Student Number ____________________

2020

HSC Assessment 1 Term 4

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 80 minutes
- Board approved calculators may be used.
- Write using black or blue pen
- All necessary working should be shown in all questions
- Write your student number at the top of every page of your answer sheets.
- NESA Reference Sheet is provided

Total marks – 68

Attempt all questions

Start each question on a new sheet of paper.

NESA Reference Sheet is provided

This paper MUST NOT be removed from the examination room

Section A Multiple Choice (5 Marks)

QUESTION 1

It is known that $(x+2)$ is a factor of the polynomial $P(x)$ and that

$$P(x) = (x^2 + x + 1) \times Q(x) + (2x + 3)$$

for some polynomial $Q(x)$. From this information alone, it can be deduced that:

(A) $Q(-2) = -\frac{1}{3}$

(B) $Q(-2) = \frac{1}{3}$

(C) $Q(2) = -1$

(D) $Q(2) = 1$

QUESTION 2

The roots of the polynomial $4x^3 + 4x - 5 = 0$ are α, β and γ.

What is the value of $(\beta + \gamma - 3\alpha)(\alpha + \beta - 3\gamma)(\gamma + \alpha - 3\beta)$?

(A) -80

(B) 80

(C) -16

(D) 16

QUESTION 3

If $\tan x = \frac{a}{b}$, then $b \cos 2x + a \sin 2x$ is equal to

(A) $\frac{a}{b}$

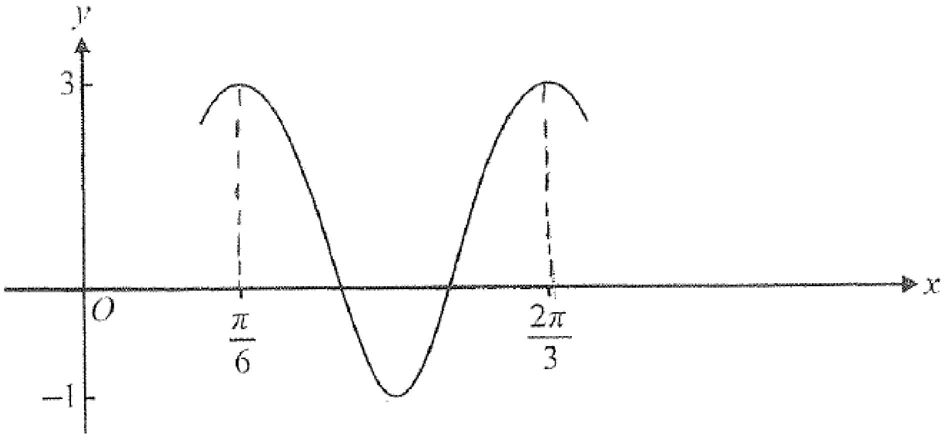
(B) $\frac{b}{a}$

(C) a

(D) b

QUESTION 4

The graph below could have the equation



(A) $y = 1 + 2cos(x - \frac{\pi}{6})$

(B) $y = 1 + 2cos(x + \frac{\pi}{6})$

(C) $y = 1 + 2cos4(x - \frac{\pi}{6})$

(D) $y = 1 + 2cos4(x + \frac{\pi}{6})$

QUESTION 5

The number of values of x in $[0, 2\pi]$ that satisfies the equation

$$(\sin x)^2 - \cos x = \frac{1}{4}$$

(A) 1

(B) 2

(C) 3

(D) 4

QUESTION 6 Start a new page **9 marks**

(a) Solve $4cosx + 3sinx = -4$ using the result $t = tan\frac{x}{2}$ for $0^0 \leq x \leq 360^\circ$ **3**

(b) Solve $cos2x + cosx + 1 = 0$ for $0 \leq x \leq 2\pi$ **3**

(c) Prove that for all integers $n \geq 1$,

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}$$ **3**

QUESTION 7 Start a new page **9 marks**

(a) The numbers α, β and γ satisfy the equations

$$\alpha + \beta + \gamma = 3$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2.$$

(i) Find the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **2**

(ii) Explain why α, β and γ are the roots of the cubic equation **1**

$$x^3 - 3x^2 + 4x - 2 = 0.$$

(b) When $-2x^2 + ax + 4$ is divided by $x - 1$ and $2x + 1$ respectively, the remainders are equal. Find the value of a. **3**

(c) Use the substitution $t = tan\frac{x}{2}$ to show that

$$secx + tanx = tan(\frac{\pi}{4} + \frac{x}{2})$$ **3**

QUESTION 8 Start a new page **9 marks**

(a) Graph the polynomial function $P(x) = x^3 + 2x^2 - 5x - 6$ **3**

(b) Given that

$$5\cos\theta - 5\sqrt{3}\sin\theta = A cos(\theta + \alpha) \ ,$$

(i) Find values of A and α where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$ **2**

(ii) Hence find the maximum value of $5\cos\theta - 5\sqrt{3}\sin\theta + 15$ **1**

(c) Prove by mathematical induction that for all integers $n \geq 0$

$3^{4n} - 1$ is divisible by 80 **3**

QUESTION 9 Start a new page **9 marks**

(a) The polynomial $P(x) = x^4 + 7x^3 + 9x^2 - 27x + C$ has a triple zero.

(i) Determine the value of the triple zero **2**

(ii) Hence, find the value of *C* **1**

(iii) Factorise $P(x)$ **1**

(iv) Sketch the graph of $y = P(x)$ **1**

(b) (i) Sketch the curve $y = \frac{1}{2}\sin(2x - \pi) + 1$ for $0 \leq x \leq 2\pi$ **2**

(ii) Hence or otherwise find all the solutions of $\frac{1}{2}\sin(2x - \pi) = -\frac{1}{2}$ in an unrestricted domain **2**

QUESTION 10 Start a new page **9 marks**

(a) (i) Prove $\sin 3x = 3\sin x - 4\sin^3 x$ **2**

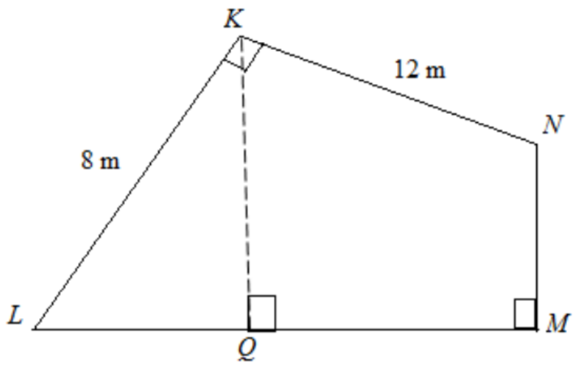
(ii) Hence or otherwise solve $3\sin x - 4\sin^3 x = \frac{\sqrt{3}}{2}$ in the domain $[0, 2\pi]$ **2**

(b) The diagram shows a pond KLMN in a park.

LK= 8 metres and KN = 12 metres.

$\angle$LKN = $\angle$LMN = 90º and $\angle$KLM = θ, where 0º < θ < 90º.

The perimeter of the pond is P metres and QK is perpendicular to LM.



(i) Show that the perimeter KLMN can be written in the form $P = a + b\cos\theta + d\sin\theta$ where $a, b,$ and d are $20, -4$ and 20 respectively **2**

(ii) Express P in the form of $a + R\sin(\theta - \beta)$, Where $R > 0$ and $0^0 < \beta < 90^0$, β to nearest minute **2**

(iii) Hence, find the value of θ (to nearest minute) when P=38 metres **1**

QUESTION 11 Start a new page **9 marks**

(a) A polynomial of degree n is given by $P(x) = x^n + ax - b$.

It is given that the polynomial has a double root at $x = a$.

Find the derived polynomial $P'(x)$ and show that $a^{n-1} = -\frac{a}{n}$ **2**

(b) Given that $81^{(sin^2x)} + 81^{(cos^2x)} = 30$

(i) Show that the above equation can be written in the form

$y^2 - 30y + 81 = 0$ where y = 81sin²x **1**

(ii) Solve $y^2 - 30y + 81 = 0$ **1**

(iii) Hence or otherwise, find all solutions of the equation

$81^{(sin^2x)} + 81^{(cos^2x)} = 30,$ where (0,π] **2**

(c) Use Mathematical Induction to prove for integers $n \geq 1$

$1(1!) + 2(2!) + 3(3!) + \ldots\ldots + n(n!) = (n+1)! - 1$ **3**

QUESTION 12 Start a new page **9 marks**

a) Given the identity $\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$.

(i) Deduce that $x = \sin\left(\frac{\pi}{10}\right)$ is one of the solutions to

$16x^5 - 20x^3 + 5x - 1 = 0$ **2**

(ii) Find the polynomial $P(x)$ such that

$(x-1)P(x) = 16x^5 - 20x^3 + 5x - 1$ **2**

(iii) Find the value of a such that $P(x) = (4x^2 + ax - 1)^2$ **1**

(iv) Hence, find an exact value for $\sin\left(\frac{\pi}{10}\right)$ **1**

(b) Prove by Mathematical Induction that $a^n - b^n$ is divisible by $a - b$ for all positive integers n. **3**

2020 HSC Assess 1 T4 Ext 1

MATHEMATICS Extension 1 : Question..MC..

Suggested Solutions	Marks	Marker's Comments

1. $P(x) = (x^2+x+1)Q(x) + (2x+3)$

$P(-2) = ((-2)^2+(-2)+1)Q(-2) + (2(-2)+3)$

$0 = 3Q(-2) - 1$

$3Q(-2) = 1$

$Q(-2) = \frac{1}{3}$ (B)

2. $\alpha+\beta+\gamma=0$

$\alpha\beta+\alpha\gamma+\beta\gamma=1$

$\alpha\beta\gamma = \frac{5}{4}$

$(\beta+\gamma-3\alpha)(\alpha+\beta-3\gamma)(\gamma+\alpha-3\beta)$

$=(\beta+\gamma+\alpha-4\alpha)(\alpha+\beta+\gamma-4\gamma)(\gamma+\alpha+\beta-4\beta)$

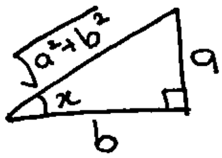
$=(0-4\alpha)(0-4\gamma)(0-4\beta)$

$=-64\alpha\beta\gamma$

$=-64\left(\frac{5}{4}\right)$

$=-80$ (A)

3. $\tan x = \frac{a}{b}$



$b\cos 2x + a\sin 2x$

$= b(\cos^2 x - \sin^2 x) + 2a\sin x\cos x$

$= b\left(\frac{b^2}{a^2+b^2} - \frac{a^2}{a^2+b^2}\right) + 2a\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right)$

$= \frac{b^3-a^2b}{a^2+b^2} + \frac{2a^2b}{a^2+b^2}$

$= \frac{b^3+a^2b}{a^2+b^2}$

$= \frac{b(b^2+a^2)}{a^2+b^2}$ (D)

$= b$

MATHEMATICS Extension 1 : Question..M.C		
Suggested Solutions	Marks	Marker's Comments
4. Period $= \frac{\pi}{2}$ $\therefore \frac{2\pi}{n} = \frac{\pi}{2}$ (C) $n = 4$ 5. $\sin^2 x - \cos x = \frac{1}{4}$ $1 - \cos^2 x - \cos x = \frac{1}{4}$ $4 - 4\cos^2 x - 4\cos x = 1$ $4\cos^2 x + 4\cos x - 3 = 0$ $\cos x = \frac{-4 \pm \sqrt{16 + 4(4)(3)}}{2 \times 4}$ $= \frac{1}{2}, -\frac{3}{2}$ $\cos x = \frac{1}{2}$ $\quad$ $\cos x = -\frac{3}{2}$ $x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\quad$ no solution. $\therefore$ 2 solutions (B)		

$$\frac{4(1-t^2)}{1+t^2} + \frac{6t}{1+t^2} = -4$$

$$4 - \cancel{4t^2} + 6t = -4 - \cancel{4t^2}$$

1 $6t = -8$

$t = -\frac{4}{3}$

$\tan\frac{x}{2} = -\frac{4}{3}$ for $0 \le \frac{x}{2} \le 180°$

1 Domain.

$\frac{x}{2} = 180 - 53°7'48.37\ldots$

$\frac{x}{2} = 126°52'11.6\ldots$

1 1 $x = 253°44'$ (nearest minute)

check 180°:

LHS $= 4(-1) + 3(0) = -4 =$ RHS so 180° is a solution

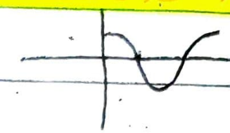
So $x = 180°, 253°44'$ (nearest minute).

$253.74°$ (2dp).

(b)

$\cos 2x + \cos x + 1 = 0 \qquad 0 \leq x \leq 2\pi$

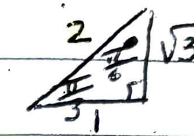
1 — $\cos^2 x - \sin^2 x + \cos x + 1 = 0$ $\qquad (2\cos^2 x - 1) + \cos x + 1 = 0.$



$\cos^2 x - (1 - \cos^2 x) + \cos x + 1 = 0$

$2\cos^2 x + \cos x = 0$

$\cos x(2\cos x + 1) = 0$



$\cos x = 0 \quad$ or $\quad 2\cos x + 1 = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \cos x = -\frac{1}{2}$

1

$x = \pi - \frac{\pi}{3}, \; \pi + \frac{\pi}{3}$

$= \frac{2\pi}{3}, \frac{4\pi}{3}$ — 1

$\therefore x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}.$

(c) RTP $1^2+3^2+5^2+\dots+(2n-1)^2 = \dfrac{4n^3-n}{3}$.

$n=1$ LHS $= (2(1)-1)^2$
$= 1$

RHS $= \dfrac{4(1)^3-(1)}{3}$
$= \dfrac{3}{3}$
$= 1$

LHS = RHS
Hence true for $n=1$

show substitution 1

Assume true for $n=k$ + prove true for $n=k+1$

$n=k$ $1^2+3^2+5^2+\dots+(2k-1)^2 = \dfrac{4k^3-k}{3}$

$n=k+1$

RTP $1^2+3^2+5^2+\dots+(2k-1)^2+(2(k+1)-1)^2 = \dfrac{4(k+1)^3-(k+1)}{3}$

LHS $= \dfrac{4k^3-k}{3} + (2(k+1)-1)^2$

$= \dfrac{4k^3-k}{3} + (2k+1)^2$

$= \dfrac{4k^3-k+3(4k^2+4k+1)}{3}$

$= \dfrac{4k^3-k+12k^2+12k+3}{3}$

$= \dfrac{4k^3+12k^2+11k+3}{3}$

$=$ RHS

RHS $= \dfrac{4(k^3+3k^2+3k+1)-k-1}{3}$

$= \dfrac{4k^3+12k^2+11k+3}{3}$

$= \dfrac{4k^3+12k^2+12k+4-k-1}{3}$

Correct assumption 1

show all steps 1

Since true for $n=1$ it is true for $n=1+1$ ie $n=2$ & so on for all positive integral $n \geq 1$

MATHEMATICS Extension 1 : Question...7...		
Suggested Solutions	**Marks**	**Marker's Comments**
a) i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$		
$1 = 3^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$		
$\therefore \alpha\beta+\alpha\gamma+\beta\gamma = \dfrac{3^2-1}{2}$		
$= 4$	1	
$\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma} = 2$		
$\dfrac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma} = 2$		
$\therefore \alpha\beta\gamma = \dfrac{4}{2}$		
$= 2$	1	
$\therefore \alpha\beta+\alpha\gamma+\beta\gamma = 4$		
$\alpha\beta\gamma = 2$		
ii) $x^3-3x^2+4x-2=0$		
let the roots be a, b, c		
sum of roots $a+b+c=3$		
from (i) $\alpha+\beta+\gamma=3$		
sum of paired roots $ab+ac+bc=4$		
from (i) $\alpha\beta+\alpha\gamma+\beta\gamma=4$		must have some
product of roots $abc=2$	1	explanation and
from (i) $\alpha\beta\gamma=2$		show $\Sigma\alpha$, $\Sigma\alpha\beta$
$\therefore \alpha=a$, $\beta=b$ and $\gamma=c$		and $\alpha\beta\gamma$.
$\therefore \alpha, \beta$ and γ are the roots of		
$x^3-3x^2+4x-2=0$		
b) let $P(x) = -2x^2+ax+4$		
$P(1) = -2(1)^2+a(1)+4$		
$= 2+a$	1	
$P(-\tfrac{1}{2}) = -2(-\tfrac{1}{2})^2+a(-\tfrac{1}{2})+4$		
$= \dfrac{7}{2}-\dfrac{a}{2}$	1	
$P(1) = P(-\tfrac{1}{2})$		
$2+a = \dfrac{7}{2}-\dfrac{a}{2}$		
$\dfrac{3}{2}a = \dfrac{3}{2}$		

$\therefore a = 1$ 1

MATHEMATICS Extension 1 : Question…7…		
Suggested Solutions	**Marks**	**Marker's Comments**
c) $\sec x + \tan x = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$		
LHS $= \sec x + \tan x$		
$= \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}$		
$= \frac{t^2+2t+1}{1-t^2}$	1	
$= \frac{(t+1)^2}{(1-t)(1+t)}$		
$= \frac{t+1}{1-t}$	1	
$= \frac{1+t}{1-1\times t}$		
$= \frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}$	1	for recognising that $\tan\frac{\pi}{4} = 1$ and writing the compound angle correctly.
$= \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$		
$=$ RHS		

MATHS EXTENSION I

HSC Assessment 1 – Term 4 (2020)

Questions 8

(a) Graph the polynomial function $P(x) = x^3 + 2x^2 - 5x - 6$

(b) Given that: $5cos\theta - 5\sqrt{3}\ sin\theta = Acos(\theta + \alpha)$,

i. Find values of A and α where A > 0 and 0 < α < $\frac{\pi}{2}$

ii. Hence find the maximum value of $5cos\theta - 5\sqrt{3}\ sin\theta + 15$

(c) Prove by mathematical induction that for all integers $n \geq 0$, $3^{4n} - 1$ **is divisible by 80**

Q8(a)

Find x-intercepts

$$P(x) = x^3 + 2x^2 - 5x \ - 6$$

$$P(1) = 1 + 2 - 5 - 6$$

$$P(-1) = -1 + 2 + 5 - 6$$

$\therefore\ x + 1$ is a factor

$$P(x) = (x + 1)\left(x^2 + x - 6\right)$$

$$= (x + 1)(x + 3)(x - 2)$$

$$\boxed{\therefore\ x - intercepts\ are\ \ -1, -3\ and\ 2}$$

1

Q8(a)

Find y-intercept

$$y\text{-intercept } @ \ x = 0$$

$$P(0) = 0^3 + 2(0)^2 - 5(0) - 6$$

$$P(0) = -6$$

$$\boxed{\therefore y - intercept\ is\ -6}$$

1

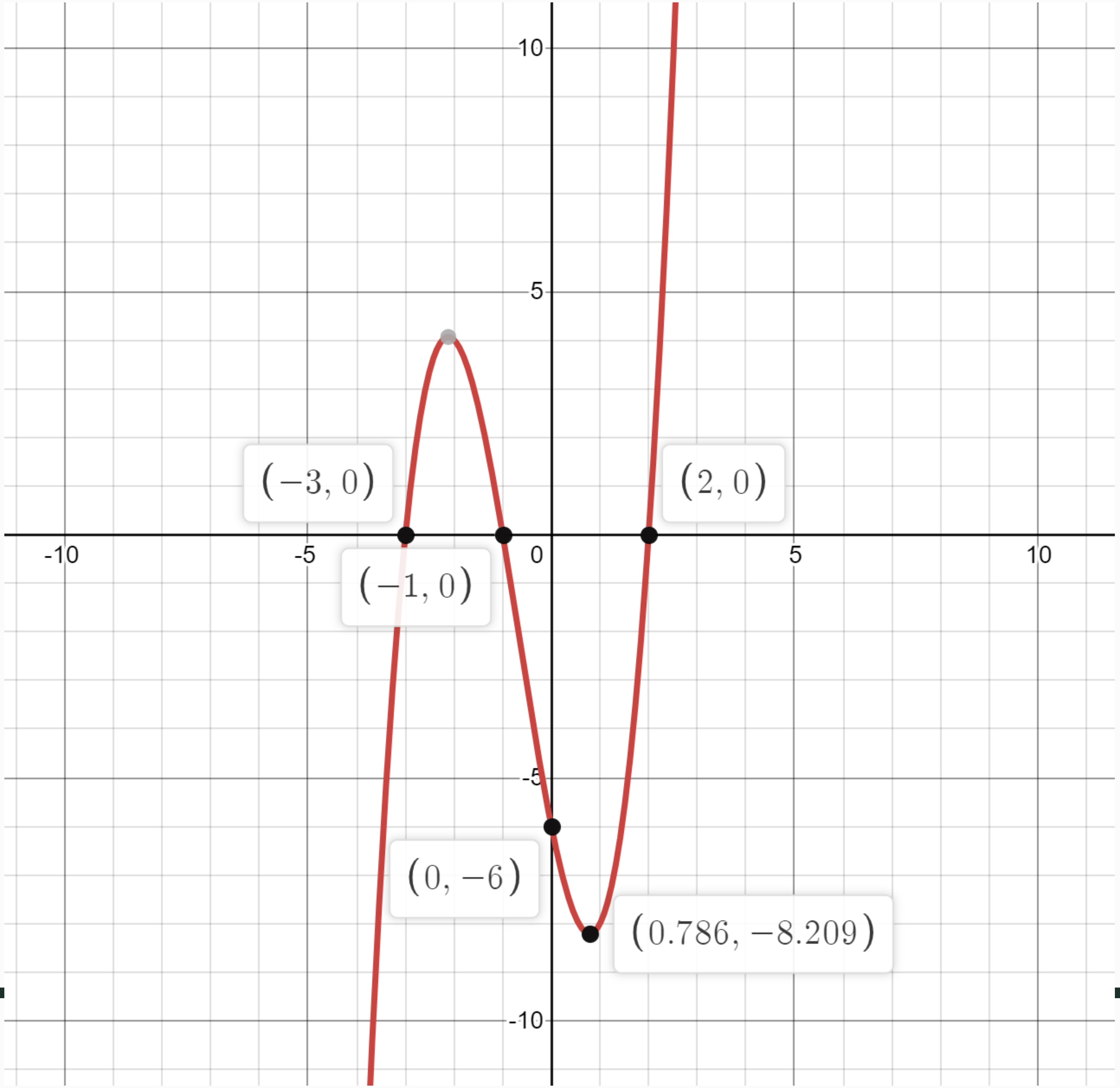
Finding additional points

$$P\left(\frac{1}{2}\right) = -\frac{63}{8}$$

$$P(-2) = 4$$

Graph

1. x-intercepts
1. y-intercepts
1. shape



Q8(b)(i)

$$5\cos\theta - 5\sqrt{3}\sin\theta = A\cos(\theta + \alpha)$$

$$5\cos\theta - 5\sqrt{3}\sin\theta = A\cos\theta\cos\alpha - A\sin\theta\sin\alpha$$

$$5 = A\cos\alpha$$

$$\cos\alpha = \frac{5}{A}$$

$$-5\sqrt{3} = -A\sin\alpha$$

$$\sin\alpha = \frac{5\sqrt{3}}{A}$$

$$A = \sqrt{5^2 + \left(5\sqrt{3}\right)^2}$$

$$= \sqrt{100}$$

$$= 10$$

Since A > 0

1

$$\tan\alpha = \frac{5\sqrt{3}}{5}$$

$$= \sqrt{3}$$

$$\alpha = \frac{\pi}{3}$$

Since $0 < \alpha < \frac{\pi}{2}$

1

Q8(b)(ii)

$$5cos\theta - 5\sqrt{3}sin\theta = 10cos\left(\theta + \frac{\pi}{3}\right) + 15$$

$$\therefore maximum\ value = 10 + 15$$

$$\boxed{= 25}$$

1

Q8(c)

Prove $3^{4n}-1$ is divisible by 80.

Test $n=0$

$$3^{4n}-1=3^{4(0)}-1$$
$$=1-1$$
$$=0 \text{ (which is divisible by 80)}$$

1

True for n = 0

Assume true for $n=k$ $(where\ k\ is\ some\ positive\ integer)$:

$$3^{4k}-1=80M \text{ (M is an integer)}$$

Prove true for $n=k+1$:

$$3^{4(k+1)}-1=3^{4k+4}-1$$
$$=3^{4k}\times 3^{4}-1$$
$$=3^{4k}\times 81-1$$
$$=(3^{4k}-1)\times 81+80$$

1

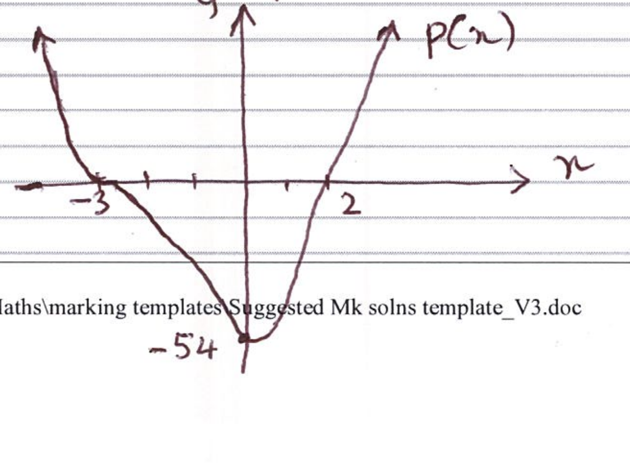
$$=80M\times 81+80$$
$$=80(81M+1)$$

1

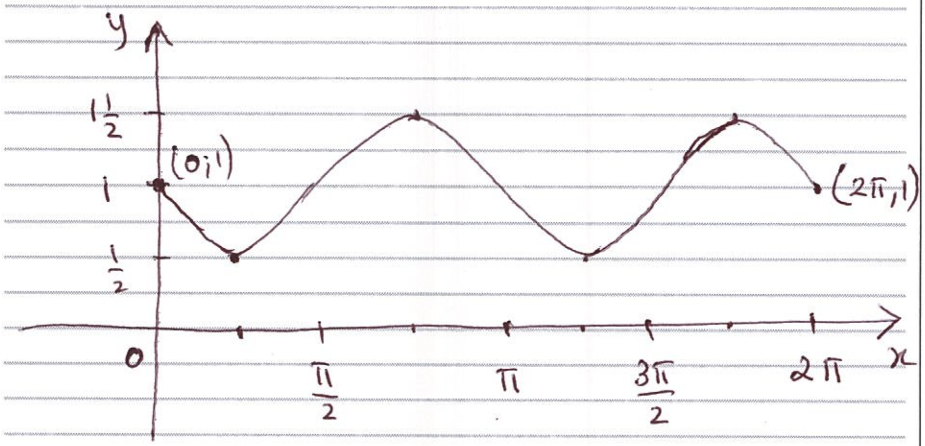
Which is divisible by 80.
True for n = k + 1

Since true for n = 0, by PMI, true for any integral ≥ 0.

MATHEMATICS Extension 1 : Question 9

Suggested Solutions	Marks	Marker's Comments
Question 9:		
(a) $p(x) = x^4 + 7x^3 + 9x^2 - 27x + C$		
(i) $p(x)$ has a triple zero		
let α is the triple zero		
$\therefore p(\alpha) = p'(\alpha) = p''(\alpha) = 0$		
$p'(x) = 4x^3 + 21x^2 + 18x - 27$		
$p''(x) = 12x^2 + 42x + 18$		
$= 6(2x+1)(x+3)$	1	
$p''(\alpha) = 0 \therefore 6(2\alpha+1)(\alpha+3) = 0$		
$\therefore \alpha = \frac{-1}{2}$ or $\alpha = -3$		
$p'\left(\frac{-1}{2}\right) = 4\left(\frac{-1}{2}\right)^3 + 21\left(\frac{-1}{2}\right)^2 + 18\left(\frac{-1}{2}\right) - 27$		
$= \frac{-125}{4} \neq 0$		
$\therefore -\frac{1}{2}$ is not a triple zero		Need to show all working to get the mark.
$p'(-3) = 4(-3)^3 + 21(-3)^2 + 18(-3) - 27$	1	
$= 0 \therefore \alpha = -3$ is the triple zero		
(ii) $p(-3) = 0 \therefore (-3)^4 + 7(-3)^3 + 9(-3)^2 - 27(-3) + C = 0$		
$C = -54$	1	
(iii) $p(x) = (x+3)^3(x-2)$		
since sum of roots $= -7$	1	
(iv) 	1	

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments
(b) (i) $y = \frac{1}{2}\sin(2x - \pi) + 1$ 	2	1 for shape 1 for max/min and y-intercept
(ii) $\frac{1}{2}\sin(2x-\pi) = -\frac{1}{2}$ $\sin(2x-\pi) = -1 = \sin\left(-\frac{\pi}{2}\right)$ $2x - \pi = -\frac{\pi}{2} + 2k\pi \quad k \in \mathbb{Z}$ $2x = \frac{\pi}{2} + 2k\pi$ $x = \frac{\pi}{4} + k\pi \quad k \in \mathbb{Z}$ Graphically, looking for solutions when $y = \frac{1}{2}$ $\therefore x = \ldots, -\frac{11\pi}{4}, -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots$	2	1 mark for some solutions (at least one) correct The two marks only for full correct solutions.

Question 10

Sunday, 29 November 2020 5:14 pm

a) (i) $\sin 3x = \sin(2x + x)$

$= \sin 2x \cos x + \cos 2x \sin x$

$= 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x$

$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

① expansion and double angle

① simplification involving Pythagorean identity

(ii) $\sin 3x = \frac{\sqrt{3}}{2}$ $\qquad 0 \le 3x \le 6\pi$

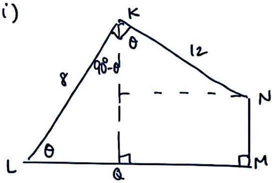
$3x = \pi/3,\ 2\pi/3,\ 7\pi/3,\ 8\pi/3,\ 13\pi/3,\ 14\pi/3$

$x = \pi/9,\ 2\pi/9,\ 7\pi/9,\ 8\pi/9,\ 13\pi/9,\ 14\pi/9$

① 4 solutions

① 6 solutions

b) (i)



Students should not introduce pronomeral without showing or explaining very clearly where it is.

$\cos\theta = \frac{LQ}{8}$ $\qquad \sin\theta = \frac{QM}{12}$

$LQ = 8\cos\theta$ $\qquad QM = 12\sin\theta$

① LQ, QM one in each triangle

$NM = 8\sin\theta - 12\cos\theta$

① NM + P

$P = 8 + 12 + 8\sin\theta - 12\cos\theta + 12\sin\theta + 8\cos\theta$

$= 20 + 20\sin\theta - 4\cos\theta$

$= 20 - 4\cos\theta + 20\sin\theta$

(ii) $R\sin(\theta - \beta) = R\sin\theta\cos\beta - R\cos\theta\sin\beta = 20\sin\theta - 4\cos\theta$

$R\cos\beta = 20$

$R\sin\beta = 4$

$R = \sqrt{16 + 400}$

$= \sqrt{416}$

① R & β

$\beta = \tan^{-1}\frac{1}{5}$

$= 11°19'$

① P

$\therefore\ P = 20 + \sqrt{416}\sin(\theta - 11°19')$

(iii) $\sqrt{416}\sin(\theta - 11^\circ 19') = 18$

$\sin(\theta - 11^\circ 19') = \frac{18}{\sqrt{416}}$

$\theta - 11^\circ 19' = 61^\circ 57'$

$\theta = 73^\circ 16'$ (1)

(or $73^\circ 15'$ using exact values)

MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
a) $P(x) = x^n + ax - b$		
$P'(x) = nx^{n-1} + a$	1	
$P'(a) = 0$ (Since $x=a$ is a double root)		
$\therefore na^{n-1} + a = 0$		
$na^{n-1} = -a$		
$a^{n-1} = \dfrac{-a}{n}$	1	
b) i $y = 81^{\sin^2 x}$		
$y = 81^{\cos^2 x} = 81^{1-\sin^2 x} = \dfrac{81}{81^{\sin^2 x}}$		
$\therefore 81^{\sin^2 x} + \dfrac{81}{81^{\sin^2 x}} = 30$		
$(81^{\sin^2 x})^2 + 81 = 30 \times 81^{\sin^2 x}$		
let $y = 81^{\sin^2 x}$		
$\therefore y^2 + 81 = 30y$		
$y^2 - 30y + 81 = 0$	1	

MATHEMATICS Extension 1 : Question ...11...

Suggested Solutions	Marks	Marker's Comments
ii) $y^2 - 30y + 81 = 0$		
$(y-27)(y-3) = 0$		
$y = 27$ or 3	1	
iii) Solve $81^{\sin^2 x} = 27$ or 3.		
$\therefore \sin^2 x = \frac{3}{4}$ or $\frac{1}{4}$		
$\therefore \sin x = \frac{\pm\sqrt{3}}{2}$ or $\pm\frac{1}{2}$	1	
$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$.	1	

MATHEMATICS Extension 1 : Question 11		
Suggested Solutions	**Marks**	**Marker's Comments**
c) Test for $n=1$,		
$LHS = 1(1!) = 1$ $RHS = (1+1)! - 1 = 2-1 = 1$		
$\therefore LHS = RHS$		
$\therefore$ The statement is true for $n=1$	1	
Assume the statement is true for $n=k$, $k \in Z^+$		
$\therefore 1(1!) + 2(2!) + \dots k(k!) = (k+1)! - 1$		
Test for $n=k+1$		
RTP: $1(1!) + 2(2!) + 3(3!) + \dots (k+1)[(k+1)!] = (k+2)! - 1$		
$LHS = [1(1!) + 2(2!) + 3(3!) + \dots k(k!)] + (k+1)[(k+1)!]$		
$= [(k+1)! - 1] + (k+1)[(k+1)!]$ (By assumption)	1	
$= (k+1)![1 + (k+1)] - 1$		
$= (k+1)!(k+2) - 1$		
$= (k+2)!$		
$\therefore$ If the statement is true for $n=k$, it is true for $n=k+1$.		
$\therefore$ The statement is true for all positive integer n by the principle of mathematical induction.	1	

Mathematics Extension 1: Question 12

Solution	Marks	Marker's Comments
a) i) Correct Method	Total=2	The question asks to prove that $x = \sin\frac{\pi}{10}$ is one of the solutions. It means:
$16x^5 - 20x^3 + 5x - 1 = 0$		① $x = \sin\frac{\pi}{10}$ is a solution
let $x = \sin\theta$		② there are more than one solution
$\therefore 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta - 1 = 0$		
$\therefore \sin 5\theta - 1 = 0$ (given $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$)	1	
$\therefore \sin 5\theta = 1$		Many students forget to add $2k\pi$ to the solution, which means they fail to prove ②. Thus, they lose 1 mark.
$5\theta = \frac{\pi}{2} + 2k\pi \quad (k \in N)$		
$\theta = \frac{\pi}{10} + \frac{2}{5}k\pi$		
when $k = 0$, $\theta = \frac{\pi}{10}$		
$\therefore x = \sin\frac{\pi}{10}$ is one of the solutions	1	
Incomplete Method		This method can only receive up to 1 mark because it fails to prove ②.
when $x = \sin\frac{\pi}{10}$		
$16x^5 - 20x^3 + 5x - 1$		
$= 16\sin^5\frac{\pi}{10} - 20\sin^3\frac{\pi}{10} + 5\sin\frac{\pi}{10} - 1$		
$= \sin(5 \times \frac{\pi}{10}) - 1$ (given $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$)	1	
$= \sin\frac{\pi}{2} - 1 = 0$		

Mathematics Extension 1: Question 12

Solution	Marks	Marker's Comments
a) ii) Method 1 – long division $(x-1)P(x) = 16x^5 - 20x^3 + 5x - 1$ $P(x) = \dfrac{16x^5 - 20x^3 + 5x - 1}{x-1}$ $$\begin{array}{r} 16x^4 + 16x^3 - 4x^2 - 4x + 1 \\ x-1 \overline{)\, 16x^5 + 0x^4 - 20x^3 + 0x^2 + 5x - 1} \\ \underline{16x^5 - 16x^4} \\ 16x^4 - 20x^3 \\ \underline{16x^4 - 16x^3} \\ -4x^3 + 0x^2 \\ \underline{-4x^3 + 4x^2} \\ -4x^2 + 5x \\ \underline{-4x^2 + 4x} \\ x - 1 \\ \underline{x - 1} \\ 0 \end{array}$$ $\therefore P(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1$	Total = 2 2	This is not a difficult question, however, many students do not receive full marks, due to careless mistakes in using the long division method. Common miskes: ① "$-4x^2$" → "$4x^2$" ② "$-4x$" → "$4x$" ③ forget to copy the "–" sign when writing the expression of $P(x)$ Partial mark: 1 mark: if the process of long division method is demonstrated and 3 or 4 terms from $16x^4+16x^3-4x^2-4x+1$ are correct.

Mathematics Extension 1: Question 12

Solution	Marks	Marker's Comments
a) ii) Method 2 – inspection By inspection, $P(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1$	2	Partial Mark: If "By inspection" or a similar statement is missing but the $P(x)$ expression is correct – 1 mark – Any answer without showing working out or metioning "inspection method" but has the correct expression only receives 1 mark.
Method 3 – factor theorem From (i), the roots of the equation are $x = \sin\frac{\pi}{10}, \sin\frac{5\pi}{10}, \sin\frac{9\pi}{10}, \sin\frac{13\pi}{10}, \sin\frac{17\pi}{10}$ $\therefore 16x^5 - 20x^3 + 5x - 1$ $= 16(x-\sin\frac{\pi}{10})(x-\sin\frac{5\pi}{10})(x-\sin\frac{9\pi}{10})(x-\sin\frac{13\pi}{10})(x-\sin\frac{17\pi}{10})$ $= 16(x-\sin\frac{\pi}{10})(x-1)(x-\sin\frac{9\pi}{10})(x-\sin\frac{13\pi}{10})(x-\sin\frac{17\pi}{10})$ $P(x) = 16(x-\sin\frac{\pi}{10})(x-\sin\frac{9\pi}{10})(x-\sin\frac{13\pi}{10})(x-\sin\frac{17\pi}{10})$	1	A few students use this method. Some forget to include the leading coefficient 16 in the factor form, thus receive 0 mark. In order to receive 2 marks for this method, the final answer needs to be in expanded form.

Mathematics Extension 1: Question 12

Solution	Marks	Marker's Comments
a) iii) $P(x) = (4x + ax - 1)^2$ $= 16x^4 + 16x^3 - 4x^2 - 4x + 1$ $(4x^2 + ax + 1)^2$ $= 16x^4 + 4ax^3 - 4x^2 + 4ax^3 + a^2x^2 - ax - 4x^2 - ax + 1$ Method 1: equate the coefficients x term $= -ax - ax = -2ax = -4x$ $a = 2$ Method 2: equate the coefficients x^3 term $= 4ax^3 + 4ax^3 = 8ax^3 = 16x^3$ $a = 2$	Total=1 1	Note: the x or x^3 term has the simplest expression of a. It is a linear expression which avoid bringing additional solution like quadratic term does. x^2 term: $-4x^2 + a^2x^2 - 4x^2 = -4x^2$ $a^2 = 4$ $a = \pm 2$ Some students use the substitution method by subbing different values of x into the equation. Because $(4x^2 + ax - 1)^2$ has a square sign for the expression of a, it brings in an additional answer.

Mathematics Extension 1: Question 12

Solution	Marks	Marker's Comments
a) iv) $P(x) = 16x^5 - 20x^3 + 5x - 1$ $= (x-1)(4x^2+2x-1)^2 = 0$ $\therefore\ x-1=0$ or $4x^2+2x-1=0$ $x=1$ or $x = \frac{-2 \pm \sqrt{2^2+4\times 4}}{8}$ $= \frac{-2 \pm \sqrt{20}}{8}$ $= \frac{-2 \pm 2\sqrt{5}}{8}$ $= \frac{-1 \pm \sqrt{5}}{4}$ $\because \sin\frac{\pi}{10} > 0$ $\therefore \sin\frac{\pi}{10} = \frac{1+\sqrt{5}}{4}$	Total=1 1	Most students do well in this question. Common mistake: $\frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{8}$

Mathematics Extension 1: Question 12

Solution	Marks	Marker's Comments
b) let $n=1$	Total=3	The question should have restricted a and b to integers only.
$a^n - b^n = a - b = 1\times(a-b)$		
$1 \in N$, therefore, the statement is true when $n=1$	1	"A is divisible by B ($A, B \in N$)" means that $A = k\times B$, $k \in N$ not k is a polynomial.
Assume the statement is true for $n=k$ ($k \in N^*$), this is,		
$a^k - b^k = p(a-b)$ $\quad (p \in N)$	1	
$a^k = b^k + p(a-b)$ ① or		→ some students write it as "$-b^k+p(a-b)$" or "$a^k-p(a-b)$"
$b^k = a^k - p(a-b)$ ②		
When $n=k+1$		
$a^{k+1} - b^{k+1} = a^k \times a - b^k \times b$		Some students do not prove the case when $n=k+1$ by using the statement from $n=k$, which is against the method of mathematical induction.
(from ①) $= [b^k + p(a-b)]\times a - b^k \times b$		
$= b^k \times a + ap(a-b) - b^k \times b$		
$= b^k(a-b) + ap(a-b)$		
$= (b^k + ap)(a-b)$	1	
∵ $b^k + ap \in N$ ∴ the statement is true when $n=k+1$		
or (from ②) $= a^k \times a - [a^k + p(a-b)]\times b$		
$= a^k \times a - a^k \times b + bp(a-b)$	1	
$= a^k(a-b) + bp(a-b)$		
$= (a^k + bp)(a-b)$		
∵ $a^k + bp \in N$ ∴ the statement is true when $n=k+1$		
∴ By mathematical induction, the		

statement is true for all positive integers n