions:

- *All* questions may be attempted
- *All* questions are of equal value
- Reference Sheet will be supplied
- Department of Education approved calculators and templates are permitted
- In every Question, show all relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or badly arranged work

Questions 1-5 are to be completed on the multiple choice sheet, then Question 6-12 to be completed on separate sheets of paper and handed in in separate bundles.
Each question must show your Candidate Number (in the top right hand corner).

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 6, Question 7, etc. Each question must show your Candidate Number.

SECTION I MULTIPLE CHOICE (5 marks)

Attempt Questions 1 – 5 (1 mark each)

Allow approximately 8 minutes for this section

1. Which of the following is a primitive function of $\cot x$?

(A) $\ln(\cos x)$
(B) $\ln(\sin x)$
(C) $\ln(\cot x)$
(D) $\ln(\tan x)$

2. What is the $\lim_{x\to 0}\frac{3\sin^2 x}{x}$ equal to?

(A) 0
(B) 1
(C) 3
(D) ∞

3. What is the value of k such that $\int_0^k \frac{6}{9+x^2}\,dx = \frac{\pi}{2}$?

(A) -1
(B) -3
(C) 3
(D) 1

4. What is the value of the term independent of x in the binomial expansion of $\left(x^2 - \frac{1}{x}\right)^9$?

(A) 9
(B) 36
(C) 84
(D) 126

5. What is the domain of the function $f(x) = \sin^{-1}(3x + 1)$?

(A) $-\frac{2}{3} \le x \le \frac{2}{3}$
(B) $-\frac{2}{3} \le x \le 0$
(C) $-\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$
(D) $-\frac{2\pi}{3} \le x \le 0$

END OF SECTION I

SECTION II EXTENDED RESPONSE (77 marks)

77 marks

Attempt Questions 6 – 12

Allow approximately 1 hour and 52 minutes for this section

QUESTION 6 (11 Marks) **COMMENCE A NEW PAGE** **MARKS**

(a) Differentiate with respect to x and simplify as far as possible: **2**

$$y = \ln\left(e^x\sqrt{\sin x}\right)$$

(b) For what values of x does the following series have a limiting sum? **3**

$$1 + \left(\frac{x-2}{x-3}\right) + \left(\frac{x-2}{x-3}\right)^2 + \left(\frac{x-2}{x-3}\right)^3 + \ldots$$

(c) Find the exact value of $\int_{0.5}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx$, using the substitution $\theta = \cos^{-1} x$. **3**

(d) Find the exact volume of the solid formed if the curve $y = \sin^{-1} x$ is rotated about the y – axis from $y = 0$ to $y = \frac{\pi}{3}$. **3**

QUESTION 7 (11 Marks) **COMMENCE A NEW PAGE** **MARKS**

(a) Consider the function $f(x) = xe^{-\frac{1}{2}x^2}$

(i) Is the function odd, even or neither. Explain. **2**

(ii) Determine any asymptotes. **1**

(iii) State the regions where the function is increasing. **1**

(iv) Sketch the graph of the function, showing all intercepts, asymptotes and turning points, if any. **3**

(b) Prove by mathematical induction that for $n \in Z^+$, **4**

$$2.2^0 + 3.2^1 + 4.2^2 + 5.2^3 + \ldots + (n+1)2^{n-1} = n.2^n$$

QUESTION 8 (11 Marks) **COMMENCE A NEW PAGE**

(a) Sketch the graph of $y = \frac{x+1}{x^2-4}$, showing any intercepts and asymptotes. **3**

(b) An inverted cone of base radius 4 cm and height 8 cm is initially filled with water. Water drips out from the vertex at a constant rate of $2\pi\ \text{cm}^3/\text{s}$.

(i) Show that the volume of the water remaining after 16 seconds is $\frac{32\pi}{3}\ \text{cm}^3$. **2**

(ii) If h is the depth of the water remaining at time t, r the radius and V the volume, show that $V = \frac{\pi h^3}{12}\ \text{cm}^3$. **1**

(iii) Hence find the rate of decrease in the depth h, of the water in the cone 16 seconds after the dripping starts, to 3 decimal places. **3**

(c) If $\frac{1}{1+3x+2x^2} = \frac{2}{1+2x} - \frac{1}{x+1}$, calculate the exact area enclosed by the curve $y = \frac{1}{1+3x+2x^2}$, the lines $y = 0$, $x = 0$ and $x = 5$. **2**

QUESTION 9 (11 Marks) **COMMENCE A NEW PAGE**

(a) (i) Show that $\sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$ **1**

(ii) Hence sketch the graph of $y = \sin x - \cos x$ for $-\pi \le x \le \pi$ **3**

(iii) Use the graph to solve $\cos x - \sin x > 0$ for $-\pi \le x \le \pi$ **1**

(b) At the beginning of the first year of joining a superannuation fund, an employee invests \$4000 at 5% p.a. compounded annually. Each year the contributions increase by 10% of the amount invested in the previous year to allow for inflation.

(i) Show that the employee is expected to receive \$342 232.99 at the end of the 20^{th} year? **3**

(ii) At the end of the 10^{th} year however, the fund rules changed, forcing the employee to invest \$4000 maximum per year. **3**

For how long, after the 10^{th} year should contributions be made to receive the same benefit initially expected at the end of the 20^{th} year?

[Assume the interest rate remains the same and the employee works past the planned retirement age at the end of 20 years]

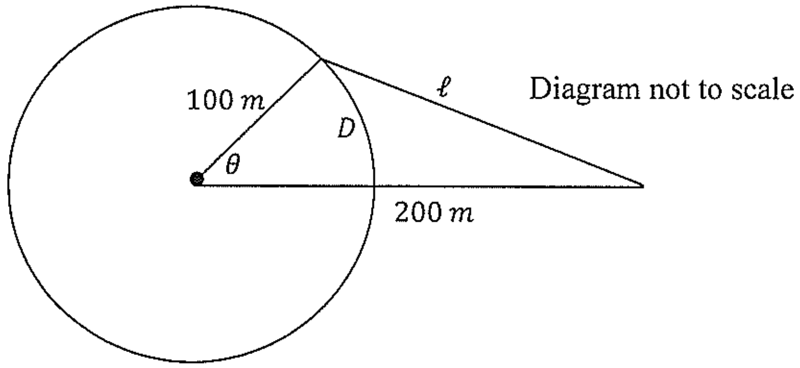
QUESTION 12 (11 Marks) **COMMENCE A NEW PAGE** **MARKS**

(a) Prove that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$ **3**

(b) Prove by mathematical induction that for all integers $n \geq 1$, **3**

$$10^n \geq 1 + 9n$$

(c) A runner sprints around a circular track of radius 100 metres at a constant speed of 7 metres per second. The runner's friend is standing at a fixed distance of 200 metres from the centre of the track, as shown in the diagram.



Let the distance between the runner and friend be ℓ, and D the distance run when the angle is θ.

(i) Show that $\ell = 100\sqrt{5 - 4\cos\left(\frac{D}{100}\right)}$ **2**

(ii) Find the rate of change of the distance between the friend and the runner when they are 200 m apart. **3**

Leave answer in exact form.

☺ **END OF EXAMINATION** ☺

QUESTION 10 (11 Marks) **COMMENCE A NEW PAGE** **MARKS**

(a) A length of channel of given depth d is to be made from a rectangular sheet of metal of width $2x$. The metal is to be bent in such a way that the cross section $ABCD$ is as shown in the figure below, with $AB + BC + CD = 2x$, and with AB and CD inclined to the line BC at an angle θ and $AB = DC$.

(i) Show that the area, A, of the cross section $ABCD$ is given by **3**

$A = 2xd + d^2(\cot\theta - 2\,\mathrm{cosec}\,\theta)$.

(ii) Hence prove that the maximum value that the area A can take, as θ varies, is $d(2x - \sqrt{3}d)$. **3**

[You may assume $\frac{d}{d\theta}(\cot\theta) = -\,\mathrm{cosec}^2\,\theta$ and
$\frac{d}{d\theta}(\mathrm{cosec}\,\theta) = -\cot\theta\,\mathrm{cosec}\,\theta$]

(iii) By considering the length of BC, show that the cross sectional area can only be made equal to the maximum value if $2d \leq \sqrt{3}x$. **2**

(b) Find the general solution to $4\sin 2x - \mathrm{cosec}\,2x = 0$. **3**

QUESTION 11 (11 Marks) **COMMENCE A NEW PAGE**

(a) (i) Show that $\frac{d}{dx}\left(\ln\left(\frac{1+\sin x}{\cos x}\right)\right) = \sec x$, **2**

(ii) Hence find the exact area bounded by the curve $y = \sec x$, the y – axis and the line $y = \sqrt{2}$. **3**

(b) (i) If $t = \tan\frac{\theta}{2}$, express $1 + \sin\theta + \cos\theta$ as a rational fraction in terms of t. **1**

(ii) Hence find $\int \frac{d\theta}{1 + \sin\theta + \cos\theta}$ **2**

(c) If the curve of $y = \sqrt{\frac{1-x}{1+x}}$ is rotated about the x – axis from $x = 0$ to $x = 1$, prove that the exact volume of the solid of revolution formed is $\ln\left(\frac{4}{e}\right)^{\pi}$ unit^3. **3**

TASK 2 TERM 1 MATHS EXTENSION 2016 JRAHS

SECTION I

1. B
2. A
3. C
4. C
5. B

$5 \times 1 = 5$

1. $\cot x = \frac{\cos x}{\sin x}$

$\therefore$ primitive of $\cot x$ is $\ln(\sin x)$

2. $\lim_{x \to 0} \frac{3\sin^2 x}{x} = 3 \lim_{x\to 0} \frac{\sin x}{x} \times \lim_{x \to 0} \sin x$

$= 3 \times 1 \times 0$

$= 0$

3. $6\int_0^k \frac{1}{3^2+x^2}\,dx = 6 \cdot \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right)\Big|_0^k = \frac{\pi}{2}$

$\Rightarrow \tan^{-1}\left(\frac{x}{3}\right)\Big|_0^k = \frac{\pi}{4}$

$\therefore$

$\tan^{-1}(k/3) - \tan^{-1} 0 = \pi/4$

$\Rightarrow k/3 = \tan \pi/4$

i.e $k = 3 \times 1 = 3$

4. $\left(x^2 - \frac{1}{x}\right)^k$

$T_{k+1} = {}^9C_k \left(x^2\right)^{9-k} \left(-\frac{1}{x}\right)^k$

$= {}^9C_k (-1)^k x^{18-3k}$

Term independent of x occurs when $18-3k=0$

i.e $k = 6$

$\therefore T_7 = {}^9C_6 (-1)^6 = 84$

5. $y = \sin^{-1}(3x+1)$

Domain : $-1 \le 3x+1 \le 1$

$-2 \le 3x \le 0$

$-\frac{2}{3} \le x \le 0$

SECTION II :

Question 6

a) $y = \ln\left(e^x \sqrt{\sin x}\right)$

$= \ln e^x + \ln\sqrt{\sin x}$

$= x \ln e + \frac{1}{2} \ln(\sin x)$

$\therefore \frac{dy}{dx} = 1 + \frac{1}{2} \cdot \frac{1}{\sin x} \cdot \cos x$ ✓ for differentiation

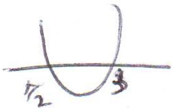
$= 1 + \frac{1}{2}\cot x$ ✓ for simplification

Y12 Ext 1 T1 2016 Q6

a) $y = \ln e^x + \frac{1}{2}\ln\sin x = x + \frac{1}{2}\ln\sin x$ 1m

$\frac{dy}{dx} = 1 + \frac{1}{2}\frac{\cos x}{\sin x}$

$= 1 + \frac{1}{2}\cot x$ # 1m



b) $r = \frac{x-2}{x-3}$

$S = \frac{1}{1-|r|}$

$|r| = \left|\frac{x-2}{x-3}\right| < 1$ 1m

$|x-2| < |x-3|$

$(x-2)^2 < (x-3)^2$ 1m

$\cancel{x^2} - 4x + 4 < \cancel{x^2} - 6x + 9$

$2x < 5$

$x < 5/2$ 1m

#

or $-(x-3)^2 < (x-2)(x-3)$

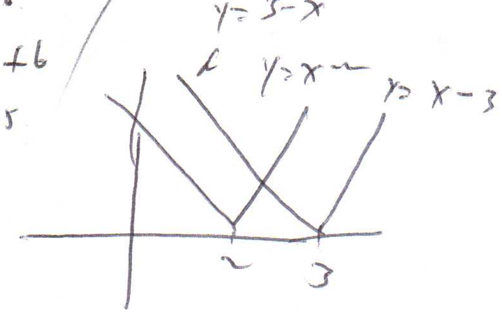
$-(x^2-6x+9) < x^2-5x+6$

$-x^2+6x-9 < x^2-5x+6$

$0 < 2x^2-11x+15$

$(2x-5)(x-3) > 0$

or



soln =

$x - 2 = 3 - x$

$2x = 5$

$x = 5/2$ #

c) $x = \cos\theta \qquad dx = -\sin\theta\, d\theta$

$1 - x^2 = \sin^2\theta \qquad x = \frac{1}{2} \quad \theta = \frac{\pi}{3}$

$x = 1 \quad \theta = 0$

$\int_{\frac{\pi}{3}}^{0} \frac{\sqrt{\sin^2\theta}}{\cos^2\theta}(-\sin\theta)\, d\theta =$

$I = -\int_{\frac{\pi}{3}}^{0} \frac{\sin^2\theta}{\cos^2\theta}\, d\theta = \int_0^{\frac{\pi}{3}} \tan^2\theta$

$= \int(\sec^2\theta - 1)$

$= \int_0^{\frac{\pi}{3}} \frac{\sin^3\theta}{\cos^2\theta}\, d\theta = \int_0^{\frac{\pi}{3}} \tan^2\theta \sin\theta\, d\theta = \int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)\sin\theta\, d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{\sin\theta}{\cos^2\theta} - \sin\theta\, d\theta = \frac{\frac{1}{\cos\theta}}{-1} + \cos\theta\Big]_0^{\frac{\pi}{3}}$

$= \frac{1}{\cos\theta} + \cos\theta\Big]_0^{\frac{\pi}{3}} = \frac{1}{\cos\frac{\pi}{3}} + \cos\frac{\pi}{3} - (1+1)$

$= 2 + \frac{1}{2} - 2 = \frac{1}{2}$ #

Yr 12 Ext T1 2016 Q6

a) $y = \ln\left(e^x\sqrt{\sin x}\right)$

$= \ln e^x + \frac{1}{2}\ln(\sin x)$ 1m

$= x + \frac{1}{2}\ln(\sin x)$

$y' = 1 + \frac{1}{2}\frac{\cos x}{\sin x} = 1 + \frac{1}{2}\cot x$ 1m

or $y' = \dfrac{\cancel{e^x}\left[\frac{1}{2}\frac{\cos x}{\sqrt{\sin x}} + \sqrt{\sin x}\right]}{\cancel{e^x}\sqrt{\sin x}}$ 1m

$= \frac{1}{2}\frac{\cos x}{\sin x} + 1 = 1 + \frac{1}{2}\cot x$ 1m

1 m for properties of $\log_e$

1 m for simplified answer

Students using the 2nd method tend to make mistakes

b) $r = \frac{x-2}{x-3}$

For limiting sum to exist $\left|\frac{x-2}{x-3}\right| < 1$ 1m

$(x-2)^2 < (x-3)^2$

$x^2 - 4x + 4 < x^2 - 6x + 9$ 1m

$2x < 5$

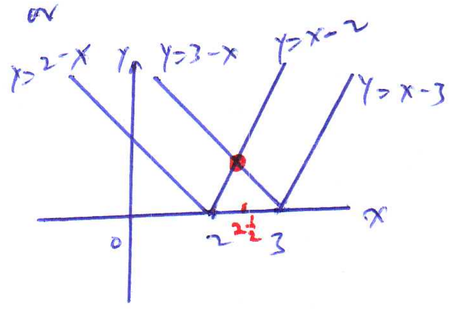
$x < 5/2$ 1 m

This is the best way

most students get the first mark $\left|\frac{x-2}{x-3}\right| < 1$

$-1 < \frac{x-2}{x-3} < 1$

or

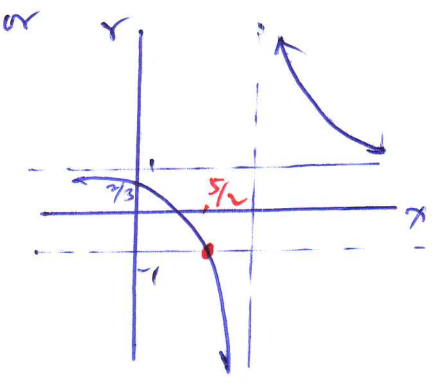


$3 - x = x - 2$

$5/2 = x$

$\therefore$ For $\left|\frac{x-2}{x-3}\right| < 1$ $\therefore x < 5/2$

or



$r = \frac{x-2}{x-3} = 1 + \frac{1}{x-3}$

see graph $-1 < \frac{x-2}{x-3} < 1$

when $1 + \frac{1}{x-3} = -1$

$\frac{1}{x-3} = -2$

$-\frac{1}{2} = x - 3$

$2\frac{1}{2} = x$

$\therefore \left|\frac{x-2}{x-3}\right| < 1$ when $x < 5/2$

Many students use this method

$\left|\frac{x-2}{x-3}\right|<1 \quad \therefore -1<\frac{x-2}{x-3}<1$ 1m

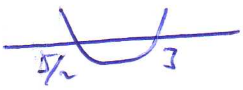
For $-1<\frac{x-2}{x-3}$:

$-(x-3)^2<(x-2)(x-3)$

$-x^2+6x-9<x^2-5x+6$

$0<2x^2-11x+15$

$0<(2x-5)(x-3)$



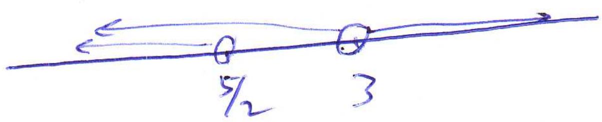
$\therefore x>3$ or $x<5/2$

For $\frac{x-2}{x-3}<1$: $(x-2)(x-3)<(x-3)^2$

$\cancel{x^2}-5x+6 < \cancel{x^2}-6x+9$

$x<3$

$\therefore x>3$ or $x<5/2$ <u>and</u> $x<3$ 1m



$\therefore x<5/2$ is the solution 1m

c) $\int_{0.5}^{1}\frac{\sqrt{1-x^2}}{x^2}dx \qquad x=\cos\theta \qquad \sqrt{1-x^2}=\sin\theta$

$dx=-\sin\theta\, d\theta$

$x=\frac{1}{2} \qquad \theta=\frac{\pi}{3}$

$x=1 \qquad \theta=0$

$=\int_{\frac{\pi}{3}}^{0}\frac{\sin\theta}{\cos^2\theta}\left[-\sin\theta\, d\theta\right]$ 1m

$=\int_{0}^{\frac{\pi}{3}}\frac{\sin^2\theta}{\cos^2\theta}d\theta = \int_{0}^{\frac{\pi}{3}}\tan^2\theta\, d\theta = \int_{0}^{\frac{\pi}{3}}\sec^2\theta-1\, d\theta$ 1m

$=\tan\theta-\theta\Big]_{0}^{\frac{\pi}{3}}=\sqrt{3}-\frac{\pi}{3}$ # 1m

Some students just multiply $(x-3)$ across would never get the right answer

Students need to be careful with the WORD 'AND' 'OR' in equalities

Some students wrote $x>3$ or $x<5/2$ <u>or</u> $x<3$ $\therefore x<5/2$ won't get the last mark

1'3

d) $V = \int_0^{\frac{\pi}{3}} r^2\, dy$ $\qquad x = \sin y$

$= \int_0^{\frac{\pi}{3}} \pi x^2\, dy = \int_0^{\frac{\pi}{3}} \pi \sin^2 y\, dy$ $\qquad$ 1m

$= \pi \int_0^{\frac{\pi}{3}} \frac{1-\cos 2y}{2}\, dy$

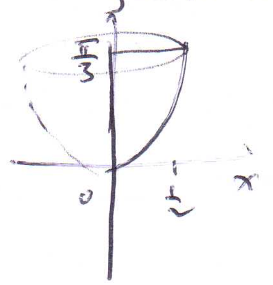
$= \frac{\pi}{2}\left[y - \frac{\sin 2y}{2}\right]_0^{\frac{\pi}{3}}$ $\qquad$ 1m

$= \frac{\pi}{2}\left[\frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2}\right]$

$= \frac{\pi^2}{6} - \frac{\pi}{2}\frac{\sqrt{3}}{4}$

$= \frac{\pi^2}{6} - \frac{\pi}{8}\sqrt{3}$ units^3 # $\qquad$ 1m

$y = \sin^{-1} x$



integrating double angle

MANY forgot $\int \cos 2y = \frac{\sin 2y}{2}$

MATHEMATICS Extension 1 : Question 7

Suggested Solutions	Marks	Marker's Comments

a) i) $f(x) = xe^{-\frac{1}{2}x^2}$

$f(-x) = -xe^{-\frac{1}{2}(-x)^2}$

$= -xe^{-\frac{1}{2}x^2}$

$= -f(x)$ — 1

$\therefore f(x)$ is an odd function — 1

ii) $y = 0$ — 1 — do not leave as $f(x) \to 0$

iii) $f'(x) = e^{-\frac{1}{2}x^2} - x^2e^{-\frac{1}{2}x^2}$

$= e^{-\frac{1}{2}x^2}(1 - x^2)$

for stationary points $f'(x) = 0$

$0 = (1 - x^2)$

$x^2 = 1$

$x = \pm 1$

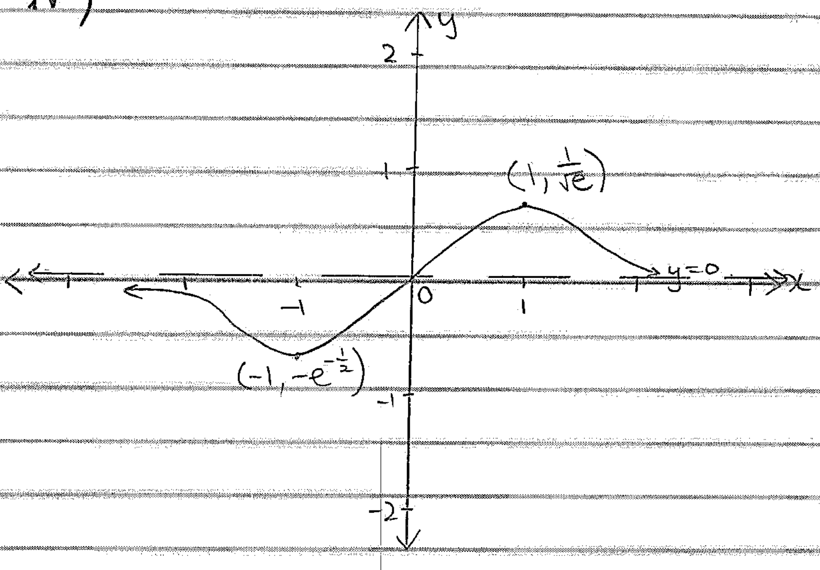
for increasing $f'(x) > 0$

$1 - x^2 > 0$

$x^2 - 1 < 0$

$-1 < x < 1$ — 1

iv)



1 intercepts & asymptote

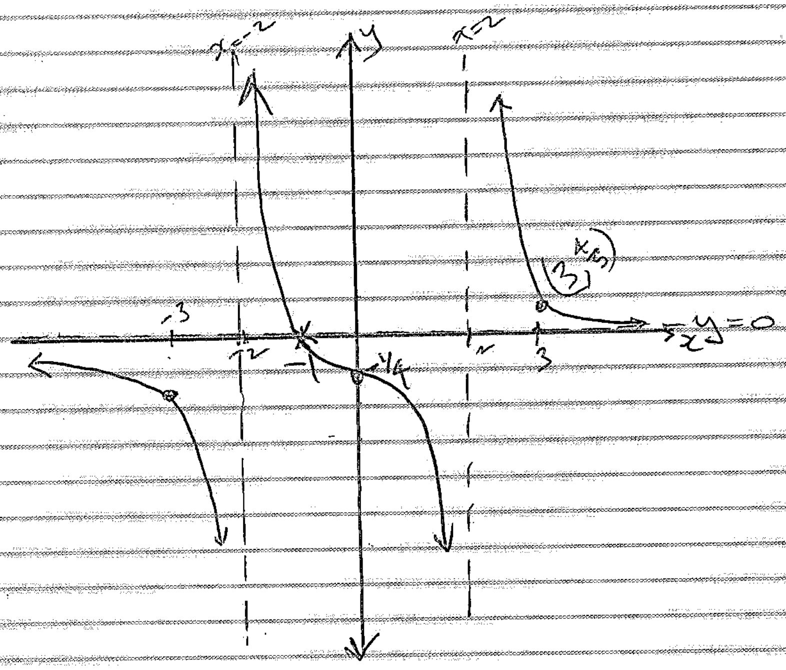
1 shape

1 turning points & differentiation

MATHEMATICS Extension 1 : Question 7

Suggested Solutions	Marks	Marker's Comments
b) $2.2^0 + 3.2^1 + 4.2^2 + \ldots + (n+1)2^{n-1} = n.2^n$		
Prove true for $n=1$		
$LHS = (1+1)2^{1-1}$		
$= 2$	1	
$RHS = 1.2^1$		
$= 2$		
$\therefore$ LHS = RHS		
$\therefore$ statement is true for $n=1$		
Assume true for $n=k$, $k \in Z^+$	1	state it is a positive integer + conclusion
ie. $2.2^0 + 3.2^1 + \ldots + (k+1)2^{k-1} = k.2^k$		
Prove true for $n=k+1$		
ie. $2.2^0 + 3.2^1 + \ldots + (k+1)2^{k-1} + (k+2)2^k = (k+1)2^{k+1}$	1	writing the statement with $n=k+1$ + using assumption
$LHS = 2.2^0 + 3.2^1 + \ldots + (k+1)2^{k-1} + (k+2)2^k$		
$= k.2^k + (k+2)2^k$ (by assumption)		
$= 2^k(k+k+2)$		
$= 2^k(2k+2)$	1	
$= 2^k.2(k+1)$		
$= 2^{k+1}(k+1)$		
$=$ RHS		
$\therefore$ statement is true for $n=k+1$		
$\therefore$ Statement is true for $n \in Z^+$ by the process of mathematical induction		

MATHEMATICS Extension 1: Question 8

Suggested Solutions	Marks	Marker's Comments
(a) 		*didn't need calculus; read the question!!! (1) correct asymptotes and intercepts (1) correct middle curve (1) both outer curves.
(b) (i) At $t=0$, $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{2}\pi (4)^2 \times 8$ $= \frac{128\pi}{3}$ units^3	(1)	
At $t = 16s$, $V = \frac{128\pi}{3} - 2\pi \times 16$ $= \frac{128\pi}{3} - 32\pi$ $= \frac{32\pi}{3}$ units^3	(1)	
(ii) $\frac{4}{8} = \frac{r}{h}$ (ratio of corresponding sides in similar triangles) $r = \frac{1}{2}h$ $\therefore V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ $= \frac{\pi}{12}h^3$	(1)	→ must mention similar triangles to get the mark.
(iii) $\frac{dV}{dh} = \frac{\pi h^2}{4}$ $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $= \frac{4}{\pi h^2} \times -2\pi$		

MATHEMATICS Extension ~~2~~ 1 Question 8 cont

Suggested Solutions	Marks	Marker's Comments
$\frac{dh}{dt} = \frac{-8\pi}{\pi h^2}$		
$= \frac{-8}{h^2}$	1	
now when $V = \frac{32\pi}{3}$, $t = 16$		
$\therefore \frac{\pi}{12}h^3 = \frac{32\pi}{3}$		* If $\frac{dV}{dt} = 2\pi$ and student doesn't mention the volume is decreasing then they lost one mark.
$h^3 = 128$		
$h = \sqrt[3]{128}$	1	
$\therefore \frac{dh}{dt} = \frac{-8}{(\sqrt[3]{128})^2}$		
$= -0.315$		
$\therefore$ decreasing at 0.315 m/s	1	
(c) $A = \int_0^5 \frac{1}{1+3x+2x^2}\,dx$		
$= \int_0^5 \frac{2}{1+2x} - \frac{1}{x+1}\,dx$ (data)		
$= \left[\ln(1+2x) - \ln(x+1)\right]_0^5$	1	
$= \ln 11 - \ln 6 - (\ln 1 - \ln 1)$		
$= \ln 11 - \ln 6 - 0$	1	
$= \ln \frac{11}{6}$ units^2		

Question 9

a) Show $\sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

(i)

$$= \sqrt{2}\left(\sin x \cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos x\right)$$

$$= \sqrt{2}\left(\sin x \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sqrt{2}\cdot\frac{1}{\sqrt{2}}\sin x - \sqrt{2}\cdot\frac{1}{\sqrt{2}}\cos x$$

$$= \sin x - \cos x$$

$$= LHS$$

✓ 1

Some found r and α.

$\sin x - \cos x = R\sin(x - \alpha)$

$R \sin x \cos x - \sin\alpha\cos$

$R\cos\alpha = 1$ and $R\sin\alpha = 1$

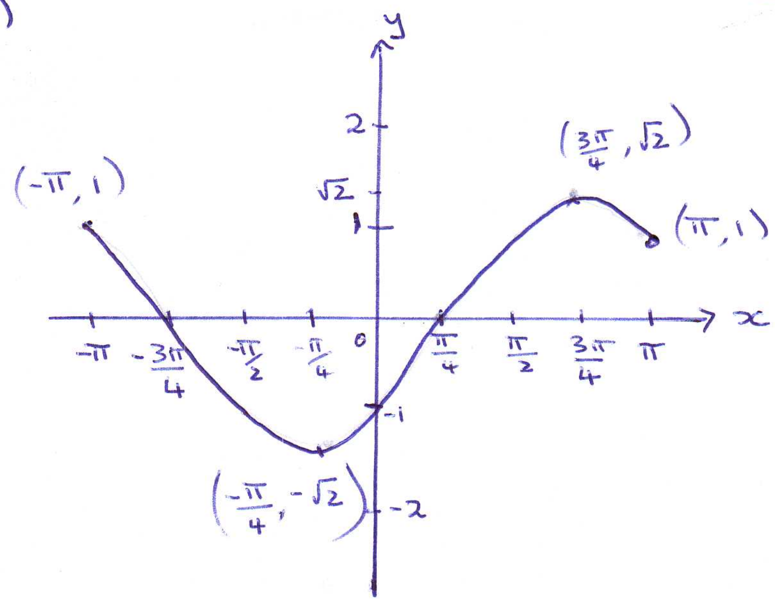
$\therefore \tan\alpha = 1$

$\therefore \alpha = \frac{\pi}{4}$

and $\sqrt{1^2 + 1^2} = \sqrt{2}$

$\therefore r = \sqrt{2}$

(ii)



3

✓ Correct domain and range with correct endpoints $(\pi, 1)$ and $(-\pi, 1)$

$D: -\pi \le x \le \pi$

$R = -\sqrt{2} \le y \le \sqrt{2}$

✓ Correct turning points and labelled with $\left(\frac{3\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{-\pi}{4}, -\sqrt{2}\right)$.

✓ Correct intercepts

x is $-\frac{3\pi}{4}$, $\frac{\pi}{4}$

y is -1

and shape of curve

(iii) $\cos x - \sin x > 0$

but we have drawn

$y = \sin x - \cos x$

$-1(\cos x - \sin x) < 0$

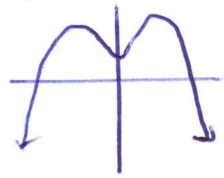
$\sin x - \cos x < 0$

From graph above, less than zero exists between x values $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$.

$\therefore \frac{-3\pi}{4} < x < \frac{\pi}{4}$

1

• Graph like



earned zero marks

- None of above was correct.

(b) (i)

Method 1

Value of investment of n years

$A_1 = 4000 \times 1.1^0 \times 1.05^{20}$
$= 4000(1.05^{20})$

$A_2 = 4000(1.05^{19})(1.1)$

$A_3 = 4000(1.05^{18})(1.1^2)$

$\vdots$

$A_{20} = 4000(1.05)(1.1^{19})$

Total $= 4000(1.05)(1.1)^{19} + 4000(1.05^2)(1.1^{18}) \ldots + \ldots 4000 \times 1.05^{20}$

$= 4000(1.05)\left[1.1^{19} + (1.1^{18})(1.05) + \ldots\right] + \ldots 4000(1.05)^{20}$

$= 4000 \times 1.05 \left[\text{GP with } a = 1.1^{19}, \; r = \frac{1.05(1.1^{18})}{(1.1)^{19}} \quad \therefore r = \frac{1.05}{1.1}\right]$

$= 4000 \times 1.05 \left[1.1^{19} \frac{\left(1 - \left(\frac{1.05}{1.1}\right)^{20}\right)}{1 - \frac{1.05}{1.1}}\right]$

$= 4200 \times 81.48404488$

$= \$342232.99$

option 2

$4000\left[1.05^{20} + 1.05^{19}(1.1) \ldots\ldots (1.05)(1.1)^{19}\right]$

$a = 1.05^{20}$
$r = \frac{1.1}{1.05}$

$4000\left((1.05^{20})\left[\frac{\left(\frac{1.1}{1.05}\right)^{20} - 1}{\frac{1.1}{1.05} - 1}\right]\right)$

Option 3

$a = 4000(1.05^{19})(1.1)$ & $r = \frac{1.1}{1.05}$

$= 4000 \times 1.05^{20} + 4000(1.05^{19})(1.1)\left[\frac{\left(\frac{1.1}{1.05}\right)^{19} - 1}{\frac{1.1}{1.05} - 1}\right]$

Most common

Method 2

Amount of fund after n years

$A_1 = 4000(1.05)$

$A_2 = [A_1 + 4000(1.1)] \times 1.05$
$= 4000(1.05^2) + 4000(1.1)(1.05)$

$A_3 = [A_2 + 4000 \times 1.1^2]1.05$
$= 4000(1.05^3) + (1.05^2)(1.1) + (1.05)(1.1^2)$

$\vdots$

$A_n = 4000(1.05^n + 1.05^{n-1}(1.1) + \ldots\ldots (1.05)(1.1^{n-1})$

$A_{20} = 4000\left(1.05^{20} + \underbrace{(1.05^{19})(1.1) + \ldots\ldots 1.05(1.1^{19})}_{\text{GP with } a = (1.05)^{19}(1.1)}\right)$

$r = \frac{1.1}{1.05}$

option 1

$4000 \times 1.05^{20} + 4000(1.05^{19})(1.1)\left[\frac{\left(\frac{1.1}{1.05}\right)^{20} - 1}{\frac{1.1}{1.05} - 1}\right]$

option 2 Most common

$4000 \times 1.05^{20}\left[\frac{\left(\frac{1.1}{1.05}\right)^{20} - 1}{\frac{1.1}{1.05} - 1}\right]$

option 3

$4000 \times 1.05^{20} + 4000(1.1)(1.05)\left[(1.05^{18})\left[\frac{\left(\frac{1.1}{1.05}\right)^{19} - 1}{\frac{1.1}{1.05} - 1}\right]\right]$

b(ii). Value of investment after 10 years.

$$A_{10} = 4000(1.05^{10})\left[\frac{\left(\frac{1.1}{1.05}\right)^{10} - 1}{\frac{1.1}{1.05} - 1}\right] = \$81\,047.72$$

$$A_{11} = [\$81047.72 + 4000] \times 1.05$$

$$A_{12} = \$81047.72(1.05^2) + 4000(1.05^2) + 4000 \times 1.05$$

$\vdots$

$$A_n = \$81\,047.72(1.05^n) + 4000(1.05)^n + \text{---------} 4000(1.05)$$

we know $A_n = \$342232.99$.

$$342232.99 = 81047.72(1.05)^n + 4000\underbrace{(1.05^n + 1.05^{n-1} \text{------} .1.05)}_{\text{GP}}$$

GP $a = 1$

$r = 1.05$.

$$342232.99 = 81047.72(1.05^n) + 4000(1.05)\left[\frac{1.05^n - 1}{1.05 - 1}\right]$$

$$342232.99 = 81047.72(1.05^n) + 84000(1.05^n - 1)$$

$$= 81047.72(1.05^n) + 84000(1.05^n) - 84000$$

$$= 165047.72(1.05^n) - 84000.$$

$$426232.99 = 165047.72(1.05^n)$$

$$\therefore 1.05^n = \frac{426232.99}{165047.72}.$$

$$1.05^n = 2.582483357.$$

$$n = \log_{1.05} 2.582483357$$

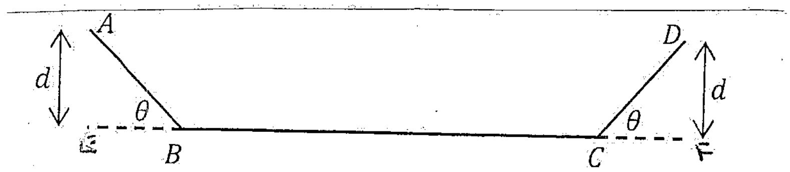
$$= 19.44554796 \text{ years}$$

Hence after a further 20 years after the 10th year, he will receive the same benefit

Ext 2 **MATHEMATICS: Question 10.....**

Suggested Solutions	Marks	Marker's Comments

(i)



$\dfrac{d}{AB} = \sin\theta$ $\qquad\qquad$ $\dfrac{d}{CF} = \tan\theta$

$AB = d\,\text{cosec}\,\theta$ $\qquad\qquad$ $CF = d\cot\theta$

$CD = d\,\text{cosec}\,\theta$ similarly $\qquad$ $EB = d\cot\theta$ $\qquad$ 1 $\qquad$ for 2 correct statements.

$BC = 2x - 2d\,\text{cosec}\,\theta$

Area = area of trapezium ABCD

$= \dfrac{d}{2}\left[BC + AD\right]$

$= \dfrac{d}{2}\left(2x - 2d\,\text{cosec}\,\theta + 2x - 2d\,\text{cosec}\,\theta + 2d\cot\theta\right)$ $\qquad$ 1 $\qquad$ 1 for area statement

$= 2xd - 2d^2\cot\theta - d^2\cot\theta$ $\qquad$ 1 $\qquad$ 1 for completing correctly.

$= 2xd + d^2(\cot\theta - 2\,\text{cosec}\,\theta)$

(ii)

$\dfrac{dA}{d\theta} = d^2\left(-\text{cosec}^2\theta + 2\cot\theta\,\text{cosec}\,\theta\right)$

$\dfrac{dA}{d\theta} = 0$ $\qquad$ $-\text{cosec}^2\theta + 2\cot\theta\,\text{cosec}\,\theta$ $\qquad$ 1 for correct differentiation and this to solve $A' = 0$

$-\text{cosec}\,\theta + 2\cot\theta = 0$ $\qquad$ $\text{cosec}\,\theta \neq 0$

$-1 + 2\cos\theta = 0$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$ $\qquad$ $0 \le x \le \frac{\pi}{2}$.

Many students didn't realise $2xd$ was a constant so disappeared when differentiating wrt θ.

θ	$\pi/4$	$\pi/3$	$\pi/2$
$A'(\theta)$	$\cdot 82d^2$ $(-\sqrt{2}+2)$	0	$-d^2$

1 $\qquad$ 1 for solving correctly and testing.

MATHEMATICS Extension 2: Question 10		
Suggested Solutions	**Marks**	**Marker's Comments**
when $\theta = \pi/3$		
$A = 2xd + d^2(\cot \pi/3 - 2\operatorname{cosec} \pi/3)$		
$= 2xd + d^2\left(\frac{1}{\sqrt{2}} - \frac{4}{\sqrt{3}}\right)$		needed this step
$= 2xd - \frac{3}{\sqrt{3}} d^2$		
$= 2xd - \sqrt{3}d^2$	1	for substituting and simplifying
(iii)		
$BC = 2x - 2d \operatorname{cosec} \theta$		
$= 2(x - d \operatorname{cosec} \pi/3)$	1	
$= 2x - \frac{4d}{\sqrt{3}}$		
Since $BC > 0$		
$2x - \frac{4d}{\sqrt{3}} > 0$		
$2x > \frac{4d}{\sqrt{3}}$	1	for inequality.
$x > \frac{2d}{\sqrt{3}}$		

MATHEMATICS Extension 2: Question 10

Suggested Solutions	Marks	Marker's Comments
b) $4\sin 2x - \operatorname{cosec} 2x = 0$		
$4\sin 2x - \dfrac{1}{\sin 2x} = 0$		
$\sin^2 2x = \dfrac{1}{4}$	1	
$\sin 2x = \pm\dfrac{1}{2}$		
$2x = \dfrac{\pi}{6}$ or $-\dfrac{\pi}{6}$	1	for solving correctly.
$x = \dfrac{n\pi}{2} \pm (-1)^n \dfrac{\pi}{12}$	1	for correct final answer

2016 Extension 1 Term 1 Task Q 11

Part (a)

(i)

$\frac{d}{dx}\left[\ln\left(\frac{1+sin}{cos\,x}\right)\right] = \frac{d}{dx}[\ln(\sec x + \tan x)]$

$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$ **(1 mark)**

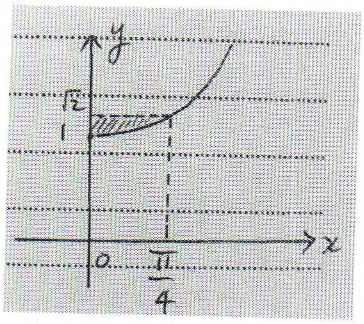
$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$

$= \sec x$ **(1 mark)**

Comments:

Differentiating correctly (using any method) earned one mark. The second mark was awarded for a clear and correct process of simplification.

(ii)



(Shaded area) = (Area of rectangle) – (Area under secant function)

$A_{shaded} = \frac{\sqrt{2}\pi}{4} - \int_0^{\frac{\pi}{4}} \sec x\, dx$ **(1 mark)**

From above, the primitive of $\sec x$ is $\ln\left(\frac{1+sin}{cos\,x}\right)$.

$A_{shaded} = \frac{\sqrt{2}\pi}{4} - \left[\ln\left(\frac{1+sin}{cos\,x}\right)\right]_0^{\frac{\pi}{4}}$

$A_{shaded} = \frac{\sqrt{2}\pi}{4} - \left[\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) - \ln\left(\frac{1+0}{1}\right)\right]$ **(1 mark)**

$A_{shaded} = \frac{\sqrt{2}\pi}{4} - \ln(\sqrt{2}+1)$ **(1 mark)**

Comments:

The first mark was awarded if the student set up the secant integral with the correct limits. If the student did not demonstrate understanding of which region was needed and over which interval integration should take place, then no marks were awarded.

The second mark was awarded if the student correctly integrated the function and made some progress towards evaluation. The third mark was awarded only if the student showed where the final answer came from. Many students had difficulty with manipulating surdic expressions, while many also neglected to consider the rectangle.

Part (b)

(i)

$1 + \sin\theta + \cos\theta = 1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$

$= \frac{t^2+1+2t+1-t^2}{1+t^2}$

$= \frac{2(1+t)}{1+t^2}$ **(1 mark)**

Comments:

Students needed to correctly substitute the t results **and** simplify the expression to earn the mark.

Part (c)

$V = \pi \int y^2 dx$

$= \pi \int_0^1 \frac{1-x}{1+x} dx$ **(1 mark)**

$= \pi \int_0^1 -1 + \frac{2}{1+x} dx$

$= \pi[-x + 2\ln(x+1)]_0^1$ **(1 mark)**

$= \pi(-1 + 2\ln 2 - (0 + 2\ln 1))$

$= \pi(\ln 4 - 1)$

$= \pi(\ln 4 - \ln e)$

$= \pi \ln\left(\frac{4}{e}\right)$

$= \ln\left(\frac{4}{e}\right)^{\pi}$ units^2 **(1 mark)**

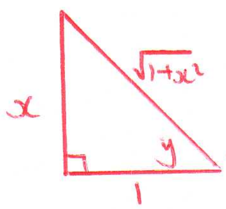
Comments:

The first mark was awarded for correctly setting up the integral for volume by rotation. The second mark was awarded for correctly integrating.

If it was not obvious how the student could transform the expression resulting from integration into the logarithm at the end, then the final mark was not awarded. Students were required to **show**.

Ext 1.

MATHEMATICS: Question 12

Suggested Solutions	Marks Awarded	Marker's Comments
12. a) let $y=\tan^{-1}x$		
$x=\tan y$		
$\frac{dx}{dy}=\sec^2 y$	1	
$=1+\tan^2 y$	1	
$=1+x^2$		
$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$	1	
or		
$\frac{dx}{dy}=\sec^2 y$	1	
$\therefore \frac{dy}{dx}=\cos^2 x$	1	
 $\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$	1	

MATHEMATICS Extension 1 : Question........

Suggested Solutions	Marks	Marker's Comments

12 a) Alternatively

$$\frac{d}{dx}(\tan^{-1}x) = \lim_{h\to 0} \frac{\tan^{-1}(x+h) + \tan^{-1}x}{h}$$

$$\tan\left[\tan^{-1}(x+h) + \tan^{-1}x\right] = \frac{(x+h) - x}{1 + xh + x^2}$$ 1

$$\therefore \frac{d}{dx}(\tan^{-1}x) = \lim_{h\to 0} \frac{\tan^{-1}\left(\frac{h}{1+xh+x^2}\right)}{h}$$ 1

$$= \lim_{h\to 0} \frac{\tan^{-1}\left(\frac{h}{1+xh+x^2}\right)}{\frac{h}{1+xh+x^2}} \times \frac{1}{1+xh+x^2}$$

$$= 1 \times \lim_{h\to 0} \frac{1}{1+xh+x^2}$$

$$= \frac{1}{1+x^2}$$ 1

Note: $\lim_{A\to 0} \frac{\tan^{-1}A}{A} = 1$

This is because as $A \to 0$, $\tan A \approx A$

$\therefore A \approx \tan^{-1}A$

MATHEMATICS Extension 1 : Question……..		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
b) RTP: $10^n \geq 1+9n$, $n \geq 1$		Note: Just writing $10 \geq 10$ is insufficient!
Let $n=1$, LHS $= 10$, RHS $= 1+9 = 10$ $\therefore$ LHS $=$ RHS $\therefore$ The result is true for $n=1$	1	
Assume the result is true for $n=k$, $k \in \mathbb{Z}^+$ $\therefore 10^k \geq 1+9k$		
RTP: $10^{k+1} \geq 1+9(k+1)$		
LHS $= 10^{k+1}$ $= 10^k \cdot 10$ $\geq 10(1+9k)$ (By assumption)	1	
$= 10+90k$ $\geq 10+9k$ $(k \geq 1)$ $= 1+9(k+1)$		Note: You only receive 2 marks if you expanded $1+9(k+1)$ wrong.
$\therefore 10^{k+1} \geq 1+9(k+1)$	1	
$\therefore$ If the result is true for $n=k$, it is true for $n=k+1$		
$\therefore$ The result is true by Mathematical Induction		

MATHEMATICS Extension 2: Question……..		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
c) i) $l^2 = 100^2 + 200^2 - 2\times 100\times 200\cos\theta$ (cos rule)		
$= 50000 - 40000\cos\theta$	1	
$D = 100\theta$	1	
$\therefore l = 100\sqrt{5 - 4\cos\frac{D}{100}}$ $(l > 0)$		
ii) $\frac{dl}{dD} = \frac{1}{2}\times 100\times\left(5 - 4\cos\frac{D}{100}\right)^{-\frac{1}{2}}\times 4\sin\frac{D}{100}\times\frac{1}{100}$		
$= \frac{2\sin\left(\frac{D}{100}\right)}{\sqrt{5 - 4\cos\left(\frac{D}{100}\right)}}$	1	
$\frac{dl}{dt} = \frac{dl}{dD}\times\frac{dD}{dt}$		Those who made D the subject to find $\frac{dD}{dt}$ got into a huge mess!
$= \frac{14\sin\left(\frac{D}{100}\right)}{\sqrt{5 - 4\cos\left(\frac{D}{100}\right)}}$		
when $l = 200 \rightarrow 200 = 100\sqrt{5 - 4\cos\left(\frac{D}{100}\right)}$		
$2 = \sqrt{5 - 4\cos\left(\frac{D}{100}\right)}$		
$4 = 5 - 4\cos\left(\frac{D}{100}\right)$		
$4\cos\left(\frac{D}{100}\right) = 1$		
$\cos\left(\frac{D}{100}\right) = \frac{1}{4}$	1	
$\frac{D}{100} = \cos^{-1}\left(\frac{1}{4}\right)$		

MATHEMATICS Extension 1 : Question........

Suggested Solutions	Marks	Marker's Comments
$\therefore \dfrac{dl}{dt} = \dfrac{14\sin\left[\cos^{-1}\frac{1}{4}\right]}{\sqrt{5-4\cos\left[\cos^{-1}\frac{1}{4}\right]}}$		This is insufficient for the last mark
$= \dfrac{14\sin\left[\cos^{-1}\frac{1}{4}\right]}{\sqrt{5-4\left(\frac{1}{4}\right)}}$		
$= 7\sin\left[\cos^{-1}\frac{1}{4}\right]$	1	
$= \dfrac{7\sqrt{15}}{4}$