Name:	
Class:	

Year 12
Assessment task 2
Term 1 2018

Mathematics extension 1

Time Allowed - 120 minutes

(plus 5 minutes Reading Time)

General Instruction:

- All questions may be attempted
- All questions are equal value
- Reference Sheet will be supplied
- Department of Education approved calculators and templates are permitted
- In every Question, show all relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or badly arranged work

Questions 1-5 are to be completed on the multiple choice sheet, then Questions 6-12 to be completed on separate sheets of paper and returned in separate bundles. Questions must show your Candidate Number (in the top right hand corner).

SECTION I
MULTIPLE CHOICE QUESTIONS

Attempt Questions 1 – 5 (1 mark each).

Allow approximately 8 minutes for this section

1. The value of the expression
$$36^{log_6 5} + 10^{1-log_{10} 2} - 3^{log_9 36}$$
is:
   A) $\dfrac{74}{3}$;
   B) 24;
   C) $\dfrac{73}{4}$;
   D) 6.

2. Which of the following is an expression for $\displaystyle\int \cos^2 2x dx$?
   A) $x - \dfrac{1}{4}\sin 4x + c$;
   B) $x + \dfrac{1}{4}\sin 4x + c$;
   C) $\dfrac{x}{2} - \dfrac{1}{8}\sin 4x + c$;
   D) $\dfrac{x}{2} + \dfrac{1}{8}\sin 4x + c$;

3. The graph of the function $f(x) = e^x \sin x$, $0 \leq x \leq \pi$, has a maximum gradient of:
   A) 1;
   B) $\frac{\pi}{2}$;
   C) e^{π};
   D) $e^{\frac{\pi}{2}}$.

4. The angle between tangents of the graph $y = 2x^3 + x$ at the points $(-1, -3)$ and $(1, 3)$ is:
   A) $\dfrac{\pi}{3}$;
   B) $\tan^{-1}(6)$;
   C) 0;
   D) $\dfrac{\pi}{2}$

5. Four functions $f_1(x) = 2\log_2 x$, $f_2(x) = \log_2 x^2$, $f_3(x) = 2\log_2 |x|$ and $f_4(x) = \dfrac{2}{\log_x 2}$ are given, find the true statement:
   A) $f_1 \neq f_2 = f_3 \neq f_4 \neq f_1$;
   B) $f_2 \neq f_1 = f_4 \neq f_3 \neq f_2$;
   C) $f_3 \neq f_1 = f_2 \neq f_4 \neq f_3$;
   D) all four functions are equal.

END OF SECTION I

SECTION II
EXTENDED RESPONSE (77 marks)

77 marks
Attempt Questions 6 – 12
Allow approximately 1 hour and 52 minutes for this section

MARKS:

QUESTION 6. (11 marks) **COMMENCE A NEW PAGE**

(a) Find

$$\lim_{x \to 0} \left(\frac{\sin 3x}{2x} \right)$$

1

(b) Using the substitution $u = 1 + e^x$, find the value of $\displaystyle\int \frac{e^{3x}}{1 + e^x} dx$. **3**

(c) Use mathematical induction to prove that $3^{2n} - 1$ is divisible by 8, for all integers $n \geq 1$. **4**

(d) A ladder 3 metres long has its top end resting against a vertical wall and its lower end on horizontal ground. The top end of the ladder is slipping down at a constant rate of 0.1 metres per second. Find the rate at which the lower end is moving away from the wall when the lower end is 1 metre from the wall. **3**

QUESTION 7. (11 marks) **COMMENCE A NEW PAGE**

(a) Consider the function

$$f(x) = \cos x + \sqrt{3} \sin x, \quad 0 \leq x \leq 2\pi$$

Given that $f(x)$ can be expressed in the form $r\cos(x - a)$, where $r > 0$ and $0 < a < \frac{\pi}{2}$:

(i) Show that the values of $r = 2$ and $a = \frac{\pi}{3}$. **2**

(ii) Find the range of the function. **1**

(iii) Find the coordinates of the y-axis intercept. **1**

QUESTION 7. (continues on next page)

QUESTION 7. (continues from previous page)

MARKS:

(iv) Find the coordinates of the x-axis intercept. **2**

(v) Graph $f(x) = \cos x + \sqrt{3}\sin x$. **2**

(vi) Find the volume of the solid formed when the region bounded by the graph of $y = f(x)$, the x-axis and y-axis is rotated abut the x-axis. **3**

QUESTION 8. (11 marks) **COMMENCE A NEW PAGE**

(a) Let $f(x) = x\log_e(x-1)$. What is the domain and range of $f(x)$? **2**

(b) (i) Find the value of a such that

$$\frac{1}{x+1} - \frac{1}{x+4} = \frac{a}{(x+1)(x+4)}.$$

1

(ii) Hence or otherwise, find the value of

$$\int_0^1 \frac{dx}{x^2+5x+4}$$

2

(c) Use substitution $x = 2\sin\theta$ to evaluate $\int_0^1 \sqrt{4-x^2}dx$. **3**

(d) (i) Find the derivative of $f(x) = \tan(x^2)$. **1**

(ii) Hence or otherwise evaluate $\int_{-a}^{a} x\sec^2(x^2)dx$. **2**

MARKS:

QUESTION 9. (11 marks) **COMMENCE A NEW PAGE**

(a) In the arithmetic sequence $T_4 = 5$. Find the value of the common difference, d, when the sum of the squares of the first three terms in the sequence is a minimum.

3

(b) Prove by mathematical induction that for all positive integer values of n,

$$\frac{1}{3}\times\frac{1}{1}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{7}\times\frac{1}{5}+...+\frac{1}{(2n+1)}\times\frac{1}{(2n-1)}=\frac{n}{2n+1}.$$

4

(c) Sketch the graph of

$$y=\frac{24}{x^2+2x-24}.$$

State the equations of any asymptotes. Find the coordinates of any axis intercepts and any turning points.

4

QUESTION 10. (11 marks) **COMMENCE A NEW PAGE**

(a) (i) Show that the equation of the tangent to the parabola $x^2 = 4Ay$ at any point $P(2Ap, Ap^2)$ is given by $px - y - Ap^2 = 0$.

2

(ii) S is the focus of the parabola and T is the point of intersection of the tangent and the y-axis. Prove that $SP = ST$.

3

(iii) Hence show that $\angle SPT$ is equal to the acute angle between the tangent and the line through P parallel to the axis of the parabola.

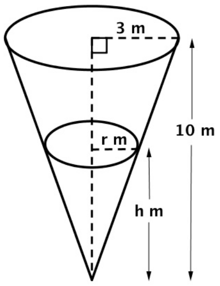
2

QUESTION 10. (continues on next page)

QUESTION 10. (continues from previous page)

MARKS:

(b)



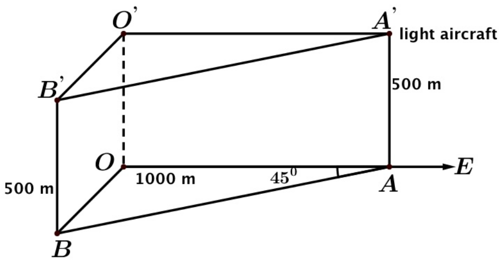
The diagram shows a conical wheat flu. The flu is being filled with wheat at the rate of 2 m^3 per minute. The height of wheat at time t minutes is h metres and the radius of the wheat's top surface is r metres.

(i) Show that $r = \dfrac{3h}{10}$. **1**

(ii) Find the rate at which the height is increasing when the height of the wheat is 8 m. (Volume of cone $= \dfrac{1}{3}\pi r^2 h$). **3**

QUESTION 11. (11 marks) **COMMENCE A NEW PAGE**

(a)



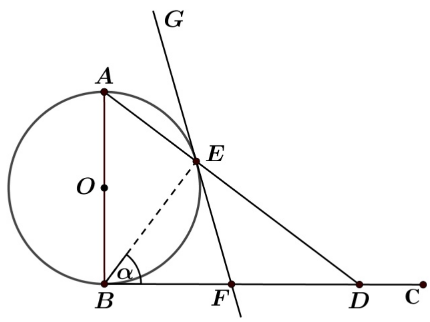
A light aircraft flying at a height of 500 m above the ground is sighted by an observer stationed at a point, O, on the ground, 1 km due west of the plane. The aircraft is flying south-west from A' (along $A'B'$) at 300 km/h.

(i) How far will it travel in 1 minute? **1**

(ii) Find its bearing from OO' at this time. **3**

(iii) What will be its angle of elevation from O at this time? **2**

(b)



In the diagram, AB is a diameter of the circle, centre O, and BC is tangential to the circle at B. The line AED intersects the circle at E and BC at D. The tangent to the circle at E intersects BC at F. Let $\angle EBF = \alpha$.

(i) Copy the diagram into your Writing Booklet labelling all appropriate information. **1**

(ii) Prove that $\angle FED = \dfrac{\pi}{2} - \alpha$. **2**

(iii) Prove that $BF = FD$ **2**

MARKS:

QUESTION 12. (11 marks) **COMMENCE A NEW PAGE**

(a) The cylinder, with the radius x and height H, is inscribed in a cone of height h.

(i) If θ is semi-vertical angle of the cone, show that the volume of cylinder is given by:

$$V = \pi(x^2h - x^3\cot\theta)$$

2

(ii) Show that for the maximum volume of the inscribed cylinder the height of the cylinder is the third of the height of the cone, $H = \dfrac{1}{3}h$.

3

(b) (i) Write down an expression for $\cos(a+b)$ and hence prove that $\cos(2q) = 1 - 2\sin^2 q$.

1

(ii) Prove the identity

$$\frac{\cos y - \cos(y+2q)}{2\sin q} = \sin(y+q).$$

2

(iii) Use mathematical induction and the result of part (ii) to prove the identity:

$$\sin q + \sin 3q + \sin 5q + \ldots + \sin(2n-1)q = \frac{1-\cos 2nq}{2\sin q}.$$

3

END OF THE TEST

Student Number [illegible]

Multiple Choice

1. B
2. D
3. D
4. C
5. A

MATHEMATICS: Question……..

Suggested Solutions	Marks Awarded	Marker's Comments

Question 6

a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{2x}$

$= \lim\limits_{x\to 0} \dfrac{\sin 3x}{3x} \times \dfrac{3}{2}$

$= \dfrac{3}{2} \times \lim\limits_{x\to 0} \dfrac{\sin 3x}{3x}$

$= \dfrac{3}{2} \times 1$

$= \dfrac{3}{2}$

1 mark for answer although students should show working out very well done

b) $\displaystyle\int \frac{e^{3x}}{1+e^{x}}\,dx$

let $u = 1+e^{x}$

$\dfrac{du}{dx} = e^{x}$

$dx = \dfrac{du}{e^{x}}$

$= \dfrac{du}{u-1}$

$\displaystyle\int \frac{e^{3x}}{1+e^{x}}\,dx$

$= \displaystyle\int \frac{(u-1)^3}{u} \times \frac{du}{u-1}$

1 mark for differentiating correctly and making correct substitution.

Note: students lost a mark for mixing variables in their integral

$= \displaystyle\int \frac{(u-1)^2}{u}\,du$

$= \displaystyle\int \frac{u^2-2u+1}{u}\,du$

$= \displaystyle\int u - 2 + \frac{1}{u}\,du$

$= \dfrac{u^2}{2} - 2u + \ln|u|$

1 mark for integrating correctly

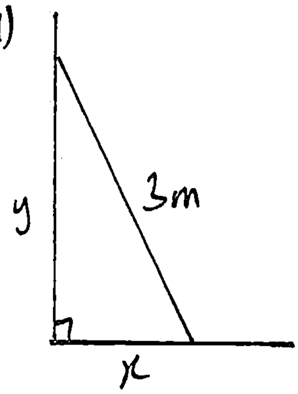
MATHEMATICS Extension 1 : Question........

Suggested Solutions	Marks Awarded	Marker's Comments
$= \frac{(1+e^x)^2}{2} - 2(1+e^x) + \ln(1+e^x) + C$		1 mark for remembering to resubstitute u.
c) when $n=1$, $3^{2\times 1}-1$ $= 9-1$ $= 8$ $= 8 \times 1$ ∴ divisible by 8 ∴ statement is true for $n=1$		1 mark for showing true for $n=1$
Assume true for $n=k$ where $k \in \mathbb{Z}$ $3^{2k}-1 = 8M$ where $M \in \mathbb{Z}$		1 mark for correct assumption 1 mark for stating $k \in \mathbb{Z}$
Prove for $n=k+1$ $3^{2(k+1)}-1$ $= 3^{2k+2}-1$ $= 9 \times 3^{2k} - 1$ $= 9 \times (8M+1) - 1$ (by assumption) $= 72M + 9 - 1$ $= 72M + 8$ $= 8(9M+1)$ $= 8P$ where $P = 9M+1$ Since $M \in \mathbb{Z}$, $9M+1 \in \mathbb{Z}$ ∴ divisible by 8 when $n=k+1$ ∴ statement is true by the principle of mathematical induction		1 mark for correct steps proving true for $n=k+1$

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks Awarded	Marker's Comments

d)



$$\frac{dy}{dt} = -0\cdot 1 \text{ m/s}$$

$$\frac{dx}{dt} = ?$$

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$$

$x^2 + y^2 = 9$ (Pythagoras)

$y^2 = 9 - x^2$

$y = \sqrt{9-x^2}$ $(y > 0)$

$$\frac{dy}{dx} = \frac{1}{2}(9-x^2)^{-\frac{1}{2}} \times (-2x)$$

$$= -\frac{x}{\sqrt{9-x^2}}$$

1 mark for getting an expression for x or y and correctly differentiating

$$\frac{dx}{dt} = -\frac{\sqrt{9-x^2}}{x} \times (-0\cdot 1)$$

when $x = 1$,

1 mark for the correct use of chain rule.

$$\frac{dx}{dt} = \frac{\sqrt{9-1^2}}{1} \times 0\cdot 1$$

$$= \frac{\sqrt{8}}{10}$$

$$= \frac{2\sqrt{2}}{10}$$

$$= \frac{\sqrt{2}}{5} \text{ m/s}$$

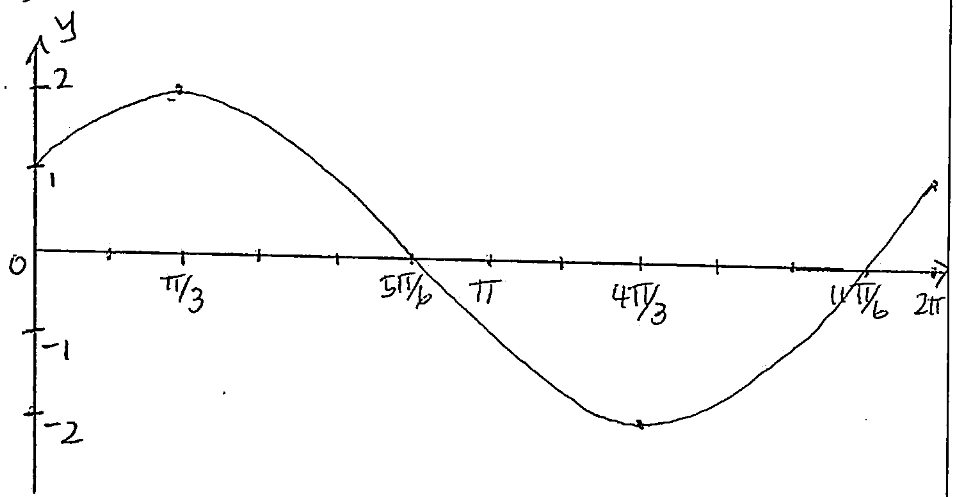
$\therefore$ the lower end is moving away at $\frac{\sqrt{2}}{5}$ m/s.

1 mark for final answer

MATHEMATICS Extension 1 : Question 7

Suggested Solutions	Marks	Marker's Comments
a) i) $\cos x + \sqrt{3}\sin x = r\cos(x-\alpha)$ $= r\cos x\cos\alpha + r\sin x\sin\alpha$ $r\cos\alpha = 1 \qquad r\sin\alpha = \sqrt{3}$ $r^2\cos^2\alpha + r^2\sin^2\alpha = 1+3$ $r^2(\cos^2\alpha + \sin^2\alpha) = 4$ $r^2 = 4$ $r = 2$ (since $r>0$)	1	showing value of r
Method 1: $\frac{r\sin\alpha}{r\cos\alpha} = \frac{\sqrt{3}}{1}$ $\tan\alpha = \sqrt{3}$ $\alpha = \pi/3$, $(0<\alpha<\pi/2)$	1	showing value of α
Method 2: sub $r=2$ into $r\cos\alpha = 1$ $\cos\alpha = 1/2$ $\alpha = \pi/3$ $(0<\alpha<\pi/2)$		
Method 3: sub $r=2$ and $\alpha = \pi/3$ into $r\cos(x-\alpha)$ $2\cos(x-\pi/3) = 2\cos x\cos\pi/3 + 2\sin x\sin\pi/3$ $= 2\cos x \times 1/2 + 2\sin x \times \sqrt{3}/2$ $= \cos x + \sqrt{3}\sin x$ $\therefore r=2$ and $\alpha = \pi/3$ satisfies		
ii) $\cos x + \sqrt{3}\sin x = 2\cos(x-\pi/3)$ range: $-2 \le y \le 2$	1	
iii) when $x=0$, $y = 2\cos(-\pi/3)$ $= 2\times 1/2$ $=1$ y-intercept is 1	1	

MATHEMATICS Extension 1 : Question 7

Suggested Solutions	Marks	Marker's Comments
iv) when $y=0$ $2\cos(x-\pi/3)=0$ $x-\pi/3=\pi/2,\ 3\pi/2$ $x=\pi/2+\pi/3,\ 3\pi/2+\pi/3$ $=5\pi/6,\ 11\pi/6$	2	1 for each value of x within given domain
v) 	1 1	3 intercepts and shape max/min points
vi) $V=\pi\int_0^{5\pi/6}(\cos x+\sqrt{3}\sin x)^2\,dx$ $=\pi\int_0^{5\pi/6}4\cos^2(x-\pi/3)\,dx$	1	correct expression for volume
$=4\pi\int_0^{5\pi/6}\frac{1}{2}(\cos 2(x-\pi/3)+1)\,dx$	1	change to $\cos 2\theta$
$=2\pi\int_0^{5\pi/6}(\cos(2x-2\pi/3)+1)\,dx$ $=2\pi\left[\frac{1}{2}\sin(2x-2\pi/3)+x\right]_0^{5\pi/6}$ $=\pi\left[\sin(5\pi/3-2\pi/3)+5\pi/3-\sin(-2\pi/3)-0\right]$ $=\pi\left[\sin\pi+5\pi/3+\sqrt{3}/2\right]$ $=\pi\left[5\pi/3+\sqrt{3}/2\right]$ cubic units	1	correct substitution and evaluation

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a) D: $x > 1$ 1

R: $y \in R$ 1

b i) $\dfrac{x+4-x-1}{(x+1)(x+4)} = \dfrac{3}{(x+1)(x+4)}$ $\therefore a = 3$ 1

ii) $\displaystyle\int_0^1 \frac{dx}{(x+1)(x+4)} = \frac{1}{3}\int\left(\frac{1}{x+1} - \frac{1}{x+4}\right)dx$

$= \frac{1}{3} \ln\left(\frac{x+1}{x+4}\right)\Big]_0^1$ 1

$= \frac{1}{3}\left(\ln \frac{2}{5} - \ln \frac{1}{4}\right)$ 1

or $\frac{1}{3}\ln\left(\frac{2\times 4}{5}\right) = \frac{1}{3}\ln\left(\frac{8}{5}\right)$

or $\frac{1}{3}\left(\ln 2 - \ln 5 + \ln 4\right)$

c) $\displaystyle\int_0^{\frac{\pi}{6}} 2\cos\theta \cdot 2\cos\theta \, d\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$x = 0, \ \theta = 0$

$x = 1, \ \theta = \frac{\pi}{6}$

$= 4\displaystyle\int_0^{\frac{\pi}{6}} \cos^2\theta \, d\theta$ 1

$= \frac{4}{2}\displaystyle\int_0^{\frac{\pi}{6}} (1 + \cos 2\theta)\, d\theta$ 1

$= 2\left(\theta + \frac{\sin 2\theta}{2}\right)\Big]_0^{\frac{\pi}{6}}$

$= 2\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$

$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$ 1 for correct integration double angle and substitute values of θ

d) $y = \tan x^2$

$y' = 2x \sec^2(x^2)$ /

$\int_{-a}^{a} x \sec^2(x^2)\,dx$

$= \frac{1}{2} \tan(x^2)\Big]_{-a}^{a}$ /

$= \frac{1}{2} \tan a^2 - \frac{1}{2} \tan(-a)^2$

$= \underline{0}$ /

or

$f(x) = x \sec^2 x^2$ is an odd function since

$f(-x) = -x \sec^2(-x)^2$

$f(-x) = -x \sec^2 x^2$

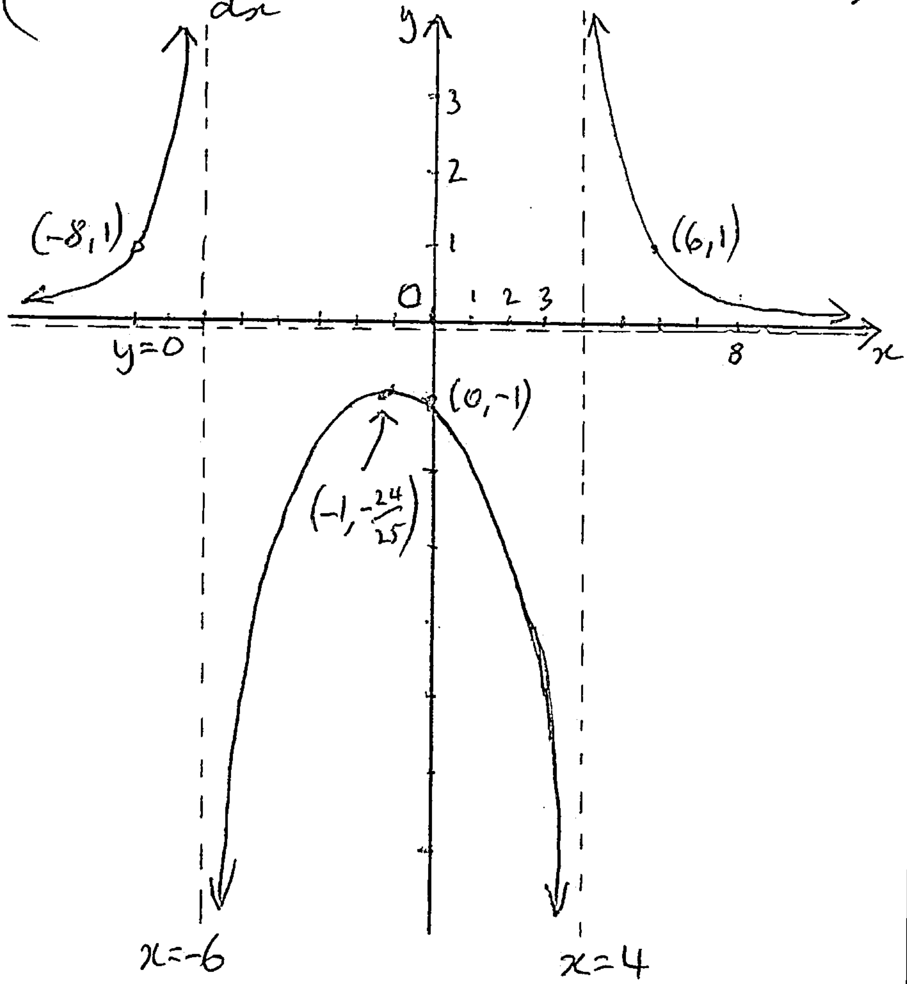
$\therefore f(x) = f(-x)$

MATHEMATICS Extension 1 : Question 9(a) Sheet 1 of 3		
Suggested Solutions	**Marks**	**Marker's Comments**
a) $T_4 = a + 3d = 5$ $\therefore \underline{a = 5 - 3d}$ Let $S = T_1^2 + T_2^2 + T_3^2$ $= a^2 + (a+d)^2 + (a+2d)^2$ $= 3a^2 + 5d^2 + 6ad$ $= 3(5-3d)^2 + 5d^2 + 6d(5-3d)$ $\underline{\underline{= 75 - 60d + 14d^2}}$ $\frac{dS}{dd} = -60 + 28d$ $= 0$ at Turning Point $\therefore d = 15/7$ " " " $\frac{d^2S}{dd^2} = 28$ > 0 $\therefore$ Concave up $\therefore S$ is a minimum when $\underline{\underline{d = 15/7}}$	 1 1 1	Rather too many people used calculus before eliminating a (or d). Such calculus is invalid as a and d vary together to satisfy $T_4 = 5$. Such people did not usually qualify for any marks. A few people substituted for d rather than a. It should have worked but the numbers were uglier.

MATHEMATICS Extension 1 : Question 9(b) 2 9 3

Suggested Solutions	Marks	Marker's Comments
<u>To Prove</u>: $\frac{1}{3}\times\frac{1}{1}+\frac{1}{5}\times\frac{1}{3}+\ldots+\frac{1}{(2n+1)}\times\frac{1}{(2n-1)}=\frac{n}{2n+1}$ <u>Step 1</u>: Check for $n=1$. $LHS=\frac{1}{3}\times\frac{1}{1}=\frac{1}{3}$ $\quad$ $RHS=\frac{1}{2+1}=\frac{1}{3}$ $LHS=RHS$ so <u>true for $n=1$.</u>	1	
<u>Step 2</u>: Assume true for $k\in\mathbb{Z}^+$ i.e. $\frac{1}{3}\times\frac{1}{1}+\frac{1}{5}\times\frac{1}{3}+\ldots+\frac{1}{(2k+1)}\times\frac{1}{(2k-1)}=\frac{k}{2k+1}$ It is required to prove for $(k+1)$. i.e. $\frac{1}{3}\times\frac{1}{1}+\ldots+\frac{1}{(2k+1)}\times\frac{1}{(2k-1)}+\frac{1}{(2k+3)}\times\frac{1}{(2k+1)}=\frac{k+1}{2k+3}$ $LHS=\frac{k}{2k+1}+\frac{1}{(2k+3)(2k+1)}$ (by <u>Assumption</u> above)	1	One for use of the word <u>Assumption</u> or something equivalent was required.
$=\frac{k(2k+3)+1}{(2k+3)(2k+1)}$ $=\frac{2k^2+3k+1}{(2k+3)(2k+1)}$ $=\frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}=\frac{(k+1)}{(2k+3)}$ as req^d.	1	
<u>Step 3</u>: Using steps 1 and 2 and invoking the Principle of Mathematical Induction, the proof is complete for $n\geq 1$.	1	Some very short cuts taken with the last step.

MATHEMATICS Extension 1 : Question 9.(c)   3 9 3

Suggested Solutions	Marks	Marker's Comments
c) $y = \dfrac{24}{(x+6)(x-4)}$, $\dfrac{dy}{dx} = \dfrac{-24(2x+2)}{(x^2+2x-24)^2}$ Vertical asymptotes $x=-6$, $x=4$ Horizontal asymptote $y=0$	1	Asymptotes (3) must be drawn and their equations must be found somewhere, preferably on the diagram.
No x intercept y intercept at $(0,-1)$	1	y intercept must be marked on diagram, preferably in coordinate form $(0,-1)$
Turning point at $(-1, -\frac{24}{25})$ (Where $\frac{dy}{dx}=0$. It is a maximum)	1	Turning point similarly $(-1, -\frac{24}{25})$
	1	Three parts of the shape basically correct. The max point must be above the intercept. (If there is no max it would be impossible to get more than 2 marks)

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MATHEMATICS Extension 1 : Question 10 1/7

Suggested Solutions	Marks Awarded	Marker's Comments

Method 1

ai) $x^2 = 4Ay$

$y = \dfrac{x^2}{4A}$

$\dfrac{dy}{dx} = \dfrac{2x}{4A} = \dfrac{x}{2A}$

when $x = 2Ap$

$\dfrac{dy}{dx} = \dfrac{\cancel{2A}p}{\cancel{2A}} = p$ — 1

Method 2

$x = 2Ap \qquad y = Ap^2$

$\dfrac{dx}{dp} = 2A \qquad \dfrac{dy}{dp} = 2Ap$

$\dfrac{dy}{dx} = \dfrac{dy}{dp} \times \dfrac{dp}{dx}$

$= \cancel{2A}p \times \dfrac{1}{\cancel{2A}} = p$ — 1

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - Ap^2 = p(x - 2Ap)$

$y - Ap^2 = px - 2A^2p$

$y - px + A^2p = 0$

ie $0 = px - y - Ap^2$
(shown as required)

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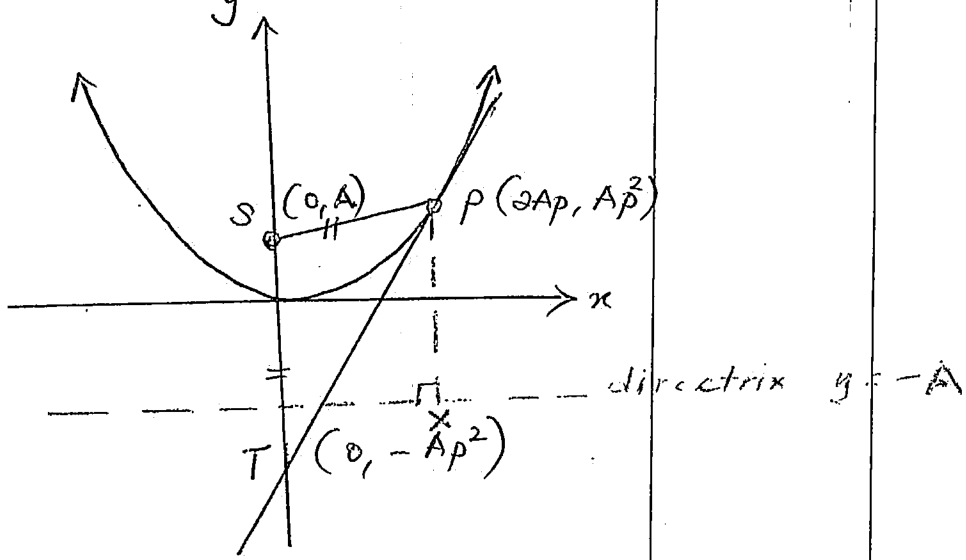
Suggested Solutions	Marks Awarded	Marker's Comments

ii)



RTP SP = ST

Method 1

from part (i)

$px - y - Ap^2 = 0$ —— i

T is on y axis ∴ $x = 0$

Thus $y = -Ap^2$

so coord T is $(0, -Ap^2)$ —— 1

d ST $= A + Ap^2$ (vertical length)

$= A(1+p^2)$ —— 1

d SP $= \sqrt{(Ap^2 - A)^2 + (2Ap - 0)^2}$

$= \sqrt{A^2p^4 - 2A^2p^2 + A^2 + 4A^2p^2}$

$= \sqrt{A^2p^4 + 2A^2p^2 + A^2}$

$= \sqrt{A^2(p^4 + 2p^2 + 1)}$

$= \sqrt{A^2(1+p^2)^2}$ > 0

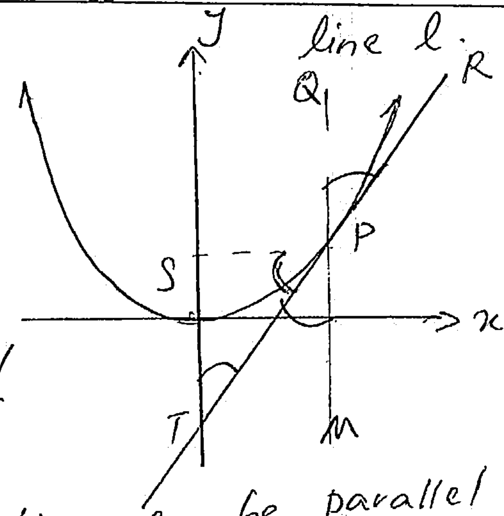
$= A(1+p^2)$ —— 1

∴ SP = ST

MATHEMATICS Extension 1 : Question 10		3/7
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
Method 2 $d\ SP = \sqrt{(Ap^2 - A)^2 + (2Ap - 0)^2}$ $= \sqrt{A^2p^4 - 2A^2p^2 + A^2 + 4A^2p^2}$ $= \sqrt{A^2p^4 + 2A^2p^2 + A^2}$ $d\ ST = \sqrt{(-Ap^2 - A)^2 + (0-0)^2}$ $= \sqrt{A^2p^4 + 2A^2p^2 + A^2}$ $= A(1+p^2)$		$= A(1+p^2)$ some stopped here * must have both eqns looking the same coz its RTP
Method 3 $SP = PX$ (by definition of parabola) (refer diagram) $\therefore SP = A + Ap^2$ $= A(1+p^2)$ $ST = A + Ap^2$ $= A(1+p^2)$ $\therefore SP = ST$		1 mark to show coord of T 1 mark for correct SP 1 mark for correct ST * and both SP and ST must be the same by correct working out

MATHEMATICS Extension 1 : Question 10		4/7
Suggested Solutions	**Marks Awarded**	**Marker's Comments**

iii)



Method 1

Let line l be parallel to y axis

from pt ii) ST = SP (1)

$\therefore \angle SPT = \angle STP$ (angles opposite equal sides in a triangle are equal)

$\angle STP = \angle TPM$ (alt angle ST // PM)

$\therefore \angle SPT = \angle TPM$)

OR .

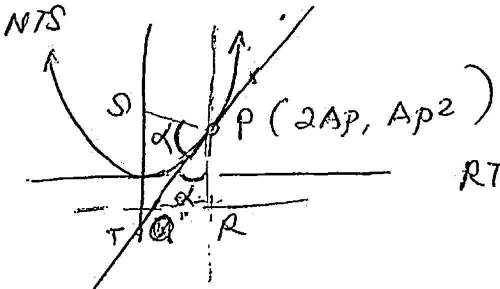
$\angle STP = \angle QPR$ (corresponding angles ST // PM)

$\therefore \angle SPT = \angle QPR$

MATHEMATICS Extension 1 : Question 10 $\quad$ 5/7

Suggested Solutions	Marks Awarded	Marker's Comments

Method 2



RTP $\angle SPT = \angle TPR$

Angle between $x = 2Ap$ and tangent

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$m_{SP} \; \frac{Ap^2 - A}{2Ap - 0} \qquad m_{PT} = \frac{Ap^2 - (-Ap^2)}{2Ap - 0}$$

$$= \frac{p^2 - 1}{2p} \qquad\qquad = p$$

$$\tan\alpha = \frac{\frac{p^2-1}{2p} - p}{1 + \left(\frac{p^2-1}{2p} \times p\right)}$$

$$= \frac{1}{p}$$

$$\angle SPT \;\therefore\; \alpha = \tan^{-1}\frac{1}{p} \qquad \text{(i)}$$

$$\angle TPR = \pi - \angle PQR - \frac{\pi}{2} \quad \text{(angle sum of } \triangle PQR)$$

$$= \frac{\pi}{2} - \angle PQR$$

MATHEMATICS Extension 1 : Question 10		6/7
Suggested Solutions	**Marks Awarded**	**Marker's Comments**

$\tan \angle TPR = \tan\left(\frac{\pi}{2} - \angle PQR\right)$

$= \cot \angle PQR$

$\tan \angle PQR = m_{TP}$

$= p$

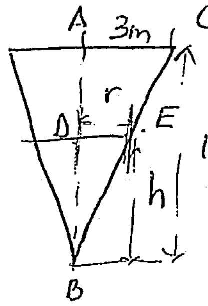
$\cot \angle PQR = \frac{1}{p}$

$\therefore \tan \angle TPR = \frac{1}{p}$ — 1

$\therefore \tan \angle SPT = \tan \angle TPR$

$\angle SPT = \angle TPR$ both angles equal.

* If used rhombus Needed to say 1 pair of adjacent side equal and 1 pair of opp sides are equal & parallel

b) 

10. $\frac{dV}{dt} = 2m^3/min$

Method 1

i) Show $r = \frac{3h}{10}$.

Must! state ($\triangle DBE \parallel\!\mid \triangle ABC$) or (state using similar triangles) to get the mark.

$\frac{h}{r} = \frac{10}{3}$

$\therefore r = \frac{3h}{10}$

Method 2 — Let $\angle DBE = \theta$

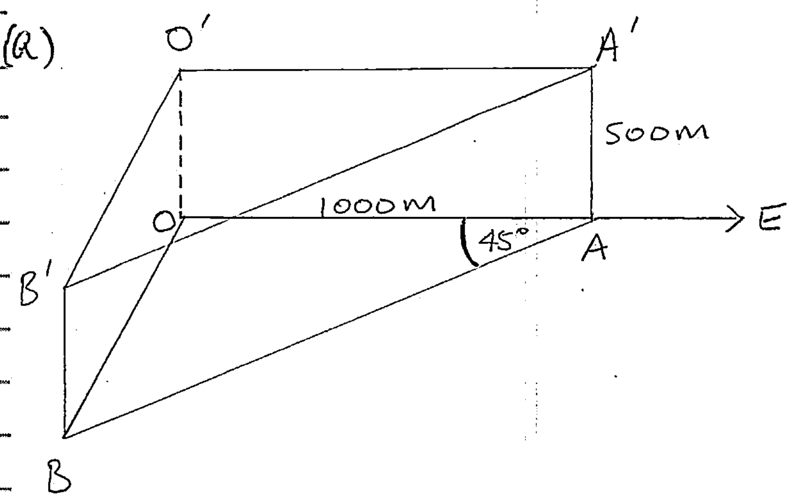
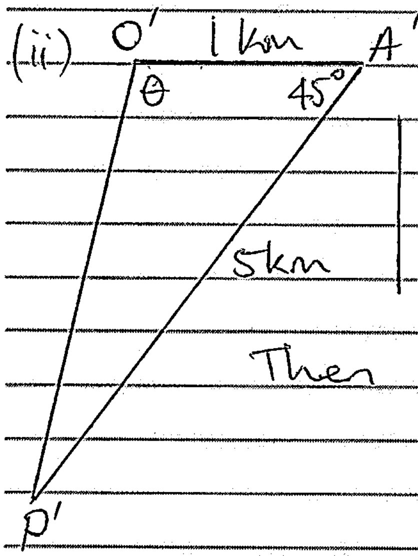
$\tan\theta = \frac{r}{h}$ — (i)

$\tan\theta = \frac{3}{10}$ — (ii)

$\therefore r = \frac{3h}{10}$

MATHEMATICS Extension 1 : Question 10		7/7
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
ii) $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h$ $= \frac{\pi h}{\cancel{3}}\left(\frac{\cancel{9}^{3}h^2}{100}\right)$ $= \frac{3\pi h^3}{100}$ — 1 $\frac{dV}{dh} = \frac{3\times 3\pi h^2}{100} = \frac{9\pi h^2}{100}$ $\frac{dh}{dt} = ?$ $\frac{dV}{dt} = 2$ $\therefore \frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$ $2 = \frac{9\pi h^2}{100}\times\frac{dh}{dt}$ $\frac{dh}{dt} = \frac{100\times 2}{9\pi h^2}$ $= \frac{200}{9\pi h^2}$ sub $h = 8$ — 1 $= \frac{\cancel{200}\ \cancel{100}\ \cancel{50}\ 25}{9\times\pi\times\cancel{8}\times 8}$ $= \frac{25}{72\pi}$ m/min — 1	final answer	* You cannot differentiate with 2 variables. - if you have done so, max mark is 1/3 * if there arent any subsequent error - you will lose a mark if your units were incorrect. - π has to be below the vinculum.

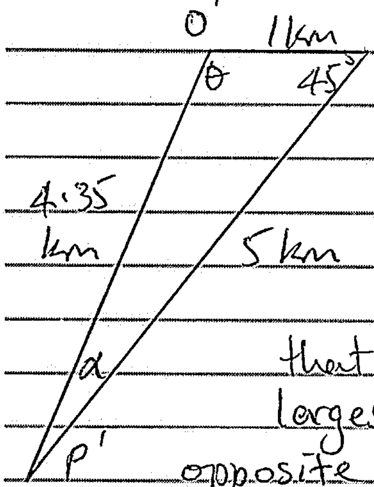
MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
(a) 		
(i) $300\ \frac{km}{h} = 300\ \frac{km}{60\ min}$ $= 5\ km/min$ $= 5000\ m/min$	1	Main error: incorrect order of magnitude.
(ii)  Let P' be position of plane along A'B' at one minute. Then $\frac{\sin\theta}{5km} = \frac{\sin 45°}{O'P'}$		Many had difficulty converting 3D problem into 2D components.

MATHEMATICS Extension 1 : Question 11		
Suggested Solutions	**Marks**	**Marker's Comments**
Now,		* Main error: mixing units / dropping units leading to incorrect quantity.
$(O'P')^2 = (1km)^2 + (5km)^2$		
$- 2(1km)(5km)\cos 45°$		
$= 26 - 10/\sqrt{2}$ km^2		
$= 26 - 5\sqrt{2}$ km^2		* Other: incorrect use of cosine rule.
So, $O'P' = \sqrt{26 - 5\sqrt{2}}$ km	1	
Hence		* Leave answers in exact form as long as possible - marker can follow; - avoid compounding errors.
$\sin\theta = \sin 45° \cdot \dfrac{5km}{\sqrt{26-5\sqrt{2}}}$		
$\Rightarrow \theta \doteqdot 54°21'$		
$\boxed{OR}$ $180° - 54°21'$ $= 125°39'$	1	* Most did not consider the possible complementary case. Must check. Also, should justify why one of two angles is chosen. →
Since $125°39' + 45° < 180°$,		
$125°39'$ is a possibility.		

MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments



Now, $\sqrt{26-5\sqrt{2}}$ km

$\doteqdot 4.35$ km.

The triangle indicates that θ should be the largest angle since θ is opposite the largest side.

If $\theta = 54°21'$, then

$\alpha = 180° - 45° - \theta$
$= 80°39'$

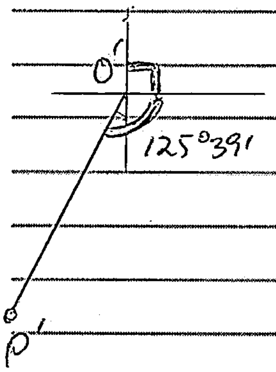
meaning $\alpha > \theta$ #

So, $\theta = 125°39'$.

Therefore, the bearing of P' from O' is

$90° + 125°39'$

$= 215°39'$ // — 1



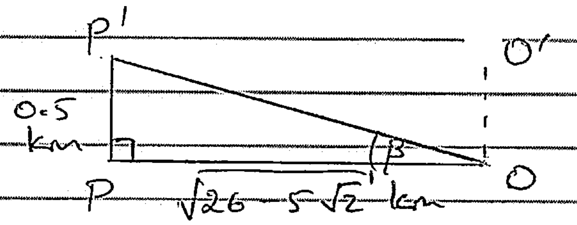
Marker's Comments:

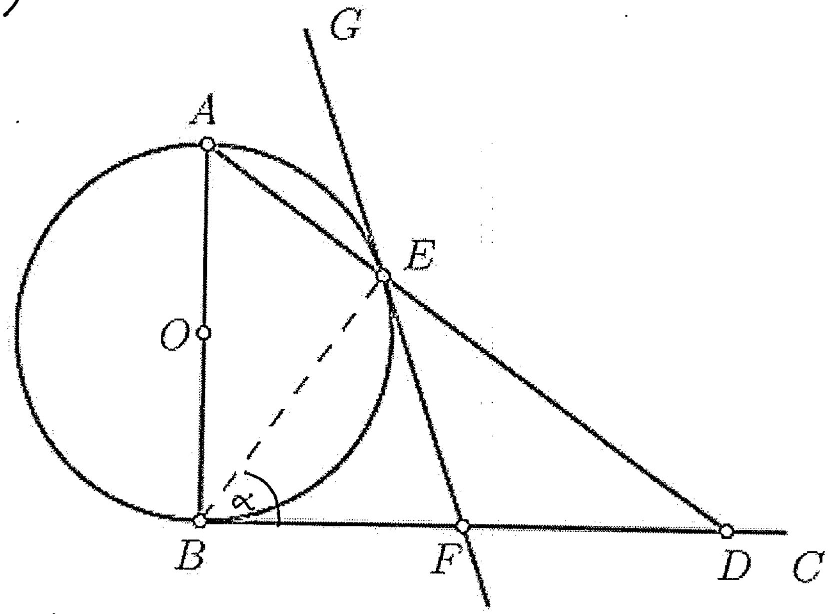
* When Δ has three sides given / determined, could use cosine rule & avoid 'trouble' with sine rule.

* Good idea to include diagrams in any geometrical proof – helps your case / makes your argument clearer.

* Some failed to recognise that 90° needed to be added for bearing.

MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
(iii) P' lies at a distance $\sqrt{26 - 5\sqrt{2}}$ km from O' & it is 0.5 km in altitude.		* Main issue: people mixing units. E.g. $\tan\beta = \frac{500}{4.35}$ not recognising that '500' is 500 m & '4.35' is 4.35 km.
So, 	1	
$\angle POP' = \beta$ will be angle of elevation.		
Then $\tan\beta = \dfrac{1/2}{\sqrt{26-5\sqrt{2}}\text{ km}}$		Either - make quantities uniform (same units) OR - Keep units in the expression.
$\Rightarrow \beta \doteqdot 6^\circ 33'$	1	

Suggested Solutions	Marks	Marker's Comments
(b) (i) 	1	
(ii) (1) Let $\angle EBF = \alpha$.		
(2) $\angle ABF = \pi/2$ (Diameter of circle is perpendicular to tangent at point of contact).	1	Mark awarded for appropriate start.
(3) $\angle ABF = \angle ABE + \angle EBF$ (adjacent angle sum)		
i.e. $\frac{\pi}{2} = \angle ABE + \alpha$ (by (1) & (2))		→
so, $\angle ABE = \frac{\pi}{2} - \alpha$		

MATHEMATICS Extension 1 : Question...11...

Suggested Solutions	Marks	Marker's Comments
(4) $\angle GEA = \angle ABE$ (angle between tangent & chord equals the angle at the circumference in the alternate segment).		* Every time you make a claim in a geometry proof, you must state why it is true.
So, $\angle GEA = \frac{\pi}{2} - \alpha$ (by (3)).		
(5) $\angle GEA = \angle FED$ (vertically opposite angles).		Don't need to justify algebraic or arithmetic operations.
Hence $\angle FED = \frac{\pi}{2} - \alpha$ (by (4)). //	1	

MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
(iii) (1) Let $\angle EBF = \alpha$.		
(2) BD is tangent (given),		
So, $\angle ABD = \frac{\pi}{2}$ (diameter of circle is perpendicular to tangent at point of contact).		
(3) Also, $\angle BAD = \angle EBF$ (angle between tangent & chord equals the angle at the circumference in the alternate segment)	1	* One mark for appropriate start.
As $\alpha = \angle EBF$ (by (1)), $\angle BAD = \alpha$.		* Don't mix degrees & radians in same expression (unless you attach <u>units</u>).
(4) $\angle BAD + \angle ABD + \angle ADB = \pi$ (angle sum plane triangle)		
So, $\alpha + \frac{\pi}{2} + \angle ADB = \pi$		
by (3), (2) respectively.		
i.e. $\angle ABD = \pi/2 - \alpha$		

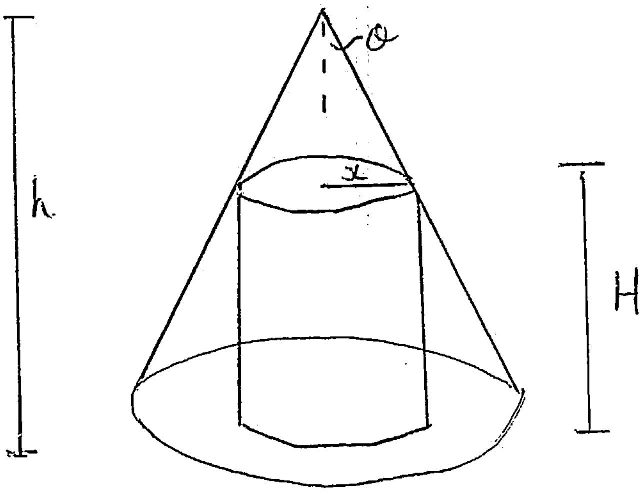
MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
(5) As $\angle FED = \frac{\pi}{2} - \alpha$ (proven in part (ii)) & $\angle ADB = \frac{\pi}{2} - \alpha$ (by (4)), $EF = FD$ (equal sides oppose equal angles in triangle)		* Much of the script (handwriting) was ILLEGIBLE. Care must be taken!
(6) Also, both BF, EF are tangents (given), so, $BF = EF$ (intercepts on tangents drawn from a common point to a circle are equal).		* Present your argument clearly; link relevant statements directly. Don't make assessor 'fish around' for facts to prove your case – that's your job.
(7) Hence by (5) & (6), $FD = EF = BF$ $\therefore BF = FD$ as required. //	1	

Mathematics Ext 1 Question 12

Suggested Solutions	Marks	Marker's Comments

12. a) i



$$V_{cylinder} = \pi x^2 H$$

$$\tan\theta = \frac{x}{h-H}$$ (1)

$$\therefore \cot\theta = \frac{h-H}{x}$$

$$h - H = x\cot\theta$$

$$H = h - x\cot\theta$$

$$\therefore V_{cylinder} = \pi x^2(h - x\cot\theta)$$ (1)

$$= \pi(x^2 h - x^3\cot\theta)$$

Alternate Method

Let the radius of cone by R

$$\frac{h-H}{h} = \frac{x}{R}$$ (corresponding sides of similar triangles) (1)

but $R = h\tan\theta$

Mathematics ________ Question ______

Suggested Solutions	Marks	Marker's Comments
$\therefore \frac{h-H}{h} = \frac{x}{h\tan\theta}$		
$\therefore (h-H)\tan\theta = x$		
$h-H = x\cot\theta$		
$H = h - x\cot\theta$		
$\therefore V_{cylinder} = \pi x^2(h - x\cot\theta)$		
$V_{cylinder} = \pi(x^2h - x^3\cot\theta)$		
ii $V = \pi(x^2h - x^3\cot\theta)$		
$\frac{dV}{dx} = \pi(2xh - 3x^2\cot\theta)$		
$\frac{dV}{dx} = 0 \Rightarrow 2xh - 3x^2\cot\theta = 0$		
$x(2h - 3x\cot\theta) = 0$		
$\therefore x = 0$ or $x = \frac{2h\tan\theta}{3}$		1
But x is a length and greater than 0.		
$\therefore x = \frac{2h\tan\theta}{3}$ only.		
$\frac{d^2V}{dx^2} = \pi(2h - 6x\cot\theta)$		
When $x = \frac{2h\tan\theta}{3}$,		
$\frac{d^2V}{dx^2} = \pi\left(2h - \frac{6\cot\theta \times 2h\tan\theta}{4}\right)$		
$= -2\pi h$		
< 0 $\therefore$ maximum at $x = \frac{2h\tan\theta}{3}$		1

Mathematics ______ Question ______		
Suggested Solutions	Marks	Marker's Comments

When $x = \frac{2h\tan\theta}{3}$

$$H = h - \left(\frac{2h\tan\theta}{3}\right)\cot\theta$$

$$= h - \frac{2h}{3}$$

$H = \frac{h}{3}$ gives a maximum ——— 1

b) i) $\cos(a+b) = \cos a\cos b - \sin a\sin b$,

When $a = b = q$,

$$\cos(2q) = \cos q\cos q - \sin q\sin q$$

$$= \cos^2 q - \sin^2 q$$

$$= (1 - \sin^2 q) - \sin^2 q$$

$$= 1 - 2\sin^2 q$$ ——— 1

ii) $\text{LHS} = \frac{\cos y - \cos(y+2q)}{2\sin q}$

$$= \frac{\cos y - (\cos y\cos 2q - \sin y\sin 2q)}{2\sin q}$$ ——— 1

$$= \frac{\cos y(1-\cos 2q) + \sin y\sin 2q}{2\sin q}.$$

$$= \frac{\cos y\, 2\sin^2 q + \sin y\, 2\sin q\cos q}{2\sin q}$$

$$= \cos y\sin q + \sin y\cos q$$ ——— 1

$$= \sin(y+q)$$

Mathematics _______ Question _____

Suggested Solutions	Marks	Marker's Comments

iii) Test for $n=1$,

$$LHS = \sin q$$

$$RHS = \frac{1-\cos 2q}{2\sin q}$$

$$= \frac{1-[1-2\sin^2 q]}{2\sin q}$$

$$= \frac{2\sin^2 q}{2\sin q}$$

$$= \sin q$$

$$= LHS$$

$\therefore$ The result is true for $n=1$ ——— 1

Assume the result is true for $n=k$, $k \in Z^+$

$$\therefore \sin q + \sin 3q + \ldots \sin(2k-1)q = \frac{1-\cos 2kq}{2\sin q}$$

Method 1

Test for $n=k+1$,

$$\therefore LHS = \sin q + \sin 3q + \ldots \sin(2k-1) + \sin(2k+1)q$$

$$= \frac{1-\cos 2kq}{2\sin q} + \sin(2kq+q) \text{ (By assumption)}$$

$$= \frac{1-\cos 2kq}{2\sin q} + \frac{\cos(2kq)-\cos(2kq+2q)}{2\sin q} \left(\text{from part ii when } y=2kq\right)$$ ——— 1

$$= \frac{1-\cos 2(k+1)q}{2\sin q}$$

$\therefore$ If the result is true for k, it is true for $k+1$,

$\therefore$ The result is true by Mathematical Induction. ——— 1

Mathematics ______ Question ______		
Suggested Solutions	Marks	Marker's Comments

(iii) Alternate Method

Test for $n=k+1$

$$LHS = \sin q + \sin 3q + \ldots \sin(2k-1) + \sin(2k+1)$$

$$= \frac{1-\cos 2kq}{2\sin q} + \sin(2kq+q) \quad \text{(By assumption)}$$

$$= \frac{1-\cos 2kq}{2\sin q} + (\sin 2kq\cos q + \cos 2kq\sin q) \qquad 1$$

$$= \frac{1-\cos 2kq + 2\sin q\cos q\sin 2kq + 2\sin^2 q\cos 2kq}{2\sin q}$$

$$= \frac{1-\cos 2kq + \sin 2q\sin 2kq + (1-\cos 2q)\cos 2kq}{2\sin q}$$

$$= \frac{1-\cos 2kq + \sin 2q\sin 2kq + \cos 2kq - \cos 2q\cos 2kq}{2\sin q}$$

$$= \frac{1-(\cos 2q\cos 2kq - \sin 2q\sin 2kq)}{2\sin q} \qquad 1$$

$$= \frac{1-\cos(2kq+2q)}{2\sin q}$$

$$= \frac{1-\cos 2(k+1)q}{2\sin q}$$

$\therefore$ If the result is true for k, it is true for $k+1$,

$\therefore$ The result is true by Mathematical Induction